USEFUL RELATIONS IN QUANTUM FIELD THEORY

In this set of notes I summarize many useful relations in Quantum Field Theory that I was sick of deriving or looking up in the "correct" conventions (see below for conventions)!

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Chapter 1

Introduction

In this note I summarize many important relations I constantly look up throughout my time working in Particle Theory and in particular calculating Feynman diagrams. I try to derive some of these relationships if the derivations are straightforward but many are just quoted. One of the most frustrating events for me is to find some formula and not know what conventions they are using. In this report I follow the Peskin and Schroeder conventions which I detail in the next sections.

1.1 Conventions

$$g_{\mu\nu} = \mathrm{diag}\{+ - - -\} \tag{1.1}$$

The gamma matrices are,

$$\gamma_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma_i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} \tag{1.2}$$

Natural units are used throughout. The covariant derivatives are given by,

$$D_\mu \equiv \partial_\mu - igT^a A^a_\mu \tag{1.3}$$

and we include a 1/2 in the hypercharge definitions such that,

$$Q = T_3 + \frac{Y}{2} \tag{1.4}$$

The higgs VEV is $v \approx 246$ GeV. We also take $e < 0$ throughout as used in Peskin and Schroeder.

At this point the notes are awfully disorganized. I hope to fix that in the future. However, if you find any errors please let me know at ajd268@cornell.edu.

Chapter 2

Classical Field Theory

2.1 Important Relations

The Euler-Lagrangian equations of motion are

$$\frac{\partial \mathcal{L}}{\partial \phi} + \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = 0 \tag{2.1}$$

The conjugate momenta of the field is given by

$$\pi = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)} \tag{2.2}$$

The Hamiltonian is given by

$$H = \int d^3x \pi \partial_0 \phi - \mathcal{L} \tag{2.3}$$

The formula for the current is

$$j^\mu(x) = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi - \mathcal{J}^\mu \tag{2.4}$$

where $\mathcal{J}^\mu$ found by finding the change in $\mathcal{L}$ through a Taylor expansion.

The energy-momentum tensor is given by

$$T^\mu{}_\nu \equiv \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\nu \phi - \mathcal{L} \delta^\mu{}_\nu \tag{2.5}$$

2.2 Free Real Scalar Field

The Klien Gordan Lagrangian for a real scalar field is

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - m^2 \phi^2 \tag{2.6}$$

Quantizing the fields gives

$$\phi(\mathbf{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\mathbf{p}}}} \left(a_{\mathbf{p}} e^{i\mathbf{p}\cdot\mathbf{x}} + a_{\mathbf{p}}^{\dagger} e^{-i\mathbf{p}\cdot\mathbf{x}} \right) \tag{2.7}$$

$$\pi(\mathbf{x}) = \int \frac{d^3p}{(2\pi)^3} (-i)\sqrt{\omega_{\mathbf{p}}2} \left(a_{\mathbf{p}} e^{i\mathbf{p}\cdot\mathbf{x}} - a_{\mathbf{p}}^{\dagger} e^{-i\mathbf{p}\cdot\mathbf{x}} \right) \tag{2.8}$$

or in an equivalent but more convenient form,

$$\phi(\mathbf{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\mathbf{p}}}} \left(a_{\mathbf{p}} + a_{-\mathbf{p}}^{\dagger} \right) e^{i\mathbf{p}\cdot\mathbf{x}} \tag{2.9}$$

$$\pi(\mathbf{x}) = \int \frac{d^3p}{(2\pi)^3} (-i)\sqrt{\omega_{\mathbf{p}}2} \left(a_{\mathbf{p}} - a_{\mathbf{p}}^{\dagger} \right) e^{i\mathbf{p}\cdot\mathbf{x}} \tag{2.10}$$

and the commutation relations are

$$\left[a_{\mathbf{p}}, a_{\mathbf{p}'}^{\dagger} \right] = (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{p}') \tag{2.11}$$

as well as

$$[\phi(\mathbf{x}), \pi(\mathbf{x}')] = i\delta^{(3)}(\mathbf{p} - \mathbf{p}') \tag{2.12}$$

$$[\phi(\mathbf{x}), \phi(\mathbf{x}')] = [\pi(\mathbf{x}), \pi(\mathbf{x}')] = 0 \tag{2.13}$$

2.3 Free Complex Scalar Field

$$\phi = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \left(a_{\mathbf{p}} e^{i\mathbf{p}\cdot\mathbf{x}} + b_{\mathbf{p}}^{\dagger} e^{-i\mathbf{p}\cdot\mathbf{x}} \right) \tag{2.14}$$

$$\phi^{\dagger} = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \left(a_{\mathbf{p}}^{\dagger} e^{-i\mathbf{p}\cdot\mathbf{x}} + b_{\mathbf{p}} e^{i\mathbf{p}\cdot\mathbf{x}} \right) \tag{2.15}$$

$$T^{\mu}{}_{\nu} = \partial^{\mu}\phi^{*}\partial_{\nu}\phi + \partial^{\mu}\phi\partial_{\nu}\phi^{*} - \delta^{\mu}{}_{\nu}\mathcal{L} \tag{2.16}$$

$$H = \int d^3x \left(\pi^{\mu}\pi + \partial_i\phi^{*}\partial_i\phi + m^2\phi^{*}\phi \right) \tag{2.17}$$

2.4 Free Dirac Field

$$\psi = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_s \left(a_{\mathbf{p}}^s e^{-ip\cdot x} u_{\mathbf{p}}^s + b_{\mathbf{p}}^{s\,\dagger} e^{ip\cdot x} v_{\mathbf{p}}^s \right) \tag{2.18}$$

2.5 Electromagnetic field

The electromagnetic field tensor,

$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu \tag{2.19}$$

can be broken down in terms of the electric and magnetic fields. We have (in natural units, of course),

$$F_{00} = F_{ii} = 0 \tag{2.20}$$

$$F_{0i} = \partial_0 A_i - \partial_i A_0 = -\partial_t A^i - \partial_i A_t = E^i \tag{2.21}$$

$$F_{ij} = \partial_i A_j - \partial_j A_i = \epsilon_{ijm} B_m = -\epsilon_{ijm} B^m \tag{2.22}$$

or in matrix form,

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix} \tag{2.23}$$

and

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} \tag{2.24}$$

The dual matrix is given by,$\tilde{F}^{\mu\nu} \equiv \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}$. We can write this in terms of components as:

$$\tilde{F}^{0i} = \frac{1}{2}\epsilon^{0ijk}F_{jk} = \frac{1}{2}\epsilon^{0ijk}\epsilon_{jkm}B_m = -B^i \tag{2.25}$$

$$\tilde{F}^{ij} = \epsilon^{ij0k}F_{0k} = -\epsilon^{ijk}E_k \tag{2.26}$$

or in matrix form,

$$\tilde{F}^{\mu\nu} = \begin{pmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z & -E_y \\ B_y & -E_z & 0 & E_x \\ B_z & E_y & -E_x & 0 \end{pmatrix} \tag{2.27}$$

2.6 Solutions of the Dirac Equation

In field theory we often use $u(p)$ and $v(p)$ as solutions to the Dirac equation (with the exponentials factored out). These obey

$$\left(\not{p} - m\right) u(p) = 0 \tag{2.28}$$

$$\left(\not{p} + m\right) v(p) = 0 \tag{2.29}$$

$$\tag{2.30}$$

which in turn imply

$$u^\dagger(p)\left(\gamma^0 \not{p} \gamma^0 - m\right) = 0 \tag{2.31}$$
$$v^\dagger(p)\left(\gamma^0 \not{p} \gamma^0 + m\right) = 0 \tag{2.32}$$
$$\tag{2.33}$$

$$\bar{u}(p)\left(\not{p} - m\right) = 0 \tag{2.34}$$
$$\bar{v}(p)\left(\not{p} + m\right) = 0 \tag{2.35}$$
$$\tag{2.36}$$

The zero momentum solutions take the form

$$u^s(0) = \sqrt{m}\begin{pmatrix} \chi^s \\ \chi^s \end{pmatrix} \tag{2.37}$$
$$v^s(0) = \sqrt{m}\begin{pmatrix} \chi^s \\ -\chi^s \end{pmatrix} \tag{2.38}$$

These can be boosted to an arbitrary momentum through

$$e^{-\frac{1}{2}\eta \hat{p}\cdot\mathbf{K}} \tag{2.39}$$

where $\eta = \sinh^{-1}\left(\frac{|\mathbf{p}|}{m}\right)$ is the rapidity, $\hat{p}$ is the unit vector of the boost, and $K^j \equiv -\frac{i}{2}\gamma^j\gamma^0$ is the boost matrix.

