
SUPERSYMMETRY LECTURE NOTES

THIS IS A SET OF LECTURE NOTES BASED PRIMARILY ON A COURSE GIVEN BY FERNANDO QUEVEDO. THE MATERIAL IS SUPPLEMENTED WITH MATERIAL FROM *Quantum Theory of Fields, Vol III* BY WEINBERG, *Modern Supersymmetry* BY JOHN TERNIN AS WELL AS A FEW OF MY OWN ADDITIONS. WE BEGIN WITH DISCUSSING THE MATHEMATICAL TOOLS INVOLVED IN SUPERSYMMETRY. WE THEN MOVE ON TO DISCUSSING THE THEORY IN DETAIL WITH A FOCUS ON $\mathcal{N} = 1$. LASTLY, WE MOVE ON TO THE MSSM AND PHENOMENOLOGY.

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2013
CORNELL UNIVERSITY

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Chapter 1

Introduction

The standard reference is Wess and Bagger. However, this book is very dry. You can trust their equations is right, however there aren't many physical motivations in the book. There are plenty of review articles that we can use as well.

1.1 Physical Motivations of SUSY

1.1.1 What do we know so far?

- Quantum Mechanics and Special Relativity - together give QFT.
We have a field. Excitations of this field give particles. From this we have learned that there are two completely different types of particles, bosons and fermions. They are basically unrelated and behave very differently from each other. Bosons have integer spin while fermions have half integer spin.
- An example of a very successful quantum field theory is the Standard Model (SM).
 - The SM describes the strong and electroweak interactions which are mediated by $s = 1$ particles (gluons, photons, W^\pm, Z).
 - The matter fields have spin 1/2 and these are the quarks and leptons
 - The Higgs boson is a spin 0 particle
 - The graviton is spin 2
- This is just one example of a QFT. There are many QFT's.
- The Basic Principle is symmetry
 - Spacetime symmetries (Lorentz, Poincare, General Coordinate Transformation)
 - Internal symmetries which transform the fields themselves without transforming space and time. In the SM the symmetry is $G_{SM} = \underbrace{SU(3)_C}_{\text{strong}} \times \underbrace{SU(2)_L \times U(1)}_{\text{Electroweak}}$

- Importance of symmetries are that

- Label particles: mass, spin, charge, colour, etc.
- Determine interactions between particles (through the “gauge principle”)
- As an example consider

$$\mathcal{L} = \partial_\mu \phi \partial^\mu \phi^* - V(|\phi|^2) \quad (1.1)$$

where ϕ is a scalar field. This Lagrangian is invariant under the transformation, $\phi \rightarrow e^{i\alpha}\phi$; $\alpha \rightarrow \text{const}$. This is called a global symmetry. If we want α to be a function of spacetime, $\alpha = \alpha(x)$ then the potential is still invariant but the kinetic term isn't,

$$\partial_\mu \phi \rightarrow e^{i\alpha} (i(\partial_\mu \alpha)\phi + \partial_\mu \phi) \quad (1.2)$$

but if we modify and have

$$D_\mu \phi \equiv \partial_\mu \phi + iA_\mu \phi \rightarrow e^{i\alpha} (\partial_\mu \phi + i(\partial_\mu \alpha)\phi + iA'_\mu \phi) \quad (1.3)$$

This is equal to $D_\mu \phi$ if $A'_\mu = A_\mu - \partial_\mu \alpha$. We get a coupling $A_\mu \phi A^\mu \phi^*$.

Similarly for $\mathcal{L} = \bar{\psi} \gamma^\mu D_\mu \psi$, we get a Dirac coupling, $\bar{\psi} A_\mu \psi$.

- Another property of symmetries that is very important is that symmetries can hide. If we take again the same Lagrangian,

$$\mathcal{L} = D_\mu \phi D^\mu \phi^* - V(|\phi|) \quad (1.4)$$

If V is a potential as $\sim |\phi|^2 + |\phi|^4$ then we have a symmetry about $\phi \rightarrow e^{i\alpha}\phi$ with the vacuum sharing the same symmetry as the potential. We have $\langle \phi \rangle = 0$. The symmetry is manifest. However, if we have the mexican hat potential then you have the same symmetry, $\phi \rightarrow e^{i\alpha}\phi$ but the minimum is at $\langle \phi \rangle \neq 0$. The symmetric point is not the minimum. The minimum is not invariant under this phase rotation. You will not see the symmetry at all. The symmetry will be hidden. This is very important since the real symmetries of nature may be much bigger than what we see because we live in a world that breaks that symmetry. This phenomena is known as spontaneous symmetry breaking. In the SM we have

$$SU(2)_L \times U(1) \xrightarrow{\langle \phi \rangle \sim 10^2 \text{ GeV}} U(1)$$

The symmetries can hide from us, but if we are clever enough we can still uncover them.

1.1.2 Problems with the SM

- Quantum Gravity
- Why? There are many why questions in the SM that we don't know. For example why is the gauge group of the Standard Model is $G_{SM} = SU(3) \times SU(2) \times U(1)$? Why there are three families of quarks and leptons? Why is it that we live in $3 + 1$ dimensions? etc.
- We have $M_{EW} \sim 10^2 \text{GeV}$, $M_{Planck} \sim 10^{19} \text{GeV}$ and so

$$\frac{M_{EW}}{M_{Planck}} \lesssim 10^{-15} \ll 1 \quad (1.5)$$

This is the standard hierarchy problem.

- There is a second hierarchy problem that is even more difficult. There is now evidence that our universe is accelerating due to a vacuum energy which is about zero,

$$\Lambda \sim 10^{-120} \quad (1.6)$$

which is completely different then the energy of the vacuum calculated by QFT.

$$\frac{M_\Lambda}{M_{EW}} \sim 10^{-15} \ll 1 \quad (1.7)$$

This is known as the cosmological constant problem.

