

Simple Derivation of the Ward Identity

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I. PREFACE

In this report we summarize a simple and elegant derivation of the Ward Identity provided by Nima Arkani-Hamed in his lecture series on Scattering Amplitudes at Cornell University.

II. POLARIZATION VECTORS

The defining equation for the vector potential without choosing a gauge is given by

$$\partial_\nu \partial^\nu A_\mu(x) = 0 \quad (1)$$

This equation is solved by the ansatz,

$$A_\mu(x) = \int \frac{d^4 p}{(2\pi)^4} \epsilon_\mu(p) e^{-ip \cdot x} \quad (2)$$

The gauge transformation of the vector potential under which physics is invariant is,

$$A'_\mu = A_\mu + \partial_\mu \alpha \quad (3)$$

Plugging in the potential as well as the Fourier transform of $\alpha = \int \frac{d^4 p}{(2\pi)^4} e^{-ip \cdot x} \tilde{\alpha}(p)$ we get

$$\int \frac{d^4 p}{(2\pi)^4} (\epsilon'_\mu - \epsilon_\mu) e^{-ip \cdot x} = \int \frac{d^4 p}{(2\pi)^4} (-ip_\mu) e^{-ip \cdot x} \tilde{\alpha} \quad (4)$$

Now multiplying by $e^{-ip' \cdot x}$ and integrating over $d^4 x$ we have

$$\int \frac{d^4 p}{(2\pi)^3} (\epsilon'_\mu - \epsilon_\mu) \delta(p - p') = -i \int \frac{d^4 p}{(2\pi)^4} p_\mu \tilde{\alpha} \delta(p - p') \quad (5)$$

$$\epsilon'_\mu(p) = \epsilon_\mu(p) - i \tilde{\alpha}(p) p_\mu \quad (6)$$

This is the equivalence of equation 3 for polarization vectors.

III. DERIVATION OF THE WARD IDENTITY

In general any amplitude with N gauge bosons will take the form,

$$\epsilon^{\mu_1} \epsilon^{\mu_2} \dots \epsilon^{\mu_N} \mathcal{M}_{\mu_1 \mu_2 \dots \mu_N} \quad (7)$$

If a theory is gauge invariant then this amplitude must be invariant under any gauge transformation. Applying equation 6 to some polarization vector i we have

$$\epsilon^{\mu_1} \epsilon^{\mu_2} \dots \epsilon^{\mu_N} \mathcal{M}_{\mu_1 \mu_2 \dots \mu_N} \rightarrow \epsilon^{\mu_1} \epsilon^{\mu_2} \dots \epsilon^{\mu_N} \mathcal{M}_{\mu_1 \mu_2 \dots \mu_N} - i \tilde{\alpha} p^{\mu_i} \mathcal{M}_{\mu_1, \dots, \mu_N} \quad (8)$$

In order for the amplitude to be gauge invariant we must have the famous Ward Identity,

$$\boxed{p^{\mu_i} \mathcal{M}_{\mu_1, \dots, \mu_N} = 0} \quad (9)$$