It is straightforward to calculate the boost matrix explicitly:

$$K_j = -\frac{i}{2}\begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = -\frac{i}{2}\begin{pmatrix} \sigma^j & 0 \\ 0 & -\sigma^j \end{pmatrix} \tag{2.40}$$

which gives the boost:

$$e^{-\frac{1}{2}\eta\hat{p}\cdot\mathbf{K}} = \exp\left(-\frac{\eta}{2}\hat{p}\cdot\begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & -\boldsymbol{\sigma} \end{pmatrix}\right) \tag{2.41}$$
$$= \begin{pmatrix} e^{-\frac{1}{2}\eta\hat{p}\cdot\boldsymbol{\sigma}} & 0 \\ 0 & e^{\frac{1}{2}\eta\hat{p}\cdot\boldsymbol{\sigma}} \end{pmatrix} \tag{2.42}$$
$$= \begin{pmatrix} \cosh\frac{\eta}{2} - \hat{p}\cdot\boldsymbol{\sigma}\sinh\frac{\eta}{2} & 0 \\ 0 & \cosh\frac{\eta}{2} + \hat{p}\cdot\boldsymbol{\sigma}\sinh\frac{\eta}{2} \end{pmatrix} \tag{2.43}$$

2.6.1 Massless Limit

Deriving the form of the equations in the massless limit is straightfoward. We have the equation

$$\gamma^\mu\partial_\mu\psi = 0 \tag{2.44}$$

Take solutions of the form $\psi = e^{ip\cdot x}u$:

$$p_\mu \gamma^\mu u = 0 \tag{2.45}$$

$$p_\mu \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} u = 0 \tag{2.46}$$

This gives two sets of equations that are completely decoupled for the left and right handed part of u. We consider the two solutions indepedently. Consider $u = \begin{pmatrix} u_+ \\ 0 \end{pmatrix}$:

$$p_\mu \bar{\sigma}^\mu u_+ = 0 \tag{2.47}$$

We now define $p_\pm \equiv p_0 \pm p_3$, $z \equiv p_1 + ip_2$, and $u_+ \equiv \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$. This gives

$$\begin{pmatrix} p_+ & \bar{z} \\ z & p_- \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = 0 \tag{2.48}$$

There are two linearly indepdent solutions to this equation.

1. If $p_- \neq 0$ then we can write $\beta = -\frac{z}{p_-}\alpha$ (including $\beta = 0$)
2. If $p_- = 0 \Rightarrow z = 0$, then β can equal anything and $\alpha = 0$.

but

$$\frac{z}{p_-} = \frac{z\sqrt{p_+}/\sqrt{p_-}}{\sqrt{p_+ p_-}} = \sqrt{p_+/p_-}e^{i\phi} \tag{2.49}$$

where we have defined

$$e^{i\phi} \equiv \frac{p_1 + ip_2}{\sqrt{p_+ p_-}} \tag{2.50}$$

With this we can write our solutions as

$$\begin{pmatrix} \sqrt{p_-} \\ -\sqrt{p_+}e^{i\phi} \\ 0 \\ 0 \end{pmatrix} \quad , \quad \begin{pmatrix} 0 \\ \sqrt{2p_0} \\ 0 \\ 0 \end{pmatrix} \tag{2.51}$$

where we have normalized our spinors to the condition $u^\dagger u = 2p_0$. For the other two linearly independent solutions we have the equation,

$$p_\mu \sigma^\mu u_- = 0 \tag{2.52}$$

$$\begin{pmatrix} p_- & -\bar{z} \\ -z & p_+ \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = 0 \tag{2.53}$$

Our two linearly independent solutions are

1. If $p_- \neq 0$ then we can write $\alpha = (\bar{z}/p_-)\beta$ (including $\alpha = 0$)
2. If $p_- = 0 \Rightarrow z = 0$, then α can equal anything and $\beta = 0$.

From before we can write

$$\frac{\bar{z}}{p_-} = \sqrt{p_+/p_-}e^{-i\phi} \tag{2.54}$$

and

$$\begin{pmatrix} 0 \\ 0 \\ \sqrt{p_+}e^{-i\phi} \\ \sqrt{p_-} \end{pmatrix} \quad , \quad \begin{pmatrix} 0 \\ 0 \\ \sqrt{2p_0} \\ 0 \end{pmatrix} \tag{2.55}$$

2.7 Sum Rules

The spin sum formula for the Dirac spinors are given by

$$\sum_s u_a^s \bar{u}_b^s = (\not{p} + m)_{ab} \tag{2.56}$$

$$\sum_s v_a^s \bar{v}_b^s = (\not{p} - m)_{ab} \tag{2.57}$$

The polarization sum rule for external vector bosons is given by

$$\sum_\lambda \epsilon_\mu(\mathbf{k};\lambda)\bar{\epsilon}_\nu(\mathbf{k};\lambda) = \begin{cases} -\eta_{\mu\nu} & \text{(massless boson)} \\ -\eta_{\mu\nu} + \frac{k_\mu k_\nu}{m^2} & \text{(massive boson)} \end{cases} \tag{2.58}$$

Chapter 3

Feynman Rules

3.1 Deriving the Feynman Rules

To properly derive the Feynman rules can be difficult. However determining the interactions is easy. The important point is to remember that the Lagrangian is a real scalar. Thus there should generally not be any i's in it. If there is a complex i then there must be an accomadating i somewhere else. Consider an arbitrary interaction:

$$\mathcal{L}_{int} = g\left(\phi_1^n \phi_2^m ...\right) \tag{3.1}$$

where the particular fields in the interaction are irrelevant. Then the Feynman rule for the interaction will just be

$$\times \quad \longrightarrow \quad ig \tag{3.2}$$

Note that the sign of the terms are conserved. Positive Lagrangian terms give positive interaction vertices. Furthermore, there is an i that comes with the term.

Now one subtely is if there is a partial derivative. The proper "replacement" rule for these is

$$\partial_\mu \to -ip_\mu \tag{3.3}$$

where p_μ is the momentum of the particle that ∂_μ is acting on.

3.2 Symmetry Factors

When using Feynman diagrams to calculate amplitudes a major difficulty in the calculation is to account for identical particles in the calculation. There can be many diagrams corresponding to the exact same process so in general we have to account for all of these. There are 3 contributing factors that result in one factor in front of the amplitude which is called the Symmetry factor.

1. Each vertex contributes a suppression factor. For example in ϕ^4 theory we typically have a 4! suppression factor for each vertex. Of course the value of these is dependent on the definition of the couplings but we define our couplings on purpose so we end up with symmetry factors on the order of unity.

2. There are different ways external particles can be arranged with each vertex. If you swap all the vertices you get the same diagram.

3. There are equivalent ways to contract the fields in the Wick expansion.

A technique to account for all of these is given by some notes I found online by Jacob Bourjaily which in turn credits Colin Morningstar [2]. The idea is as follows. Let n be the number of verteices of a diagram, η be the coupling constant suppression factor, and r the *multiplicity* of a diagram. Then the Symmetry factor is given by

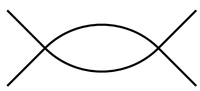
$$S = \frac{n!\,(\eta)^n}{r} \tag{3.4}$$

and the amplitude is given by

$$\mathcal{M} = \frac{1}{S}\mathcal{M}_{1\text{ diagram}} \tag{3.5}$$

The multiplicity of a diagram is the number of different contractions in the Wick expansion, or the number of ways to connect all the external lines to the vertices. This can be found by first drawing out the edges of each external line and points coming out of a vertex. Then count the number of ways the lines can be connected.

As an example we consider the “fish” diagram in ϕ^4 theory,



First we draw the edges,

Start with the initial lines. There are eight ways to connect the first line to a vertex. Then since the initial lines and final lines need to be kept as such, there are only 3 ways to connect the second initial line. Continuing on and counting the number of ways each line can be connected we have

$$r = (8)(3)(4)(3)(2)(1) \tag{3.6}$$

This gives a symmetry factor of

$$S = \frac{2!\,(4!)^2}{(8)(3)(4)(3)(2)(1)} = 2 \tag{3.7}$$

Chapter 4

Lagrangians and Feynman rules

4.1 Standard Model

The Standard Model charges are summarized below:

doublet	$T_3 = \sigma_3/2$	Y	$Q = T_3 + \frac{Y}{2}$
$\begin{pmatrix} \nu_e \\ e^- \end{pmatrix}_L$	$\begin{cases} \frac{1}{2} \\ -\frac{1}{2} \end{cases}$	$\begin{cases} -1 \\ -1 \end{cases}$	$\begin{cases} 0 \\ -1 \end{cases}$
$\begin{pmatrix} u \\ d \end{pmatrix}_L$	$\begin{cases} \frac{1}{2} \\ -\frac{1}{2} \end{cases}$	$\begin{cases} \frac{1}{3} \\ \frac{1}{3} \end{cases}$	$\begin{cases} \frac{2}{3} \\ -\frac{1}{3} \end{cases}$
singlets			
e_R	0	-2	-1
u_R	0	$\frac{4}{3}$	$\frac{2}{3}$
d_R	0	$-\frac{2}{3}$	$-\frac{1}{3}$
Higg's sector			
$\begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix}$	$\begin{cases} \frac{1}{2} \\ -\frac{1}{2} \end{cases}$	$\begin{cases} 1 \\ 1 \end{cases}$	$\begin{cases} 1 \\ 0 \end{cases}$

4.2 ϕ^4

The Higg's Lagrangian is given, in my conventions, by,

$$\mathcal{L} = |D_\mu \phi|^2 + \mu^2 \phi^\dagger \phi - \lambda(\phi^\dagger \phi)^2 \tag{4.1}$$

or a potential,

$$V(\phi) = -\mu^2 \phi^\dagger \phi + \lambda(\phi^\dagger \phi)^2 \tag{4.2}$$

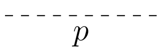
The physical Higgs is given by $\phi = (0, \frac{1}{\sqrt{2}}(h+v))$. Inserting this into the above I find,

$$V(h) = -\frac{\mu^2}{2}(h+v)^2 + \frac{\lambda}{4}(h+v)^4 \tag{4.3}$$

The linear term vanishes if $v^2 = \mu^2/\lambda$. In this case we can write the potential as,

$$V(h) - V_{\text{min}} = \lambda v^2 h^2 + \lambda v h^3 + \frac{\lambda}{4} h^4 \tag{4.4}$$

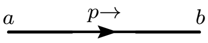
As always the scalar propagator is



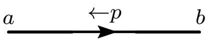
$$\rightarrow \frac{i}{p^2 - m_h^2 + i\epsilon}$$

where, in the normalization above, $m_h^2 = 2\lambda v^2$.