1.1.3 Beyond SM

- Experiments such as the LHC are looking for new particles
- We may have more symmetries
 - Internal: $G \rightarrow G_{SM} \rightarrow SU(3) \times U(1)$
 - Spacetime symmetries
 - * More dimensions
 - * Supersymmetry
 - * Beyond QFT - string theory (needs supersymmetry)

Chapter 2

Supersymmetry Algebra and Representations

2.1 Poincare Symmetry and Spinors (Review)

Poincare: Given x^μ ,

$$x'^\mu = \underbrace{\Lambda^\mu{}_\nu x^\nu}_{\text{Lorentz}} + \underbrace{a^\mu}_{\text{Translation}} \quad (2.1)$$

The Lorentz transforms obey,

$$\Lambda^T \eta \Lambda = \eta, \eta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad (2.2)$$

We work with the isochronous Lorentz group. Which is the Lorentz group connected to the identity which is denoted

$$SO(3,1)^\uparrow \quad (2.3)$$

though we will often just omit the up arrow in our notation. The algebra consists of generators, $M^{\mu\nu}, P^\sigma$ such that

$$[P^\mu, P^\nu] = 0 \quad (2.4)$$

$$[M^{\mu\nu}, P^\gamma] = i(P^\mu \eta^{\nu\gamma} - P^\nu \eta^{\mu\gamma}) \quad (2.5)$$

$$[M^{\mu\nu}, M^{\rho\sigma}] = i(M^{\mu\sigma} \eta^{\nu\rho} + M^{\nu\rho} \eta^{\mu\sigma} - M^{\mu\rho} \eta^{\nu\sigma} - M^{\nu\sigma} \eta^{\mu\rho}) \quad (2.6)$$

e.g. A representation for the generators are

$$(M^{\rho\sigma})^\mu{}_\nu = i(\eta^{\rho\mu} \delta^\sigma{}_\nu - \eta^{\sigma\mu} \delta^\rho{}_\nu) \quad (2.7)$$

As an exercise you can multiply these matrices and see if they satisfy the algebra.

2.2 Properties of the Lorentz Group

$SO(3, 1) \sim SU(2) \times SU(2)$ (Locally) since we can define

$$J_i \equiv \frac{1}{2} \epsilon_{ijk} M_{jk} \quad (\text{rotations}) \quad (2.8)$$

$$K_i \equiv M_{0i} \quad (\text{Lorentz Boosts}) \quad (2.9)$$

where $i, j, k \in 1, 2, 3$. We can now define

$$A_i \equiv \frac{1}{2} (J_i + iK_i) \quad (2.10)$$

$$B_i \equiv \frac{1}{2} (J_i - iK_i) \quad (2.11)$$

which satisfy

$$[A_i, A_j] = i\epsilon_{ijk} A_k \quad (2.12)$$

$$[B_i, B_j] = i\epsilon_{ijk} B_k \quad (2.13)$$

$$[A_i, B_j] = 0 \quad (2.14)$$

We have two $SU(2)$ algebras. However, A and B are non-Hermitian in general. So A and B do not truly represent spin operators. However note the fact that representations are labeled by “spins” (A, B) but physical spin $J = A + B$. Under Parity ($x^0 \rightarrow x^0$, $x^i \rightarrow -x^i$) we have $J_i \rightarrow J_i$, $K_i \rightarrow -K_i$ (this is found by looking at equations 2.8 and 2.9 and by how many spatial indices each of J and K have). Under Parity we then have $A \leftrightarrow B$. This is one important property of $SO(3, 1)$.

Another important property is that

$$SO(3, 1) \cong SL(2, \mathbb{C}) \quad (2.15)$$

$SL(2, \mathbb{C})$ is the group of 2×2 matrices with unit determinant. The symbol \cong represents homomorphic. It is not isomorphic because the map is not one-to-one, but in fact its two-to-one as we will see.

Since

$$\bar{x} \equiv x_\mu e^\mu = \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad (2.16)$$

and

$$\tilde{x} \equiv x_\mu \sigma^\mu = \begin{pmatrix} x_0 + x_3 & x_1 - ix_2 \\ x_1 + ix_2 & x_0 - x_3 \end{pmatrix} \quad (2.17)$$

where σ^μ are the Pauli matrices. The matrix and vector carry the same information. The interesting thing is that under $SO(3, 1)$,

$$|x|^2 = x_0^2 - x_1^2 - x_2^2 - x_3^2 \quad (2.18)$$

is invariant but under $SL(2, \mathbb{C})$

$$\tilde{x} \rightarrow \tilde{x}' = N\tilde{x}N^\dagger \quad (2.19)$$

where $N \in SL(2, \mathbb{C})$ and

$$\det \tilde{x} = \det \tilde{x}' = x_0^2 - x_1^2 - x_2^2 - x_3^2 \quad (2.20)$$

However this map is not one-to-one but two-to-one since:

$$N = \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \rightarrow \Lambda = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (2.21)$$

$SL(2, \mathbb{C})$ is closer the algebra and hence defines the representations of the algebra and is a universal covering group. This is the group that is simply connected.

2.3 Representations of $SL(2, \mathbb{C})$

The fundamental representation is

$$\psi'_\alpha = N_\alpha{}^\beta \psi_\beta \quad (2.22)$$

where $\alpha, \beta = 1, 2$ with N being a 2×2 matrix. The conjugate representation is

$$\bar{\chi}'_{\dot{\alpha}} = N^*_{\dot{\alpha}}{}^{\dot{\beta}} \bar{\chi}_{\dot{\beta}} \quad (2.23)$$

with $\dot{\alpha}, \dot{\beta} = 1, 2$. These two representations are independent. There are two contravariant representations

$$\psi'^{\alpha} = \psi^\beta (N^{-1})_\beta{}^\alpha \quad (2.24)$$

$$\bar{\chi}^{\dot{\alpha}} = \bar{\chi}^{\dot{\beta}} (N^{-1,*})_{\dot{\beta}}{}^{\dot{\alpha}} \quad (2.25)$$

2.4 Invariant Tensors

We already know that

$$\eta_{\mu\nu} = (\eta^{\mu\nu})^{-1} \quad (2.26)$$

is invariant under $SO(3, 1)$. Here in $SL(2, \mathbb{C})$ the invariant tensor is

$$\epsilon^{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = -\epsilon_{\alpha\beta} \quad (2.27)$$

It is invariant because

$$\epsilon'^{\alpha\beta} = \epsilon^{\rho\sigma} N_\rho{}^\alpha N_\sigma{}^\beta = \det N \epsilon^{\alpha\beta} = \epsilon^{\alpha\beta} \quad (2.28)$$

Now we can raise and lower indices as we wish with $\epsilon^{\alpha\beta}$. In particular,

$$\psi^\alpha = \epsilon^{\alpha\beta} \psi_\beta \quad (2.29)$$

$$\bar{\chi}^{\dot{\alpha}} = \epsilon^{\dot{\alpha}\dot{\beta}} \bar{\chi}_{\dot{\beta}} \quad (2.30)$$

where we have defined $\epsilon^{\dot{\alpha}\dot{\beta}}$ to be the same as $\epsilon^{\alpha\beta}$. The contravariant representations are not independent. They can be obtained from the fundamental by raising and lowering the indices with $\epsilon^{\alpha\beta}$.

The other object that can be considered here is something that has mixed $SL(2, \mathbb{C})$ and $SO(3, 1)$ indices,

$$(\sigma^\mu)_{\alpha\dot{\alpha}} \quad , \quad \sigma^\mu = \{\mathbb{1}, \boldsymbol{\sigma}\} \quad (2.31)$$

We know that

$$(x_\mu \sigma^\mu)_{\alpha\dot{\alpha}} \rightarrow N_\alpha{}^\beta (x_\nu \sigma^\nu)_{\beta\dot{\gamma}} N^*_{\dot{\alpha}}{}^{\dot{\gamma}} = \Lambda_\mu{}^\nu x_\nu \sigma^\mu_{\alpha\dot{\alpha}} \quad (2.32)$$

From here we see that

$$\sigma^\mu_{\alpha\dot{\alpha}} = N_\alpha{}^\beta \sigma^\nu_{\beta\dot{\gamma}} (\Lambda^\mu{}_\nu)^{-1} N^*_{\dot{\alpha}}{}^{\dot{\gamma}} \quad (2.33)$$

which is invariant. Similar relations hold for $\bar{\sigma}$,

$$(\bar{\sigma}^\mu)^{\dot{\alpha}\alpha} \equiv \epsilon^{\alpha\beta} \epsilon^{\dot{\alpha}\dot{\beta}} \sigma^\mu_{\beta\dot{\beta}} = (\mathbb{1}, -\boldsymbol{\sigma}) \quad (2.34)$$

2.5 Generators of $SL(2, \mathbb{C})$

$$(\sigma^{\mu\nu})_\alpha{}^\beta \equiv \frac{i}{4} (\sigma^\mu \bar{\sigma}^\nu - \sigma^\nu \bar{\sigma}^\mu)_\alpha{}^\beta \quad (2.35)$$

$$(\bar{\sigma}^{\mu\nu})^{\dot{\alpha}}{}_{\dot{\beta}} \equiv \frac{i}{4} (\bar{\sigma}^\mu \sigma^\nu - \bar{\sigma}^\nu \sigma^\mu)^{\dot{\alpha}}{}_{\dot{\beta}} \quad (2.36)$$

Both of $\sigma^{\mu\nu}$ and $\bar{\sigma}^{\mu\nu}$ satisfy the Lorentz algebra which implies that

$$\psi_\alpha \rightarrow \left(e^{-\frac{i}{2} \omega_{\mu\nu} \sigma^{\mu\nu}} \right)_\alpha{}^\beta \psi_\beta \quad (2.37)$$

$$\bar{\chi}^{\dot{\alpha}} \rightarrow \left(e^{\frac{i}{2} \omega_{\mu\nu} \bar{\sigma}^{\mu\nu}} \right)^{\dot{\alpha}}{}_{\dot{\beta}} \bar{\chi}^{\dot{\beta}} \quad (2.38)$$

ψ is the left handed spinor. You can check that $(A, B) \rightarrow (1/2, 0)$. $\bar{\chi}^{\dot{\alpha}}$ is a right handed spinor, and you can check that $(A, B) \rightarrow (0, 1/2)$. Parity exchanges A and B which implies that $(0, 1/2) \leftrightarrow (1/2, 0)$.

For $(1/2, 0) : J_i = \frac{1}{2} \sigma_i$ and $K_i = -\frac{i}{2} \sigma_i$. For the $(0, 1/2)$ we have $J_i = \frac{1}{2} \sigma_i$ and $K_i = \frac{i}{2} \sigma_i$

There are some useful relations for our generators and the Pauli matrices,

$$\sigma^\mu \bar{\sigma}^\nu + \sigma^\nu \bar{\sigma}^\mu = 2\eta^{\mu\nu} \mathbb{1} \quad (2.39)$$

$$\text{Tr} \sigma^\mu \bar{\sigma}^\nu = 2\eta^{\mu\nu} \quad (2.40)$$

$$(\sigma^\mu)_{\alpha\dot{\alpha}} (\bar{\sigma}_\mu)^{\beta\dot{\beta}} = 2\delta_\alpha^\beta \delta_{\dot{\alpha}}^{\dot{\beta}} \quad (2.41)$$

$$\sigma^{\mu\nu} = \frac{1}{2i} \epsilon^{\epsilon\mu\rho\sigma} \sigma_{\rho\sigma} \quad (\text{self-dual}) \quad (2.42)$$