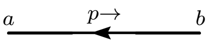
4.3 Spinor QED



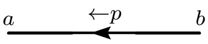
$$= \frac{i(\not{p} + m)_{ab}}{p^2 - m^2 + i\epsilon} = \left(\frac{i}{\not{p} - m + i\epsilon} \right)_{ab} \tag{4.5}$$



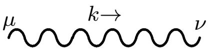
$$= \frac{i(-\not{p} + m)_{ab}}{p^2 - m^2 + i\epsilon} = \left(\frac{i}{-\not{p} - m + i\epsilon} \right)_{ab} \tag{4.6}$$



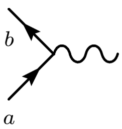
$$= \frac{i(-\not{p} + m)_{ab}}{p^2 - m^2 + i\epsilon} = \left(\frac{i}{-\not{p} - m + i\epsilon} \right)_{ab} \tag{4.7}$$



$$= \frac{i(\not{p} + m)_{ab}}{p^2 - m^2 + i\epsilon} = \left(\frac{i}{\not{p} - m + i\epsilon} \right)_{ab} \tag{4.8}$$



$$= -\frac{i}{k^2 + i\epsilon} \left(g_{\mu\nu} - (1 - \xi) \frac{k_\mu k_\nu}{k^2} \right) \tag{4.9}$$



$$= -ieQ \left(\gamma^\mu\right)_{ab} \quad \text{(+ momentum conservation)} \tag{4.10}$$

(4.11)

where $e > 0$ and $Q = -1$ for the electron.

Furthermore, incoming and outgoing photons gives,

ϵ_μ

ϵ_μ^*

4.4 Scalar QED

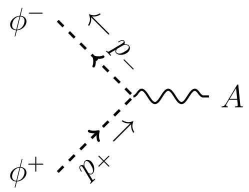
The Lagrangian for scalar QED is given by

$$\mathcal{L} = (D^\mu \phi)^\dagger D_\mu \phi - m^2 \phi^\dagger \phi - \frac{\lambda}{4}(\phi^\dagger \phi)^2 - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \tag{4.12}$$

where $D^\mu = \partial_\mu - ieA_\mu$. The vertices are given by

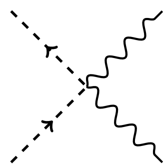
$$\mathcal{L}_{int} = -ieA_\mu \left((\partial^\mu \phi^\dagger)\phi - \phi^\dagger (\partial^\mu \phi) \right) + e^2 A_\mu A^\mu \phi^\dagger \phi - \frac{\lambda}{4}(\phi^\dagger \phi)^2 \tag{4.13}$$

Depending on the relative directions of the lines going into vertex and the type of particle we get a different vertex factor. This gives the rule:



$$\rightarrow \quad -ie\,(p_- - p_+)$$

and



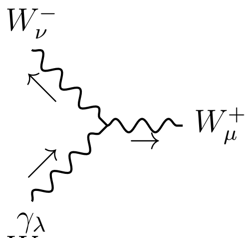
$$\rightarrow \quad 2ie^2 g_{\mu\nu}$$

4.5 Electroweak Interactions

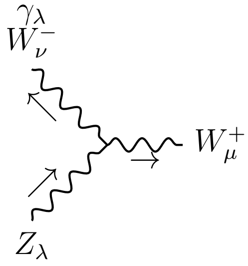
The $W^\pm \equiv (W_1 \mp iW_2)/\sqrt{2}$ boson interactions are given by,

$$\bar{\psi} i \overbrace{D_\mu}^{\partial_\mu - i\frac{1}{2}\sigma^a W^a} \gamma^\mu \psi = g\bar{\psi}\gamma_\mu \frac{1}{2} \begin{pmatrix} \cdot & \sqrt{2}W^+ \\ \sqrt{2}W^- & \cdot \end{pmatrix} \psi + ... = \frac{g}{\sqrt{2}} \bar{\psi} \gamma^\mu P_L W_\mu^+ \psi \tag{4.14}$$

The triple gauge vertices are,



$$\rightarrow -ie(g^{\mu\nu}(k_- - k_+)^\lambda - g^{\nu\lambda}(q+k_-)^\mu + g^{\lambda\mu}(q+k_+)^\nu)$$



$$\rightarrow -igc_W(g^{\mu\nu}(k_- - k_+)^\lambda - g^{\nu\lambda}(q+k_-)^\mu + g^{\lambda\mu}(q+k_+)^\nu)$$

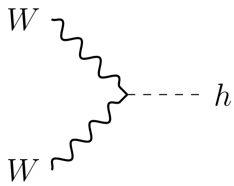
The higgs interactions are found through,

$$\phi^\dagger \left(\frac{g'}{2} Y B_\mu + g T^a W^a_\mu\right) \left(\frac{g'}{2} Y B^\mu + g T^b W^{b\mu}\right) \phi \tag{4.15}$$

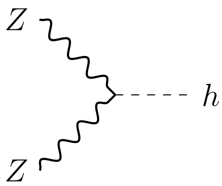
$$= \frac{1}{4} \begin{pmatrix} . & \phi_0/\sqrt{2} \end{pmatrix}^\dagger \begin{pmatrix} . & . \\ . & 2g^2 W^+ W^- + (g'YB - gW_3)^2 \end{pmatrix} \begin{pmatrix} . \\ \phi_0/\sqrt{2} \end{pmatrix} \tag{4.16}$$

$$= \frac{g^2}{4}\phi_0^2 W^+ W^- + \frac{e^2}{4c_W^2 s_W^2}\phi_0^2 \left(\frac{1}{2} Z_\mu Z^\mu\right) \tag{4.17}$$

which gives a higgs-$W^+ - W^-$ Lagrangian (there is an additional factor of 2 since each higgs in ϕ_0 can get a VEV)[1],



$\rightarrow \frac{ig^2 v}{2}$



$\rightarrow i\frac{e^2 v}{2c_W s_W}$

The masses of the vector bosons are obtained by taking $\phi_0 \to v$:

$$m_W^2 = \frac{g^2 v^2}{4}, \quad m_Z^2 = \frac{e^2 v^2}{4c_W^2 s_W^2} \tag{4.18}$$

The left fermion-Z interactions are derived from,

$$\bar{\psi}\gamma_\mu \left(\frac{g'}{2} B^\mu Y + g T^a W^{a\,\mu}\right) P_L \psi = \bar{\psi}\gamma_\mu \frac{1}{2} \begin{pmatrix} g'YB + gW_3 & . \\ . & g'YB - gW_3 \end{pmatrix} P_L \psi \tag{4.19}$$

The weinberg angle rotates into the A, Z basis. An angle of 0 corresponds to hypercharge being fully aligned with charge (i.e., $B_\mu = A_\mu$):

$$\begin{pmatrix} B_\mu \\ W_\mu^3 \end{pmatrix} = \begin{pmatrix} c_W & -s_W \\ s_W & c_W \end{pmatrix} \begin{pmatrix} A_\mu \\ Z_\mu \end{pmatrix} \tag{4.20}$$

[1]We assume $\phi = h + v$

and the couplings are related to the electroweak couplings by,

$$|e| = g' c_W = g s_W \quad |e| = \sqrt{g'^2 + g^2} \tag{4.21}$$

and in our conventions,

$$Q = T_3 + \frac{Y}{2} \tag{4.22}$$

This corresponds to,

$$g' Y B_\mu \pm g W_\mu^3 = e \left\{ (Y \pm 1) A_\mu + \frac{1}{s_W c_W} \left(\pm 1 + s_W^2 \left(-Y \mp 1 \right) \right) Z_\mu \right\} \tag{4.23}$$

The up-type and down-type fermion interactions are,

$$\mathcal{L}_{NC}^{up-type} = \left\{ eQA_\mu + \frac{e}{c_W s_W} \left(\frac{1}{2} - Q s_W^2 \right) Z_\mu \right\} \bar{u} \gamma^\mu P_L u \tag{4.24}$$

$$\mathcal{L}_{NC}^{down-type} = \left\{ eQA_\mu - \frac{e}{c_W s_W} \left(\frac{1}{2} + Q s_W^2 \right) Z_\mu \right\} \bar{d} \gamma^\mu P_L d \tag{4.25}$$

The right handed couplings are

$$\mathcal{L}_{\psi_R \psi_R Z} + ... = \bar{\psi} \gamma_\mu g' B^\mu Q P_R \psi = -\frac{e Q s_W}{c_W} \bar{\psi} \gamma_\mu P_R \psi Z^\mu + ... \tag{4.26}$$

We now compute the interactions of the goldstones with the vector bosons. These arise from the higgs kinetic term:

$$(D_\mu H)^\dagger D^\mu H = \left((\partial_\mu - i g T^a W_\mu^a - i \frac{g}{2} B_\mu) H \right)^\dagger \left((\partial_\mu - i g T^a W_\mu^a - i \frac{g'}{2} B_\mu) H \right) \tag{4.27}$$

$$= |\partial_\mu H|^2 + \partial_\mu H^\dagger \left(-i g T^a W_\mu^a - i g' \frac{1}{2} B_\mu \right) H + H^\dagger \left(i g T^a W_\mu^a + i \frac{g'}{2} B_\mu \right) \partial^\mu H$$
$$+ \left| \left(g T^a W_\mu^a + \frac{g'}{2} B_\mu \right) H \right|^2 \tag{4.28}$$

$$= |\partial_\mu H|^2 - \frac{i}{2} \left[\partial_\mu H^\dagger \begin{pmatrix} gW + g'B & g\sqrt{2} W^+ \\ \sqrt{2} g W^- & -gW^3 + g'B \end{pmatrix}^\mu H - H^\dagger \left(... \right)_\mu \partial^\mu H \right]$$
$$+ \frac{1}{4} H^\dagger \left(... \right)_\mu \left(... \right)^\mu H \tag{4.29}$$