2.6 Product of Spinors

We have the product of left-handed spinors,

$$\chi\psi \equiv \chi^\alpha\psi_\alpha = -\chi_\alpha\psi^\alpha \quad (2.43)$$

Similarly for right handed spinors,

$$\bar{\chi}\bar{\psi} \equiv \bar{\chi}_{\dot{\alpha}}\bar{\psi}^{\dot{\alpha}} = -\bar{\chi}^{\dot{\alpha}}\bar{\psi}_{\dot{\alpha}} \quad (2.44)$$

In particular,

$$\psi^2 = \psi^\alpha\psi_\alpha = \epsilon^{\alpha\beta}\psi_\beta\psi_\alpha = \psi_2\psi_1 - \psi_1\psi_2 \quad (2.45)$$

So the square of the spinor is the difference above. This is one motivation to choose these numbers to be anticommuting. If they were commuting we would have $\psi^2 = 1$. We choose the components of ψ to be anti commuting numbers, or *Grassman numbers*: $\psi_1\psi_2 = -\psi_2\psi_1$. Which gives

$$\chi\psi = \psi\chi \quad (2.46)$$

We also have

$$\psi_\alpha\psi_\beta = \frac{1}{2}\epsilon_{\alpha\beta}(\psi\psi) \quad (2.47)$$

which leads the Fierz identity,

$$(\theta\psi)(\theta\psi) = -\frac{1}{2}(\psi\psi)(\theta\theta) \quad (2.48)$$

and

$$\psi\sigma^{\mu\nu}\chi = -(\chi\sigma^{\mu\nu}\psi) \quad (2.49)$$

We define the dagger such that

$$(\theta\psi)^\dagger = \bar{\theta}\bar{\psi} \quad (2.50)$$

and

$$(\psi\sigma^\mu\bar{\chi})^\dagger = \chi\sigma^\mu\bar{\psi} \quad (2.51)$$

$$(\psi_\alpha)^\dagger \equiv \bar{\psi}_{\dot{\alpha}} \quad (2.52)$$

$$\bar{\psi}^{\dot{\alpha}} = \psi_\beta^*(\sigma^0)^{\beta\dot{\alpha}} \quad (2.53)$$

where the lower identity is just a way to control the indices ($\sigma^0 = \mathbb{1}$)

2.7 Connection to Dirac Spinors

We define

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \quad (2.54)$$

then

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \quad (2.55)$$

That means that these Gamma matrices defined in this way are a representation of the Dirac algebra (Clifford Algebra for Dirac matrices). We also have

$$\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} -\mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix} \quad (2.56)$$

This matrix has eigenvalues of ± 1 which is defined as the *chirality*. We work in the Weyl representation, where the chirality of a state is manifest.

Furthermore, you can define,

$$\Sigma^{\mu\nu} \equiv \frac{i}{4}\gamma^{\mu\nu} = \begin{pmatrix} \sigma^{\mu\nu} & 0 \\ 0 & \bar{\sigma}^{\mu\nu} \end{pmatrix} \quad (2.57)$$

These are the generators of the Lorentz group. Dirac spinors,

$$\Psi \equiv \begin{pmatrix} \psi_\alpha \\ \bar{\chi}^{\dot{\alpha}} \end{pmatrix} \quad (2.58)$$

Notice

$$\gamma^5\Psi = \begin{pmatrix} -\psi_\alpha \\ \bar{\chi}^{\dot{\alpha}} \end{pmatrix} \quad (2.59)$$

So we can define projection operators,

$$P_{R/L} \equiv \frac{1}{2}(1 \pm \gamma^5) \quad (2.60)$$

such that

$$P_L\Psi = \begin{pmatrix} \psi_\alpha \\ 0 \end{pmatrix} \quad (2.61)$$

$$P_R\Psi = \begin{pmatrix} 0 \\ \bar{\chi}^{\dot{\alpha}} \end{pmatrix} \quad (2.62)$$

Finally the conjugate is given by

$$\bar{\Psi} \equiv (\chi^\alpha \quad \bar{\psi}_{\dot{\alpha}}) = \Psi^\dagger \gamma^0 \quad (2.63)$$

We have the column conjugate,

$$\Psi^c \equiv \begin{pmatrix} \chi_\alpha \\ \bar{\psi}^{\dot{\alpha}} \end{pmatrix} = C\bar{\Psi}^T \quad (2.64)$$

where

$$C \equiv \begin{pmatrix} \epsilon_{\alpha\beta} & 0 \\ 0 & \epsilon^{\dot{\alpha}\dot{\beta}} \end{pmatrix} \quad (2.65)$$

takes particles to anti-particles. If Ψ satisfies the Dirac equation for a given charge then Ψ^c satisfies the same equation with opposite charge.

We can have majorana spinors, $\psi_\alpha = \chi_\alpha$ where

$$\Psi = \begin{pmatrix} \psi_\alpha \\ \bar{\psi}^\alpha \end{pmatrix} = \Psi^c \quad (2.66)$$

We can always write a Dirac spinor as a sum of Majorana spinors.

$$\Psi = \Psi_{M1} + i\Psi_{M2} \quad (2.67)$$

We can ask the question whether there is a spinor that is both Weyl and Majorana in 4 dimensions. This is not possible since if it's Weyl then one of its components is zero, however by definition Majorana particles have the same value for both left and right components. Hence these spinors are zero.

Chapter 3

SUSY Algebra

3.1 History of SUSY

- In the 1960's there was a lot of confusion. There were many things going on at this time. People were putting particles in multiplets through their isospin. These combined particles of the same spin into multiplets of $SU(3)_f$. People began thinking why not mix particles from different spins into their own larger multiplets.
- In 1967: Coleman-Mandula released a paper that introduced what they called *No-go theorem*. They tried to look for the most general symmetries for scattering amplitudes. They concluded that the most general symmetry of the S-matrix is the direct product of

$$\underbrace{\text{Poincare}}_{P^\mu, M^{\mu\nu}} \otimes \overbrace{\text{internal}}^{T^a} \quad (3.1)$$

- 1971: Golfand and Lickman extended Poincare algebra to include spinor generators, Q_α , $\alpha = 1, 2$.
Ramond, Neveu + Schwarz, Gervais + Sakita discussed SUSY in 2D (from string theory)
- 1973: Volkov and Akulov were trying to understand why the neutrino was a massless particle. They tried to think of neutrinos as Goldstone particles, $m = 0$. They discovered how to break supersymmetry.
- 1974: Wess + Zumino began SUSY field theories in 4D. This paper is considered the breakthrough of SUSY and implied how it can be used in field theories.

In 1975 there was an interesting paper by Haag, Lopuszanski and Sohnius where they took studied the Coleman-Mandula (CM) theorem and generalized the theorem to include spinor generators. We will now build the algebra as these people did about 30 years ago. The Poincare generators are $P^\mu, M^{\mu\nu}$. In principle there can be any other

generators of any number of indices but they are ruled out by the CM theorem. For SUSY we will add generators in the $(\frac{1}{2}, 0)$ representation which we call

$$Q_\alpha^A \quad \begin{matrix} A=1,2,\dots,N \\ \alpha=1,2 \end{matrix} \quad (3.2)$$

and we will also add operators in the $(0, \frac{1}{2})$ representation given by

$$\bar{Q}_{\dot{\alpha}}^A \quad \begin{matrix} A=1,2,\dots,N \\ \dot{\alpha}=1,2 \end{matrix} \quad (3.3)$$

We will not prove it, however these are the only generators you can add. The proof is given in Weinberg. These operators will obey different types of relations we are used to, since typically Lie groups obey commutation relations. However, spinor relations obey what's known as a Graded Algebra,

$$\mathcal{O}_a \mathcal{O}_b - (-1)^{\eta_a \eta_b} \mathcal{O}_b \mathcal{O}_a = i C_{ab}^c \mathcal{O}_c \quad (3.4)$$

for which $\eta_a = 0$ for \mathcal{O}_a bosonic generator and $\eta_a = 1$ for \mathcal{O}_a fermionic generator. η_a is called a grading. For bosons we have the standard commutator. For 2 fermions we have an anticommutator.

The SUSY generators are $P^\mu, M^{\mu\nu}, Q_\alpha^A, \bar{Q}_{\dot{\alpha}}^A, A = 1, \dots, N$. $N = 1$ is called a simple SUSY. $N > 1$ is called extended SUSY.

We start with $N = 1$. We know

$$[P^\mu, P^\nu] \quad (3.5)$$

$$[P^\mu, M^{\rho\sigma}] \quad (3.6)$$

$$[M^{\mu\nu}, M^{\rho\sigma}] \quad (3.7)$$

We need to find

1. $[Q_\alpha, M^{\mu\nu}]$
2. $[Q_\alpha, P^\mu]$
3. $\{Q_\alpha, Q_\beta\}$
4. $\{Q_\alpha, \bar{Q}_{\dot{\alpha}}\}$
5. $[Q_\alpha, T_i]$

where T_i are internal symmetry generators (e.g. $U(1)$). Because of CM we know that the T_i commute with the $P^\mu, M^{\mu\nu}$ but we don't know that holds for the spinor generators.

We begin by trying to find the first one. To find the commutator of Q and M we need to use the fact that Q_α is a spinor. Hence it must transform under the spinor representation of the Lorentz Group. We know that

$$Q'_\alpha = \left(e^{-\frac{i}{2} \omega_{\mu\nu} \sigma^{\mu\nu}} \right)_\alpha^\beta Q_\beta \quad (3.8)$$

$$\approx \left(1 - \frac{i}{2} \omega_{\mu\nu} \sigma^{\mu\nu} \right)_\alpha^\beta Q_\beta \quad (3.9)$$

But Q is not only a spinor it is also as an operator. Q as an operator also transforms under Lorentz Transformation. As an operator it transforms as follows

$$Q'_\alpha = U^\dagger Q_\alpha U \quad , \quad U = e^{-\frac{i}{2}\omega_{\mu\nu}M^{\mu\nu}} \quad (3.10)$$

Note that we wrote the Lorentz transformations in two different ways. This is the standard way to derive the relations. You can similarly do this to get the algebra from the $M^{\mu\nu}$'s and P^μ 's. We can write

$$Q'_\alpha = \left(1 + \frac{i}{2}\omega_{\mu\nu}M^{\mu\nu}\right) Q_\alpha \left(1 - \frac{i}{2}\omega_{\mu\nu}M^{\mu\nu}\right) \quad (3.11)$$

Comparing with our other expression for Q'_α we have

$$-\frac{i}{2}\omega_{\mu\nu}(\sigma^{\mu\nu})_\alpha{}^\beta Q_\beta = \frac{i}{2}\omega_{\mu\nu}[M^{\mu\nu}, Q_\alpha] - \frac{1}{4}\omega_{\mu\nu}\omega_{\rho\sigma}M^{\mu\nu}Q_\alpha M^{\rho\sigma} \quad (3.12)$$

Dropping terms proportional to ω^2 we have

$$\boxed{[Q_\alpha, M^{\mu\nu}] = (\sigma^{\mu\nu})_\alpha{}^\beta Q_\beta} \quad (3.13)$$

where we have used the fact that the equation must hold for all $\omega_{\mu\nu}$.