The first term is just a kinetic term. We first simply the term in square brackets.

$$[...] = \partial_\mu \begin{pmatrix} \pi^- & \frac{1}{\sqrt{2}} (h - i\pi^0) \end{pmatrix} \begin{pmatrix} gW + g'B & g\sqrt{2} W^+ \\ \sqrt{2} g W^- & -gW^3 + g'B \end{pmatrix}^\mu \begin{pmatrix} \pi^+ \\ \frac{1}{\sqrt{2}} (h + i\pi^0) \end{pmatrix} - h.c. \tag{4.30}$$

- $\underline{\pi^0\pi^0 Z}$:

$$-gW^3 + g'B = -\frac{e}{s_w c_w} Z \tag{4.31}$$

$\Rightarrow$

$$\Delta\mathcal{L} = -\frac{1}{2}\frac{e}{s_w c_w}\left[\left(\partial_\mu \pi^0\right)\pi^0 Z^\mu - \left(\partial_\mu \pi^0\right)\pi^0 Z^\mu\right] = 0 \tag{4.32}$$

Thus there is not $\pi^0\pi^0 Z_\mu$ interaction. This is comforting as the SM doesn't have a triple-Z interaction. This can be traced back to the fact that the Goldstones form a triplet of $SU(2)_L$ and so the interactions are parameterized by, $\epsilon_{abc}W_a W_b W_c$ which doesn't have a ZZZ interaction from the structure of the Levi-Cevita tensor.

- $\underline{h\pi^0 Z}$:

$$\Delta\mathcal{L} = -\frac{1}{2}\frac{e}{s_w c_w} i\left[(\partial_\mu h)\pi^0 Z^\mu - h\left(\partial_\mu \pi^0\right) Z^\mu\right] - h.c. = -i\frac{e}{s_w c_w}\left[\partial_\mu (h\pi^0) Z^\mu\right] \tag{4.33}$$

- $\underline{hhZ}$:

$$\Delta\mathcal{L} = -\frac{1}{2}\frac{e}{s_w c_w}\left(\partial_\mu h\right) h Z^\mu - h.c. = 0 \tag{4.34}$$

- $\underline{\pi^-\pi^+ Z, \pi^-\pi^+ A}$:

$$\Delta\mathcal{L} = (\partial_\mu \pi^-)\pi^+\left(gW^3 + g'B\right)^\mu - h.c. \tag{4.35}$$

$$= e\left[(\partial_\mu \pi^-)\pi^+\right]\left(2A^\mu + \frac{1}{s_w c_w}\left(1 - 2s_w^2\right) Z^\mu\right) - h.c. \tag{4.36}$$

- $\underline{\pi^\pm W^\mp h}$:

$$\Delta\mathcal{L} = g\left(\partial_\mu \pi^-\right) W^{+\,\mu} h + g\left(\partial_\mu h\right)\pi^+ W^{-\,\mu} - h.c. \tag{4.37}$$

$$= gh\left(\partial_\mu \pi^+ W^{+\,\mu} - \partial_\mu \pi^+ W^{-\,\mu}\right) + g\left(\partial_\mu h\right)\left[\pi^- W^{+\,\mu} - \pi^+ W^{-\,\mu}\right] \tag{4.38}$$

$$= g\left[\partial_\mu (h\pi^-) W^{+\,\mu}\right] - h.c. \tag{4.39}$$

Now lets consider the terms arising from the $\left|H^\dagger\left(...\right)^2 H\right|^2$ term. We have,

$$|...|^2 = \frac{1}{4}H^\dagger\begin{pmatrix} a^2 + 2g^2 W_\mu^- W^\pm & g\sqrt{2}(a+b)\cdot W^+ \\ g\sqrt{2}(a+b)\cdot W^+ & b^2 + 2g^2 W^+\cdot W^- \end{pmatrix} H \tag{4.40}$$

where $a_\mu \equiv e\left(2A_\mu + \frac{1}{s_w c_w}\left(1 - 2s_w^2\right) Z_\mu\right)$ and $b_\mu = -\frac{e}{c_w s_w} Z_\mu$. The terms arising from this contribution are given below:

- $\underline{\pi^+\pi^-, A, Z, W}$:

$$\Delta\mathcal{L} = \frac{e^2}{4}\pi^+\pi^-\left[\left(2A_\mu + \frac{1}{s_w c_w}\left(1 - 2s_w^2\right) Z_\mu\right)^2 + 2g^2 W_\mu^+ W^{-\,\mu}\right] \tag{4.41}$$

- $\underline{(\pi^0)^2, h^2, W^2, Z^2}$:

$$\Delta\mathcal{L} = \frac{1}{4}\frac{1}{2}\left(h^2 + \pi^{0\,2}\right)\left[\left(\frac{e}{s_w c_w}\right)^2 Z_\mu Z^\mu + 2g^2 W_\mu^+ W^{-\,\mu}\right] \tag{4.42}$$

- $\underline{\pi^-(h,\pi^0)Z, W^+}$: Top right:

$$\Delta\mathcal{L} = \frac{1}{4}\pi^-\left(h + i\pi^0\right)\left(g(a+b)\cdot W^{+\,\mu}\right) \tag{4.43}$$

$$= \frac{e^2}{2}\left[\left(h + i\pi^0\right)\pi^-\left(A_\mu - t_w Z_\mu\right)W^{+\,\mu}\right] \tag{4.44}$$

The bottom left contribution is similar. The sum gives,

$$\frac{e^2}{2}\left[\left(A_\mu - t_w Z_\mu\right)\left((h+i\pi^0)\pi^- W^{+\,\mu} + \left(h - i\pi^0\right)\pi^+ W^{-\,\mu}\right)\right] \tag{4.45}$$

4.6 Yukawa couplings

The yukawa interactions are given by,

$$\mathcal{L}_{\rm yuk} = -y_d^{ij}\bar{Q}^i_{L,0}\phi d^j_R - y_u^{ij}(\bar{Q}^i_{L,0}\epsilon\phi)u^j_R + \text{h.c.} \tag{4.46}$$

where $Q_{L,0}$ is the field prior to rotation to the physical basis (the right-handed fields are already implicitly rotated and absorbed into the yukawa couplings). The different terms can be rewritten:

$$u^i_L \equiv U_u^{ij} u^j_{L,0} \qquad d^i_L \equiv U_d^{ij} d_{L,0} \tag{4.47}$$

$$V_{ij} \equiv (U_u^\dagger U_d)_{ij} \tag{4.48}$$

$$m_u = U_u^\dagger y_u\sqrt{2} \quad m_d \equiv U_d^\dagger y_d\sqrt{2} \tag{4.49}$$

Then we can write,

$$\begin{aligned}\mathcal{L}_{\rm yuk} = -\frac{g/\sqrt{2}}{m_W}\Bigg\{ &\begin{pmatrix} (Vm_d)^{ji}\,\bar{u}^i_L & \bar{d}^j_L \end{pmatrix}\begin{pmatrix} \pi^+ \\ (h+i\pi^0)/\sqrt{2}\end{pmatrix} d_{R,j} \\ &+ \begin{pmatrix} \bar{u}^j_L & (V^\dagger m_u)^{ji}\bar{d}^i_L \end{pmatrix}_{ij}\begin{pmatrix} (h-i\pi^0)/\sqrt{2} \\ -\pi^- \end{pmatrix} u_{R,j}\Bigg\} + \text{h.c.}\end{aligned} \tag{4.50}$$

From here we can read off the Feynman rules. Here we check one entry with the literature:

$$\mathcal{L} \supset \frac{g}{\sqrt{2}} V^\dagger_{jk}\frac{m_{u,ki}}{m_W}\bar{d}^i_L u^j_R \pi^- + \text{h.c.} \tag{4.51}$$

This is in agreement with the Feynman rules displayed in [4]

4.7 CKM Matrix

The CKM matrix in the Wolfenstein parametrization is

$$V_{CKM} = \begin{pmatrix} 1-\frac{\lambda^2}{2} & \lambda & A\lambda^3(\rho - i\eta) \\ -\lambda & 1-\frac{\lambda^2}{2} & A\lambda^2 \\ A\lambda^3(1-\rho-i\eta) & -A\lambda^2 & 1 \end{pmatrix} \tag{4.52}$$

4.8 Supersymmetry

In this section we summarize the supersymmetric Feynman rules for a gauge theory. We continue to take the Peskin and Schroeder convention for the sign in the covariant derivatives. This comes at the come of deviating from the convention used in Martin's notes by a minus sign for the gauge charge. For a field charged under a gauge group the D term is given by,

$$\int d^4\theta \Phi^\dagger e^V \Phi = i\psi^\dagger \bar{\sigma}^\mu D_\mu \psi + \sqrt{2}g(\phi^\dagger T^a \psi)\lambda^a + ... \tag{4.53}$$

It is often useful to have the form of this expression in the case of a field charged under $SU(2)\times U(1)$. We denote the field by u_L, d_L but they refer to up-type and down-type fields respectfully. We have,

$$\int d^4\theta \Phi^\dagger e^V \Phi = \sqrt{2}g\left(\phi^\dagger T^a\psi\right)\tilde{W}^a + \sqrt{2}g'\frac{Y}{2}\phi^\dagger \tilde{B}\psi + h.c. + ... \tag{4.54}$$

$$= \begin{pmatrix} \tilde{u}_L^\dagger & \tilde{d}_L^\dagger \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} g\tilde{W}_0 + Yg\tilde{B} & \sqrt{2}g\tilde{W}^+ \\ \sqrt{2}g\tilde{W}^- & g'Y\tilde{B} - g\tilde{W}_0 \end{pmatrix} \begin{pmatrix} u_L \\ d_L \end{pmatrix} + h.c. \tag{4.55}$$

$$= \tilde{u}_L^\dagger \frac{1}{\sqrt{2}}\left(g\tilde{W}_0 + Yg\tilde{B}\right)u_L + \tilde{d}_L^\dagger \frac{1}{\sqrt{2}}\left(gY\tilde{B} - g\tilde{W}_0\right)d_L + g\left(\tilde{u}_L^\dagger \tilde{W}^+ d_L + \tilde{d}_L^\dagger \tilde{W}^- u_L\right) \tag{4.56}$$