We now need to derive the second relation, the one with P^μ and Q_α . The commutator $[Q_\alpha, P^\mu]$ must be linear in the P 's, M 's and Q 's. We know already that the Q_α is anticommuting and P^μ is commuting so the right hand side must also be anticommuting. We cannot have just M 's or just P 's. The only thing you can write down that satisfies the conditions is

$$[Q_\alpha, P^\mu] = c\sigma_{\alpha\dot{\alpha}}^\mu \bar{Q}^{\dot{\alpha}} \quad (3.14)$$

where c is a constant. The only problem left is to fix the constant. Still we need to know the complex conjugate of this expression which gives,

$$[\bar{Q}^{\dot{\alpha}}, P^\mu] = c^*(\bar{\sigma}^\mu)^{\dot{\alpha}\beta} Q_\beta \quad (3.15)$$

Exercise! (Hint: take adjoint and use $(Q_\alpha)^\dagger = \bar{Q}_{\dot{\alpha}}$ and $(\sigma^\mu Q)_\alpha^\dagger = (\bar{Q}\sigma^\mu)_{\dot{\alpha}}$).

To find the constant we use the Jacobi identity which says that

$$[A, [B, C]] + [C, [A, B]] + [B, [C, A]] = 0 \quad (3.16)$$

Taking $A = P^\mu$, $B = P^\nu$, and $C = Q_\alpha$ gives

$$[P^\mu, [P^\nu, Q_\alpha]] + \cancel{[Q_\alpha, [P^\mu, P^\nu]]} + [P^\nu, [Q_\alpha, P^\mu]] = 0 \quad (3.17)$$

$$-c\sigma_{\alpha\dot{\alpha}}^\nu [P^\mu, \bar{Q}^{\dot{\alpha}}] + c\sigma_{\alpha,\dot{\alpha}}^\mu [P^\nu, \bar{Q}^{\dot{\alpha}}] = 0 \quad (3.18)$$

$$|c|^2 \sigma_{\alpha\dot{\alpha}}^\nu (\bar{\sigma}^\mu)^{\dot{\alpha}\beta} Q_\beta - |c|^2 \sigma_{\alpha\dot{\alpha}}^\mu (\bar{\sigma}^\nu)^{\dot{\alpha}\beta} Q_\beta = 0 \quad (3.19)$$

$$|c|^2 \underbrace{(\sigma^\nu \bar{\sigma}^\mu - \sigma^\mu \bar{\sigma}^\nu)_{\alpha\dot{\alpha}}^\beta}_{\neq 0} Q_\beta = 0 \quad (3.20)$$

and hence $c = 0$ and we have

$$[Q_\alpha, P^\mu] = 0 \quad (3.21)$$

This can be derived insightfully by just realizing that Q_α is a spinor which are invariant under spatial translations, and hence the commutation relation must be zero.

We need to derive two more relations. First we need $\{Q_\alpha, Q_\beta\}$. Because of the index structure of the Q 's we must have

$$\{Q_\alpha, Q^\beta\} = k (\sigma^{\mu\nu})_\alpha{}^\beta M_{\mu\nu} \quad (3.22)$$

We need to find the constant k . The left hand side commutes with P^μ (as we just found). The right side does not commute with P^μ (the Lorentz generators do not commute with P^μ). The only way this can be true is if

$$\{Q_\alpha, Q_\beta\} = 0 \quad (3.23)$$

To finish we need to find $\{Q_\alpha, \bar{Q}_{\dot{\beta}}\}$. Q_α is a $(1/2, 0)$ object while $\bar{Q}_{\dot{\beta}}$ is a $(0, 1/2)$ spinor. So on the right hand side we must have a $(1/2, 1/2)$ object which is P^μ . So

$$t \sigma_{\alpha\dot{\alpha}}^\mu P_\mu \quad (3.24)$$

There is no way to fix t . All the games that we played above will not work here. So by convention we take $t = 2$ and we have the final relation

$$\{Q_\alpha, \bar{Q}_{\dot{\beta}}\} = 2 \sigma_{\alpha\dot{\beta}}^\mu P_\mu \quad (3.25)$$

So the anticommutator of two spinors gives a translation. The away to think about this is $Q_\alpha |fermion\rangle = |boson\rangle$ and $\bar{Q}_{\dot{\alpha}} |boson\rangle = |fermion\rangle$ and if we act on a boson twice as

$$Q_\alpha \bar{Q}_{\dot{\alpha}} |boson\rangle = [boson] \quad (\text{translated}) \quad (3.26)$$

There is one thing we have yet to discuss which is with the internal symmetries, $[Q_\alpha, T_i]$. Usually as in the case for the CM theorem,

$$[Q_\alpha, T_i] = 0 \quad (3.27)$$

except that the supersymmetry algebra has what is called an automorphism which changes

$$Q_\alpha \rightarrow e^{i\lambda} Q_\alpha \quad (3.28)$$

$$\bar{Q}_{\dot{\alpha}} \rightarrow e^{-i\lambda} \bar{Q}_{\dot{\alpha}} \quad (3.29)$$

where you multiply Q by a phase and \bar{Q} by an opposite phase, the whole algebra doesn't change since in the commutation and anticommutation relations we found above Q only has non-trivial commutation relations when it comes with an \bar{Q} . This is called R symmetry (it is a $U(1)$ symmetry). We have

$$[Q_\alpha, R] = Q_\alpha \quad (3.30)$$

$$[\bar{Q}_{\dot{\alpha}}, R] = \bar{Q}_{\dot{\alpha}} \quad (3.31)$$

3.2 Representations of the Poincare Group

We now try to get a better understanding of state labelling. First consider the non-relativistic labelling of particles with spin. Recall that the rotation group, $\{J_i : i = 1, 2, 3\}$ satisfies

$$[J_i, J_j] = i\epsilon_{ijk}J_k \quad (3.32)$$

with a Casimir operator (an operator that commutes with all the generators)

$$J^2 = \sum_{i=1}^3 J_i^2 \quad (3.33)$$

The Casimir operator labels the irreducible representations by the eigenvalues, $j(j+1)$ of J^2 . Within these representations, you can diagonalize one of the operators, J_3 to get the eigenvalues $\{-j, -j+1, \dots, j-1, j\}$. The states can then be labelled as $|j, j_3\rangle$.