4.8.1 Triplet

Above we considered the additional EW interactions for a field in the fundamental representation. Now we extend this to fields in the adjoint representation. For an adjoint field we have the new interactions,

$$\sqrt{2}g\left(\phi^\dagger T^a \psi\right)\tilde{W}^a + g\psi^\dagger T_a \bar{\sigma}^\mu \psi W_\mu^a \tag{4.57}$$

There are no additional $\tilde{B}$ terms since fields in the adjoint representation of a $U(1)$ aren't charged under the group. We start by considering the first term. Using $T^a_{bc} = if_{abc} = i\epsilon_{abc}$ we have,

$$\sqrt{2}g\left(\phi^\dagger T^a\psi\right)\tilde{W}^a = \sqrt{2}ig\phi_b^\dagger \epsilon_{bac}\psi_c \tilde{W}^a \tag{4.58}$$

$$= \sqrt{2}g\begin{pmatrix} \phi_1^\dagger & \phi_2^\dagger & \phi_3^\dagger \end{pmatrix} \begin{pmatrix} 0 & -\tilde{W}^3 & \tilde{W}^2 \\ \tilde{W}^3 & 0 & -\tilde{W}^1 \\ -\tilde{W}^2 & \tilde{W}^1 & 0 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} \tag{4.59}$$

We want to write the fields according to their charged basis. For this we use the transformations,

$$\begin{pmatrix} \psi_- \\ \psi_+ \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & +i \\ 1 & -i \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \quad \begin{pmatrix} \phi_- \\ \phi_+ \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & +i \\ 1 & -i \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \tag{4.60}$$

We define,

$$U \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i & 0 \\ 1 & -i & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{4.61}$$

and so,

$$i\sqrt{2}g \begin{pmatrix} \phi_-^\dagger & \phi_+^\dagger & \phi_0^\dagger \end{pmatrix} UAU^\dagger \begin{pmatrix} \psi_- \\ \psi_+ \\ \psi_0 \end{pmatrix} \tag{4.62}$$

where,

$$A \equiv \begin{pmatrix} 0 & -\tilde{W}^0 & -\frac{i}{\sqrt{2}}(\tilde{W}^- - \tilde{W}^+) \\ \tilde{W}^0 & 0 & -\frac{1}{2}\left(\tilde{W}^- + \tilde{W}^+\right) \\ \frac{i}{\sqrt{2}}\left(\tilde{W}^- - \tilde{W}^+\right) & \frac{1}{\sqrt{2}}\left(\tilde{W}^- + \tilde{W}^+\right) & 0 \end{pmatrix} \tag{4.63}$$

Multiplying through gives,

$$-\sqrt{2}g \begin{pmatrix} \phi_-^\dagger & \phi_+^\dagger & \phi_0^\dagger \end{pmatrix} \begin{pmatrix} \tilde{W}^0 & 0 & -\tilde{W}^- \\ 0 & -\tilde{W}^0 & \tilde{W}^+ \\ -\tilde{W}^+ & \tilde{W}^- & 0 \end{pmatrix} \begin{pmatrix} \psi_- \\ \psi_+ \\ \psi_0 \end{pmatrix} \tag{4.64}$$

The simpilfication for the second term is identical with $g \to g/\sqrt{2}$, $\phi \to \psi$, and $\tilde{W} \to W$,

$$g\psi^\dagger T_a \bar{\sigma}^\mu \psi W_\mu^a = -g \begin{pmatrix} \psi_-^\dagger & \psi_+^\dagger & \psi_0^\dagger \end{pmatrix} \begin{pmatrix} W_\mu^0 & 0 & -W_\mu^- \\ 0 & -W_\mu^0 & W_\mu^+ \\ -W_\mu^+ & W_\mu^- & 0 \end{pmatrix} \bar{\sigma}_\mu \begin{pmatrix} \psi_- \\ \psi_+ \\ \psi_0 \end{pmatrix} \tag{4.65}$$

To rewrite this in terms on the physical gauge bosons we use, $W_\mu^0 = s_W A_\mu + c_W Z_\mu$.

Chapter 5

Polarized Calculations

5.1 Polarization and Spin

For some reason a thorough discussion of polarization calculations is missing from the popular Quantum Field Theory books. The discussion here is an amalgam of what I've found from Peskin, Srednicki, as well as Bjorken and Drell.

Consider a particle with a spinor $u(\mathbf{p}, s)$ or $v(\mathbf{p}, s)$ which is at rest. It's polarization in the rest from of the particle is what we often call its spin. We denote this polarization as some 3-vector $\boldsymbol{\lambda}$. For example if the particle is polarized along the z axis then $\boldsymbol{\lambda} = (0, 0, 1)$. We form a four-vector to represent its "spin". We denote the rest frame spin vector by s_r^μ. Now what should the first component of the four-vector be? In the rest frame there is no other degree of freedom for the spin. We set s_r^0 to zero.

To boost back to the lab frame we apply a Lorentz Transformation in the $-\mathbf{p}$ direction onto the four-vector. Recall in matrix form the transformation matrices applied onto a vector $(t, \mathbf{r})$:

$$t' = \gamma \left(t - \mathbf{r} \cdot \mathbf{v}\right) \tag{5.1}$$

$$\mathbf{r}' = \mathbf{r} + \left(\frac{\gamma - 1}{v^2}\mathbf{r} \cdot \mathbf{v} - \gamma t\right) \mathbf{v} \tag{5.2}$$

The spatial component of the spin in the lab frame is

$$\begin{aligned}
\mathbf{s} &= \boldsymbol{\lambda} + \left(-\frac{\gamma-1}{v^2}\boldsymbol{\lambda}\cdot\mathbf{v} - 0\right)(-\mathbf{v}) \\
&= \boldsymbol{\lambda} + \left(\frac{\gamma\boldsymbol{\lambda}\cdot(\gamma m\mathbf{v})}{m\gamma} - \frac{\boldsymbol{\lambda}\cdot\gamma m\mathbf{v}}{\gamma m}\right)\mathbf{v}\gamma m \\
&= \boldsymbol{\lambda} + \left(\frac{(\gamma-1)\boldsymbol{\lambda}\cdot\mathbf{p}}{E^2(\gamma^2-1)}\gamma^2\right)\mathbf{p} \\
&= \boldsymbol{\lambda} + \frac{\mathbf{p}(\boldsymbol{\lambda}\cdot\mathbf{p})}{E^2 m(E+m)}E^2 \\
&= \boldsymbol{\lambda} + \frac{\mathbf{p}(\boldsymbol{\lambda}\cdot\mathbf{p})}{m(E+m)}
\end{aligned} \tag{5.3}$$

and the time component is

$$\begin{aligned}
s^0 &= \gamma\boldsymbol{\lambda}\cdot\mathbf{v}m/m \\
&= \frac{\mathbf{p}\cdot\boldsymbol{\lambda}}{m}
\end{aligned} \tag{5.4}$$

So we have

$$s^\mu = \left(\frac{\mathbf{p}\cdot\boldsymbol{\lambda}}{m}, \boldsymbol{\lambda} + \frac{\mathbf{p}(\mathbf{p}\cdot\boldsymbol{\lambda})}{m(m+E)}\right) \tag{5.5}$$

In the case that the spin is measured along the direction of motion (i.e. $\hat{p} \parallel \hat{\lambda}$) we have

$$s^\mu = \frac{1}{m}\left(|\mathbf{p}|, \hat{p}E\right) \tag{5.6}$$

Note that

$$\begin{aligned}
s^2 &= \frac{(\mathbf{p}\cdot\boldsymbol{\lambda})^2}{m^2} - \lambda^2 - 2\frac{(\mathbf{p}\cdot\boldsymbol{\lambda})^2}{m(E+m)} - \frac{\mathbf{p}^2(\mathbf{p}\cdot\boldsymbol{\lambda})^2}{m(m+E)} \\
&= (\mathbf{p}\cdot\boldsymbol{\lambda})^2\left(\frac{1}{m^2} - \frac{2}{m(E+m)} - \frac{E^2-m^2}{m^2(m+E)^2}\right) - 1 \\
&= (\mathbf{p}\cdot\boldsymbol{\lambda})^2\left(\frac{E^2+2mE+m^2-2Em-2m^2-E^2+m^2}{m^2(E+m)^2}\right) - 1 \\
&= -1
\end{aligned} \tag{5.7}$$

and

$$\begin{aligned}
s_\mu p^\mu &= \frac{\mathbf{p}\cdot\boldsymbol{\lambda}E}{m} - \mathbf{p}\cdot\boldsymbol{\lambda} - \frac{(E^2-m^2)(\mathbf{p}\cdot\boldsymbol{\lambda})}{m(m+E)} \\
&= (\mathbf{p}\cdot\boldsymbol{\lambda})\left(\frac{E}{m} - 1 - \frac{E-m}{m}\right) \\
&= (\mathbf{p}\cdot\boldsymbol{\lambda})\frac{E-m-E+m}{m} \\
&= 0
\end{aligned} \tag{5.8}$$

For a general spin vector the spin projection operator is (see for example [1])

$$\Sigma(s) = \frac{1+\gamma \not{s}}{2} \tag{5.9}$$

This operator obeys

$$\Sigma(s)u(p,s) = u(p,s) \tag{5.10}$$
$$\Sigma(s)v(p,s) = v(p,s) \tag{5.11}$$
$$\Sigma(-s)u(p,s) = \Sigma(-s)v(p,s) = 0 \tag{5.12}$$

5.2 Calculational Tricks

When doing a calculation with some polarized particles there are some useful tricks that can be implemented to simplify the math. The key result that can be used to derive all the following relations is

$$u(\mathbf{p},\boldsymbol{\lambda})\bar{u}(\mathbf{p},\boldsymbol{\lambda}) = \left(\not{p}+m\right)\overbrace{\frac{1+\gamma^5\not{s}}{2}}^{\text{Spin Projection Operator}} \tag{5.13}$$
$$v(\mathbf{p},\boldsymbol{\lambda})\bar{v}(\mathbf{p},\boldsymbol{\lambda}) = \left(\not{p}-m\right)\frac{1+\gamma^5\not{s}}{2} \tag{5.14}$$

With this we can now find a variety of important relations (we suppress the polarization and momentum dependence in u:

$$\begin{aligned}
\bar{u}\gamma^\mu u &= \mathrm{tr}\left(\bar{u}\gamma^\mu u\right) \\
&= \mathrm{tr}\left(\gamma^\mu u\bar{u}\right) \\
&= \frac{1}{2}\mathrm{tr}\left(\gamma^\mu\left(\not{p}+m\right)\left(1+\gamma^5\not{s}\right)\right) \\
&= \frac{1}{2}\mathrm{tr}\left(\gamma^\mu\not{p}\right) \\
&= 2p^\mu
\end{aligned} \tag{5.15}$$

$$\begin{aligned}
\bar{u}\gamma^\mu\gamma^5 u &= \mathrm{tr}\left(\bar{u}\gamma^\mu\gamma^5 u\right) \\
&= \mathrm{tr}\left(\gamma^\mu\gamma^5 u\bar{u}\right) \\
&= \frac{1}{2}\mathrm{tr}\left(\gamma^\mu\gamma^5\left(\not{p}+m\right)\left(1+\gamma^5\not{s}\right)\right) \\
&= \frac{1}{2}\mathrm{tr}\left(+m\gamma^\mu\not{s}\right) \\
&= 2ms^\mu
\end{aligned} \tag{5.16}$$

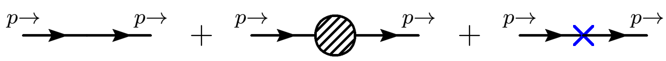
Chapter 6

Renormalization

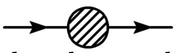
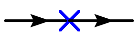
Renormalization schemes is subtle topic with a lot of depth. Here we just present the bare-bones needed to do calculations

6.1 On-Shell Renormalization

There are two renormalization conditions to consider. First corresponds to the mass renormalization. Consider the full particle propagator (with the external lines amputated, i.e. ignore the external legs contribution):

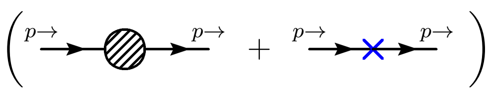


(6.1)

where  represents a sum of loop corrections and  is the counterterm. It can be shown that the Green function corresponding the above computation are (in the case of $\phi - 4$ theory)

$$\langle \Omega | \phi(x) \phi(y) | \Omega \rangle = \frac{i}{p^2 - m_p^2 + \Sigma(p^2) + i\epsilon} \tag{6.2}$$

where $\Sigma(p^2)$ is the sum of the amputated diagrams above. There are two shift factors here. There is a shift in the pole and there is a also a shift in the amplitude of this factor. The first renormalization condition is that when the incoming particle is on-shell ($p^2 = m_p^2$), the loop contribution (Σ) is zero. In other words



$$\Bigg|_{p^2=m_p^2} = 0 \tag{6.3}$$

This makes sence since should give the propagator for an on-shell particle.

The second renormalization condition is for the normalization of the Green's function not to change. We can rewrite equation as

$$\langle\Omega|\phi(x)\phi(y)|\Omega\rangle = \frac{i}{p^2 - m_p^2 + \Sigma(m^2) + (p^2 - m^2)\frac{d\Sigma}{dp^2}\Big|_{p^2=m_p} + i\epsilon} \tag{6.4}$$

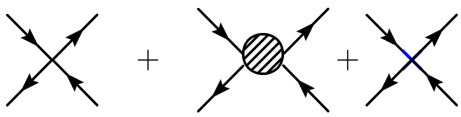
$$= \frac{i}{(p^2 - m_p^2)\left(1 + \frac{d\Sigma}{dp^2}\Big|_{p^2=m_p}\right) + \Sigma(m^2) + i\epsilon} \tag{6.5}$$

$$\tag{6.6}$$

where we only keep terms of order $p^2 - m^2$ since we are considering the conditions on Σ near the pole (these are on-shell conditions after all!). Thus we require (in $\phi - 4$ theory):

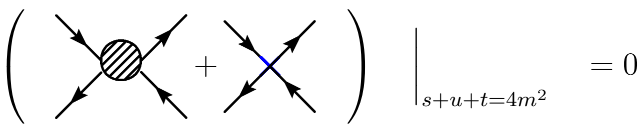
$$\frac{d\Sigma}{dp^2}\Big|_{p^2=m^2} = 0 \tag{6.7}$$

The third renormalization condition is for the coupling. The sum of the 4 vertex diagrams (this can of course be done for any types of couplings but we consider 4 external legs for concreteness).



(6.8)

The renormalization condition is



(6.9)

where λ is the renormalized couplings.

Chapter 7

Quantum Mechanics

7.1 Commutation Relations

$$[x, p] = i\hbar \tag{7.1}$$

7.2 Quantum Harmonic Oscillator

The creation annhillation operators are defined by

$$a \equiv \frac{m\omega}{2\hbar}\left(\hat{x} + \frac{i}{m\omega}\hat{p}\right) \tag{7.2}$$

$$a^\dagger = \sqrt{m\omega}2\hbar\left(x - \frac{i}{m\omega}p\right) \tag{7.3}$$

or

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(a + a^\dagger) \tag{7.4}$$

$$\hat{p} = i\sqrt{m\omega\hbar}2(a^\dagger - a) \tag{7.5}$$

which give

$$a^\dagger |n\rangle = \sqrt{n+1}\,|n+1\rangle \tag{7.6}$$

$$a\,|n\rangle = \sqrt{n}\,|n-1\rangle \tag{7.7}$$

We also have,

$$\langle 0|a^n a^{\dagger^n}|0\rangle = n! \tag{7.8}$$

This is easiest to see by starting with $n = 1$ and going on recursively to higher n.

Chapter 8

Special Relativity

In special relativity we have covariant (x_μ) and contravariant (x^μ) vectors. Contravariant vectors have positive spatial indices:

$$x^\mu = (x_0, \mathbf{x}) \tag{8.1}$$

while covariant vectors have negative spatial indices:

$$x_\mu = (x_0, -\mathbf{x}) \tag{8.2}$$

The derivatives have the opposite sign convention,

$$\partial_\mu \equiv (\partial_0, \nabla) \tag{8.3}$$
$$\partial^\mu \equiv (\partial_0, -\nabla) \tag{8.4}$$

Chapter 9

Phase space

The differential cross section of a two-body collision is given by,

$$d\sigma = \frac{1}{2E_A 2E_B |v_A - v_B|} \int d\Pi |\mathcal{M}|^2 \tag{9.1}$$

where E_i and v_i are the incoming particle energies and velocities, while we define,

$$d\Pi \equiv \left(\prod_f \frac{d^3 p_f}{(2\pi)^3} \frac{1}{2E_f} \right) (2\pi)^4 \delta^{(4)}(\sum_i p_i - \sum_f p_f) \tag{9.2}$$

If the incoming particles are identical particles we have, $E_A = E_B = E_{\rm cm}/2$, and $v_A - v_B = 2|\mathbf{p}_i|/E_{\rm cm}$, which gives,

$$d\sigma = \frac{1}{8E_{\rm cm}|\mathbf{p}_i|} \int d\Pi |\mathcal{M}|^2 \tag{9.3}$$

where $\mathbf{p}_i$ is the momenta of the incoming particles.