This is not sufficient for relativistic physics since here we want to relativistic operators, such as P^μ and $M^{\mu\nu}$ to label our states. Recall that in the Poincare group there are two Casimir operators. One of these involves the Pauli-Lubanski vector,

$$W_\mu = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}P^\nu M^{\rho\sigma} \quad (3.34)$$

where $\epsilon_{0123} = -\epsilon^{0123} = +1$. The Poincare Casimirs are then given by

$$C_1 = P^\mu P_\mu, \quad C_2 = W^\mu W_\mu \quad (3.35)$$

It is straightforward to check that these both commute with the generators of the Poincare group. We can label our states with the eigenvalues of our two Casimirs,

$$|m, w\rangle \quad (3.36)$$

However, that is not all. Just as we used j_3 to label our states above, we can also look for generators that commute with the momenta. **[Q 1: Don't these generators need also to commute with W_μ ?]** These tell you which are sensible labels for your states. The largest group that leave your momenta invariant are known as the Little Group. Consider a massive particle. You can always go to the rest frame and then you have

$$P^\mu = (m, 0, 0, 0) \quad (3.37)$$

In this case the little group is equal to the rotation group in three dimensions. In this frame it is easy to calculate what the Pauli-Lubanski vector is,

$$W_0 = \frac{1}{2}\epsilon_{0\nu\rho\sigma}P^\nu M^{\rho\sigma} \quad (3.38)$$

Since the momentum is zero in this frame we have $W_0 = 0$ in the rest frame. Now consider W_i . In this case we have

$$W_i = \frac{1}{2}\epsilon_{i\nu\rho\sigma}P^\nu M^{\rho\sigma} \quad (3.39)$$

As before the only non-trivial component of P_μ is P_0 so

$$W_i = \frac{1}{2}\epsilon_{i0jk}P^0M^{jk} = \dots = -mJ_i \quad (3.40)$$

where the above requires some playing with the epsilon symbols. So we have $W^2 = m^2J^2$. Clearly, J_i and J^2 commute with W_μ, P_μ, W^2 , and P^2 . So the w in the labelling of the states can be interpreted as the j of the multiplet and we can use j_i as our quantum number. So for massive particles we can label the representations of the Poincaré group by,

$$|m, j; p^\mu, j_3\rangle \quad (3.41)$$

For massless particles, it is a bit more complicated. You can choose the momenta to be

$$P^\mu = (E, 0, 0, E) \quad (3.42)$$

Naively anything that rotations x and y is the Little Group. However, this is not quite all. The group is actually bigger than that. The symmetry transformations of the Pauli-Lubanski vector define the Little Group. [Q 2: Why?] One can go ahead and calculate the components of the Pauli-Lubanski vector in this case,

$$W_0 = EJ_3 \quad (3.43)$$

$$W_1 = -E(J_1 - K_2) \quad (3.44)$$

$$W_2 = -E(J_2 + K_1) \quad (3.45)$$

$$W_3 = -EJ_3 \quad (3.46)$$

Now W_i forms a “Euclidean group in 2 dimension”,

$$[W_1, W_2] = 0 \quad (3.47)$$

$$[W_3, W_1] = -iEW_2 \quad (3.48)$$

$$[W_3, W_2] = iEW_1 \quad (3.49)$$

These in principle defines the Little Group. The difficulty is that it has an infinite dimensional representation. That means that the particle has an extra label. However, we don't know of any particle with this property. The way to fix that is to just consider a finite subset. The solution for this is to set $W_1 = W_2 = 0$. Then since W_0 and W_3 commute we have $SO(2)$ rotations as a subgroup. By looking at equations (3.43)-(3.46) it's easy to see that $W_\mu = \lambda P_\mu$, where we call λ the helicity of the particle. The states are then labelled as

$$|0, 0; p^\mu, \lambda\rangle \quad (3.50)$$

We often omit the first two zeros for brevity. Under CPT the states transform to $|p^\mu, -\lambda\rangle$. This can be understood as follows. Parity inverts the momenta and time reversal inverts the spin and momenta of the particle. So the net result is a flip in the helicity. The relation,

$$e^{2\pi i\theta J_z} |p^\mu, \lambda\rangle = e^{2\pi i\lambda} |p^\mu, \lambda\rangle = \pm |p^\mu, \lambda\rangle \quad (3.51)$$

hence we require $\lambda = 0, \pm\frac{1}{2}, \pm 1, \dots$ e.g. $\lambda = 0$. This is a consequence of the Lorentz Group being double covered. In the SM we have

- $\lambda = 0 \rightarrow$ Higgs
- $\lambda = \pm\frac{1}{2} \rightarrow$ quarks and leptons
- $\lambda = \pm 1 \rightarrow$ photons, gluons
- $\lambda = \pm 2 \rightarrow$ graviton

Notice that we defined all the particles that we know of in terms of the massless multiplets. The reason for this is that all the particles are fundamentally massless but gain mass through the Higgs mechanism.

Chapter 4

$\mathcal{N} = 1$ Supersymmetry Representations

We want to look for multiplets to better understand our theories and find which states are degenerate. For example, in Hydrogen finding the different multiplets (corresponding to different L values) gives a deeper understanding of the system than just knowing that some states have given energies. It tells you why the energy states are structured in the way that they are. As a relevant example consider the $SU(2)$ multiplets of the weak interaction. Here the multiplets combine the electron and the neutrino into a doublet (since the weak force is broken their symmetry isn't manifest in the end). As a last example consider the $SU(3)_C$ multiplets of QCD. Here the different colored particles are put in a multiplet (the red, green, and blue up quark for example).