The differential decay rate is,

$$d\Gamma = \frac{1}{2m_A} \int d\Pi |\mathcal{M}|^2 \tag{9.4}$$

For a two particle final state the phase space is just,

$$\int d\Pi_2 = \int \frac{d\Omega_{\rm cm}}{4\pi} \frac{1}{8\pi} \frac{2|\mathbf{p}|}{E_{\rm cm}} \tag{9.5}$$

where $|\mathbf{p}|$ is the magnitude of the momentum of either outgoing particle (they are equal in the cm frame), and $E_{\rm cm} = \sqrt{s}$ is the cm energy. In the massless limit this is just, [Q 1: There is factor of 2 problem floating around....]

$$\int d\Pi_2 = \frac{1}{8\pi} \int \frac{d\Omega_{\rm cm}}{4\pi} \tag{9.6}$$

Finally for $2 \to 2$ process with all massless particles we have,

$$d\sigma = \frac{1}{8\pi s} \int \frac{d\Omega \mathrm{cm}}{4\pi} |\mathcal{M}|^2 \tag{9.7}$$

If we have a $2 \to 2$ process with the outgoing particles having equal mass then we can rewrite the phase space as,

$$\frac{1}{4\pi s} \int_{-\frac{s}{4}(1+\beta)^2}^{-\frac{s}{4}(1-\beta)^2} dt \tag{9.8}$$

where $\beta \equiv \sqrt{1 - \frac{4m^2}{s}}$ and t is the Mandelstam variable. In the massless limit this takes the form,

$$\frac{1}{4\pi s} \int_{-s}^{0} dt \tag{9.9}$$

The three particle phase space for unpolarized particles with two massless particles $(i = 1, 2)$ and one massive $(i = 3)$ is,

$$\int d\Pi_3 = \frac{s}{128\pi^3} \int_0^{1-\alpha^2} dx_1 \int_{1-\alpha^2-x_1}^{1-(\alpha^2/(1-x_1))} dx_2 \tag{9.10}$$

where $\alpha \equiv m_3/\sqrt{s}$, $s \equiv (\sum p_i)^2$, and $x_i \equiv 2E_i/\sqrt{s}$ is the momentum fraction of particle i. For $m_3 \to 0$ we have the better known result,

$$\int d\Pi_3 = \frac{s}{128\pi^3} \int_0^1 dx_1 \int_{1-x_1}^{1} dx_2 \tag{9.11}$$

Chapter 10

Mathematics

10.1 Anticommuting Matrices

10.1.1 Sigma Matrices and Levi Civita Tensors

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{10.1}$$

They obey the commutation relations,

$$[\sigma_a, \sigma_b] = 2i\epsilon_{abc}\sigma_c \tag{10.2}$$

and the anticommutation relations,

$$\{\sigma_a, \sigma_b\} = 2\delta_{ab} \tag{10.3}$$

Furthermore any product of Pauli matrices can be written as

$$\sigma_a\sigma_b = i\epsilon_{abc}\sigma_c + \delta_{ab} \tag{10.4}$$

They also obey,

$$\sigma_i^2 = 1 \tag{10.5}$$
$$(\sigma^a)^* = i\sigma_2\sigma^a i\sigma_2 \tag{10.6}$$
$$\mathrm{Tr}\sigma_i = 0 \tag{10.7}$$
$$\det \sigma_i = -1 \tag{10.8}$$

It is a common practice to exponentiate a linear combination of these matrices. We derive the general formula below,

$$e^{i\theta_i\sigma_i} = 1 + i\theta_i\sigma_i - \frac{1}{2}\theta_i\theta_j\sigma_i\sigma_j - ... \tag{10.9}$$
$$= \left(1 - \frac{1}{2}(\theta_i\sigma_i)^2 + ...\right) + i\left(\theta_i\sigma_i - \frac{1}{3!}(\theta_i\sigma_i)^3 + ...\right) \tag{10.10}$$

but

$$\theta_i\theta_j(\sigma_i\sigma_j) = \theta_i\theta_j(i\epsilon_{ijk}\sigma_k + \delta_{ij}) \tag{10.11}$$
$$= \theta^2 \tag{10.12}$$

and

$$(\theta_i\sigma_i)^3 = \theta^2\theta_i\sigma_i \tag{10.13}$$

so

$$e^{i\theta_i\sigma_i} = \cos\theta + i\frac{\theta_i\sigma_i}{\theta}\sin\theta \tag{10.14}$$

$$\sigma^\mu\bar{\sigma}^\nu + \sigma^\nu\bar{\sigma}^\mu = 2\eta^{\mu\nu} \tag{10.15}$$

We also have

$$\mathrm{tr}\left\{\sigma^\mu\bar{\sigma}^\nu\right\} = 2\eta^{\mu\nu} \tag{10.16}$$
$$(\sigma^\mu)_{\alpha\dot{\alpha}}(\bar{\sigma}_\mu)^{\dot{\beta}\beta} = 2\delta_\alpha{}^\beta\delta_{\dot{\alpha}}{}^{\dot{\beta}} \tag{10.17}$$

Furthremore,

$$\sigma^{\mu\nu} = \frac{1}{2i}\epsilon^{\mu\nu\rho\sigma}\sigma_{\rho\sigma} \tag{10.18}$$

We have

$$\epsilon_{0123} = -\epsilon^{0123} = +1 \tag{10.19}$$

and

$$\epsilon_{ijk}\epsilon^{imn} = \delta_j^m\delta_k^n - \delta_j^n\delta_k^m \tag{10.20}$$
$$\epsilon_{jmn}\epsilon^{imn} = 2\delta_j^i \tag{10.21}$$
$$\epsilon_{ij}\epsilon^{in} = \delta_h^n \tag{10.22}$$

Furthermore, we have

$$\epsilon^{\sigma\mu\nu\rho}\epsilon_{\sigma\alpha\beta\gamma} = 6\delta_\alpha^{[\mu}\delta_\beta^\nu\delta_\gamma^{\rho]} \tag{10.23}$$
$$\epsilon^{\mu\nu\rho\sigma}\epsilon_{\rho\sigma\alpha\beta} = -4\delta_\alpha^{[\mu}\delta_\beta^{\nu]} \equiv -4\left(\delta_\alpha^\mu\delta_\beta^\nu - \delta_\alpha^\nu\delta_\beta^\nu\right) \tag{10.24}$$

where the square brackets denote all possible permutations of μ, ν, ρ. Each permutation has a negative sign if it is an odd permutation.

The three dimensional extension of the Pauli matrices are:

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{10.25}$$

10.1.2 Gamma Matrices

In the Weyl basis the Gamma matrices can be written

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \tag{10.26}$$

where $\sigma^\mu = \{\mathbb{1}, \boldsymbol{\sigma}\}$ and $\bar{\sigma}^\mu = \{\mathbb{1}, -\boldsymbol{\sigma}\}$. They obey the defining commutation relation

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \tag{10.27}$$

This leads to many important relations. For example

$$\gamma^\mu\gamma_\mu = \frac{1}{2}\{\gamma^\mu\gamma_\mu + \gamma^\mu\gamma_\mu\} \tag{10.28}$$
$$= \frac{1}{2}\left\{2g^\mu_\mu\right\} \tag{10.29}$$
$$= 4 \tag{10.30}$$

$$\gamma^\mu\gamma^\alpha\gamma_\mu = \gamma^\mu\left(2g^\alpha_\mu - \gamma_\mu\gamma^\alpha\right) \tag{10.31}$$
$$= 2\gamma^\alpha - 4\gamma^\alpha \tag{10.32}$$
$$= -2\gamma^\alpha \tag{10.33}$$

In the Weyl basis the Gamma-5 matrix takes the form

$$\gamma^5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \tag{10.34}$$

and has the properties that

$$(\gamma^5)^\dagger = \gamma^5 \tag{10.35}$$
$$(\gamma^5)^2 = \mathbb{1} \tag{10.36}$$
$$\left\{\gamma^5, \gamma^\mu\right\} = 0 \tag{10.37}$$

The γ^5 matrix can be used to form projection operators:

$$P_L = \frac{1-\gamma^5}{2} \tag{10.38}$$
$$P_R = \frac{1+\gamma^5}{2} \tag{10.39}$$

Furthermore we have,

$$\not{a}\not{b} = a \cdot b - ia_\mu\sigma^{\mu\nu}b_\nu \tag{10.40}$$

and in particular,

$$\not{a}\not{a} = a^2 \tag{10.41}$$

Trace Technology

$$
\begin{aligned}
(\bar{u}\gamma^\mu v)^* &= v^\dagger \gamma^{\mu\dagger} && (10.42)\\
&= \bar{v}\gamma_0 \gamma^{\mu\dagger}\gamma_0 u && (10.43)\\
&= \bar{v}\gamma_0(\gamma_0\gamma^\mu\gamma_0)\gamma_0 u && (10.44)\\
(\bar{u}\gamma^\mu v)^* &= \bar{v}\gamma^\mu u && (10.45)
\end{aligned}
$$

We have

$$
\begin{aligned}
\mathrm{tr}\,(\gamma^\mu\gamma^\nu) &= \frac{1}{2}\mathrm{tr}\,(\gamma^\mu\gamma^\nu + \gamma^\mu\gamma^\nu) && (10.46)\\
&= \frac{1}{2}\mathrm{tr}\,(\gamma^\mu\gamma^\nu + 2\eta^{\mu\nu} - \gamma^\nu\gamma^\mu) && (10.47)\\
&= \frac{1}{2}\mathrm{tr}\,(\gamma^\mu\gamma^\nu + 2\eta^{\mu\nu} - \gamma^\mu\gamma^\nu) && (10.48)\\
&= \frac{1}{2}\mathrm{tr}\,(2\eta^{\mu\nu}) && (10.49)\\
&= 4\eta^{\mu\nu} && (10.50)
\end{aligned}
$$

where we have used the cyclic property of traces.

Furthermore we have

$$
\begin{aligned}
&\mathrm{tr}\left(\gamma^5\right) = 0 && (10.51)\\
&\mathrm{tr}\left(\gamma^\mu\gamma^\nu\gamma^5\right) = 0 && (10.52)\\
&\mathrm{tr}\left(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^5\right) = -4i\epsilon^{\mu\nu\rho\sigma} && (10.53)
\end{aligned}
$$

10.2 Complete the Square

To complete the square we want to take an expression of the form $ax^2 + bx + c$ to $d(x+e)^2 + f$. Expanding the second form gives

$$dx^2 + 2dex + de^2 + f \qquad (10.54)$$

first off, clearly $a = d$. So we make the identifications, $b = 2ae, ae^2 + f = c$. However, we typically want the inverted form of these equations. So we have

$$
\begin{aligned}
d &= a && (10.55)\\
e &= \frac{b}{2a} && (10.56)\\
f &= c - \frac{b^2}{4a} && (10.57)
\end{aligned}
$$

10.3 Degrees of Freedom in a Matrix

The number of degrees of freedom in a unitary matrix are found below. The number of free parameters in a general complex matrix is $2N^2$. Unitarity implies that $U^\dagger U = 1$. We define the elements of U as $a_{ij} + ib_{ij}$. Then unitarity implies

$$(a_{ij} + ib_{ij})(a_{ji} - ib_{ji}) = \delta_{ij} \tag{10.58}$$

which gives two equations,

$$a_{ij}a_{ji} + b_{ij}b_{ji} = \delta_{ij} \tag{10.59}$$

$$b_{ij}a_{ji} - a_{ij}b_{ji} = 0 \tag{10.60}$$

The first equation is symmetric under interchanging $i \leftrightarrow j$ and gives

$$1 + 2 + ... + N = \frac{N(N+1)}{2} \tag{10.61}$$

conditions (just think of it as a matrix equation and the independent equations making up a top right triangle of the matrix).

The second equation is antisymmetric under changing $i \leftrightarrow j$ (the component of $i = j$ vanishes trivially as doesn't offer an extra constraint). This gives

$$1 + 2 + ... + N - 1 = \frac{N(N-1)}{2} \tag{10.62}$$

conditions.

Thus the number of free parameters in a Unitary matrix is

$$2N^2 - N^2 = N^2 \tag{10.63}$$

10.4 Feynman Parameters

Two denometers can be combined as

$$\frac{1}{AB} = \int_0^1 dx \frac{1}{[xA + (1-x)B]^2} \tag{10.64}$$

n denomenators can be combined as

$$\frac{1}{A_1 A_2 ... A_n} = \int_0^1 dx_1\, dx_2 ...\, dx_n \delta(\sum_i x_i - 1) \frac{(n-1)!}{[x_1 A_1 + ... + x_n A_n]^n} \tag{10.65}$$

This is done below for a common two denomenators:

$$\begin{aligned}\frac{1}{(\ell-p)^2-m_1^2+i\epsilon}\frac{1}{\ell^2-m_2^2+i\epsilon} &= \int dx \frac{1}{\left[x((\ell-p)^2-m_1^2+i\epsilon)+(1-x)(\ell^2-m_2^2+i\epsilon)\right]^2} \\ &= \int dx \frac{1}{\left[\ell^2-2\ell px+p^2x+(m_2^2-m_1^2)x-m_2^2+i\epsilon\right]^2} \\ &= \int dx \frac{1}{\left[(\ell-px)^2+p^2x(1-x)+(m_2^2-m_1^2)x-m_2^2+i\epsilon\right]^2} \\ &= \int dx \frac{1}{\left[(\ell-px)^2-\Delta+i\epsilon\right]^2} \end{aligned} \tag{10.66}$$

where we have defined $\Delta \equiv -p^2x(1-x)+(m_1^2-m_2^2)x+m_2^2$.

10.5 n - sphere

The surface area of a unit $n-sphere$ is

$$S_n = (n+1)\frac{\pi^{(n+1)/2}}{\Gamma\left(\frac{n+1}{2}+1\right)} \tag{10.67}$$

Note in this notation a sphere in our 3D world corresponds to $S_2 = 4\pi$.

10.6 Integrals

10.7 Sample Loop Integral

One common integral (usually in $\phi-4$ theory) is the following

$$\int d^4\ell \frac{1}{\ell^2-\Delta+i\epsilon} \tag{10.68}$$

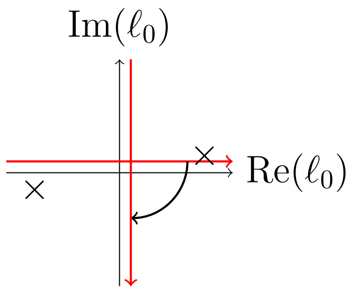
We perform this integral here to refer to this result in the future. We assume $\phi-4$ so we can work with the cutoff. The integral can be split as follows

$$= \int d^3\ell d\ell_0 \frac{1}{\ell_0^2-\ell^2-\Delta+i\epsilon}$$

The poles are at

$$\begin{aligned}\ell_0^2-\ell^2-\Delta+i\epsilon = 0 \\ \ell_0 = \pm\left(\sqrt{\ell^2+\Delta}-i\epsilon\right)\end{aligned}$$

The positions of the poles are shown below



We can apply a Wick rotation such that $\ell_0 \to -i\ell_0$. This gives

$$
\begin{aligned}
&= i \int d\ell d\ell_0 \frac{1}{-\ell_0^2 - \ell^2 - \Delta + i\epsilon} \\
&= -i \int d^4\ell_E \frac{1}{\ell_E^2 + \Delta}
\end{aligned}
$$

where we define a Eulidean four vector, $\ell_E = (\ell_0, \ell)$ with $\ell_E^2 \equiv \ell_0^2 + \ell^2$. Since we are now far from the poles we also omit the $i\epsilon$ factors.

$$
\begin{aligned}
&= -i2\pi^2 \int dk_E \frac{k_E^3}{k_E^2 + \Delta} \\
&= -i\frac{2\pi^2}{\Delta} \int_0^{\Lambda} \Delta^{1/2}\, dx \frac{\Delta^{3/2} x^3}{x^2 + 1} \\
&= -i2\pi^2\Delta \int_0^{\Lambda/m} dx \frac{x^3}{x^2 + 1} \\
&= -i\pi^2\Delta \left(\frac{\Lambda^2}{\Delta} - \log\left(1 + \Lambda^2/\Delta\right) \right) \\
&= -i\pi^2 \left(\Lambda^2 - \Delta \log\left(\frac{\Delta + \Lambda^2}{\Delta} \right) \right)
\end{aligned}
\tag{10.69}
$$

10.7.1 Dimensional Regularization and the Gamma Function

There are two particularly useful integrals when using dim-reg([3], pg. 251):

$$
\int \frac{d^d\ell_E}{(2\pi)^d} \frac{1}{(\ell_E^2 + \Delta)^n} = \frac{1}{(4\pi)^{d/2}} \frac{\Gamma\left(n - \frac{d}{2}\right)}{\Gamma(n)} \Delta^{d/2-n} \tag{10.70}
$$

$$
\int \frac{d^d\ell_E}{(2\pi)^d} \frac{\ell_E^2}{(\ell_E^2 + \Delta)^2} = \frac{1}{(4\pi)^{d/2}} \frac{d}{2} \frac{\Gamma\left(n - \frac{d}{2} - 1\right)}{\Gamma(n)} \Delta^{d/2-n+1} \tag{10.71}
$$

The Gamma function is the generalization of the factorial. It obeys the relationships

$$
\Gamma(n) = (n-1)! \quad , \tag{10.72}
$$

$$\Gamma(n+1) = n\Gamma(n) \quad , \tag{10.73}$$

and

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \tag{10.74}$$

When using "dim reg" to regular your integrals it is often useful to expand $\Gamma(\epsilon)$ to first order:

$$\Gamma(\epsilon) = \frac{1}{\epsilon} - \gamma + \frac{1}{12}(6\gamma^2 + \pi^2)\epsilon + \mathcal{O}(\epsilon^2) \tag{10.75}$$

$$\Gamma(-1+\epsilon) = -\frac{1}{\epsilon} + (\gamma - 1) \tag{10.76}$$

where $\gamma \approx 0.577$.

To expand any other terms of the form Δ^ϵ one can use

$$\begin{aligned} \Delta^\epsilon &= e^{\log \Delta^\epsilon} & (10.77) \\ &= e^{\epsilon \log \Delta} \\ &= 1 + \epsilon \log \Delta & (10.78) \end{aligned}$$

10.7.2 d-dimensional integrals in Minkowski space

$$\int \frac{d^d\ell}{(2\pi)^d} \frac{1}{(\ell^2 - \Delta)^n} = \frac{(-1)^n i}{(4\pi)^{d/2}} \frac{\Gamma\left(n - \frac{d}{2}\right)}{\Gamma(n)} \left(\frac{1}{\Delta}\right)^{n-\frac{d}{2}} \tag{10.79}$$

$$\int \frac{d^d\ell}{(2\pi)^d} \frac{\ell^2}{(\ell^2 - \Delta)^n} = \frac{(-1)^{n-1} i}{(4\pi)^{d/2}} \frac{d}{2} \frac{\Gamma\left(n - \frac{d}{2} - 1\right)}{\Gamma(n)} \left(\frac{1}{\Delta}\right)^{n-\frac{d}{2}-1} \tag{10.80}$$

$$\int \frac{d^d\ell}{(2\pi)^d} \frac{\ell^\mu \ell^\nu}{(\ell^2 - \Delta)^n} = \frac{(-1)^{n-1} i}{(4\pi)^{d/2}} \frac{g^{\mu\nu}}{2} \frac{\Gamma\left(n - \frac{d}{2} - 1\right)}{\Gamma(n)} \left(\frac{1}{\Delta}\right)^{n-\frac{d}{2}-1} \tag{10.81}$$

$$\int \frac{d^d\ell}{(2\pi)^d} \frac{(\ell^2)^2}{(\ell^2 - \Delta)^n} = \frac{(-1)^n i}{(4\pi)^{d/2}} \frac{d/(d+2)}{4} \frac{\Gamma\left(n - \frac{d}{2} - 2\right)}{\Gamma(n)} \left(\frac{1}{\Delta}\right)^{n-\frac{d}{2}-2} \tag{10.82}$$

$$\int \frac{d^d\ell}{(2\pi)^d} \frac{\ell^\mu \ell^\nu \ell^\rho \ell^\sigma}{(\ell^2 - \Delta)^n} = \frac{(-1)^n i}{(4\pi)^{d/2}} \frac{\Gamma\left(n - \frac{d}{2} - 2\right)}{\Gamma(n)} \left(\frac{1}{\Delta}\right)^{n-\frac{d}{2}-2} \times \frac{1}{4}\left(g^{\mu\nu}g^{\rho\sigma} + g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}\right) \tag{10.83}$$

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