First we need to look for the Casimirs of the SUSY algebra. The Casimir, $C_1 = P_\mu P^\mu$, is still a Casimir for SUSY. This is easy to see since P^μ commutes with the SUSY generators. However, the second Casimir that we were using, is no longer a Casimir for SUSY. That means that spin is no longer a good label within a multiplet. That means you can have particles of different spin within a multiplet. C_2 is no longer a Casimir. But there is another operator we can define instead. The new Casimir is a bit complicated. It is given by

$$\tilde{C}_2 \equiv C_{\mu\nu} C^{\mu\nu} \quad (4.1)$$

where

$$C_{\mu\nu} = B_\mu P_\nu - B_\nu P_\mu \quad (4.2)$$

$$B_\mu = W_\mu - \frac{1}{4} \bar{Q}_{\dot{\alpha}} (\bar{\sigma}_\mu)^{\dot{\alpha}\beta} Q_\beta \quad (4.3)$$

The corresponding quantum number for this Casimir is often called the **Superspin**. Before we start building the multiplets of SUSY we first prove a general result for all the multiplets.

The claim is as follows: In any supersymmetric multiplet the number of bosons (n_B) is equal to the number of fermion (n_F),

$$n_B = n_F \quad (4.4)$$

From this statement arises the idea that for every particle in the SM there exists a corresponding particle with opposite spin.

The proof is as follows. Consider an operator, $(-1)^F$ (abbreviated $(-)^F$) known as the fermion number which is defined through

$$(-)^F |B\rangle, \quad (-)^F |F\rangle = -|F\rangle \quad (4.5)$$

where $|B\rangle$ is defined as a boson and $|F\rangle$ a fermion. Its easy to see that $(-)^F$ anticommutes with Q_α . Consider the action of the operator on a fermion:

$$(-)^F Q_\alpha |F\rangle = (-)^F |B\rangle = |B\rangle \quad (4.6)$$

but

$$-Q_\alpha (-)^F |F\rangle = +|B\rangle \quad (4.7)$$

since an analogous result can be shown for the operator acting on a fermion we have

$$\{(-)^F, Q_\alpha\} = 0 \quad (4.8)$$

We first define the trace operator in the multiplet space. It is given by

$$\text{Tr} \{\mathcal{O}\} = \sum_n \langle n | \mathcal{O} | n \rangle \quad (4.9)$$

where the sum is over all the states in the multiplet (bosons and fermions)

Now consider the following object (here by trace we mean summing over all states within a multiplet),

$$\text{Tr} \{(-)^F \{Q_\alpha, \bar{Q}_\beta\}\} = \text{Tr} \{(-)^F Q_\alpha \bar{Q}_\beta + (-)^F \bar{Q}_\beta Q_\alpha\} \quad (4.10)$$

We now use the anticommuting property we showed above for the first terms and the cyclic property of the trace for the second,

$$\text{Tr} \{(-)^F \{Q_\alpha, \bar{Q}_\beta\}\} = \text{Tr} \{-Q_\alpha (-)^F \bar{Q}_\beta + (-)^F Q_\alpha \bar{Q}_\beta\} \quad (4.11)$$

$$= 0 \quad (4.12)$$

On the other hand the computation can be done more explicitly using $\{Q_\alpha, \bar{Q}_\beta\} = 2(\sigma^\mu)_{\alpha\beta} P_\mu$,

$$\text{Tr} \{(-)^F \{Q_\alpha, \bar{Q}_\beta\}\} = \text{Tr} \{(-)^F 2(\sigma^\mu)_{\alpha\beta} P_\mu\} \quad (4.13)$$

We fix P^μ to be the same for a single multiplet (we are using it as one of the labels of the multiplets) so we have

$$\text{Tr} \{(-)^F \{Q_\alpha, \bar{Q}_\beta\}\} = (\sigma^\mu)_{\alpha\beta} P_\mu \text{Tr} \{(-)^F 2\} \quad (4.14)$$

All the preceding objects are clearly non-zero and thus we know that

$$\text{Tr}(-)^F = 0 \quad (4.15)$$

but

$$\text{Tr}(-)^F = \sum_{i=1}^{n_B} \langle B_i | (-)^F | B_i \rangle + \sum_{j=1}^{n_F} \langle F_j | (-)^F | F_j \rangle \quad (4.16)$$

$$= n_B - n_F \quad (4.17)$$

since this is equal to zero we have proven our result,

$$n_B = n_F \quad (4.18)$$

This is true for any multiplet. We call these multiplets a **supermultiplet**.

4.1 Massless Multiplets

We now try to find what types of particles are in multiplets in SUSY. Without SUSY all objects in the same multiplet have the same spin. As well will see this will no longer be the case with supersymmetry.

Consider the massless supermultiplet. We work in the frame that,

$$P_\mu = (E, 0, 0, E) \quad (4.19)$$

For this we have the first Casimir to be zero,

$$C_1 = P_\mu P^\mu = 0 \quad (4.20)$$

Plugging in this value of P_μ one can show that

$$\tilde{C}_2 = 2m^2 Y_i Y^i = 0 \quad (4.21)$$

where

$$Y_i \equiv J_i - \frac{1}{4m} \bar{Q} \bar{\sigma}_i Q \quad (4.22)$$

$$= J_i - \frac{1}{4m} \bar{Q}_{\dot{\beta}} \bar{\sigma}_i^{\dot{\beta}\alpha} Q_\alpha \quad (4.23)$$

So the representation is labelled by two zeros, which is not very enlightening. The states in the representation are only labelled by their momenta and helicity,

$$|P^\mu, \lambda\rangle \quad (4.24)$$

The difference from multiplets in the SM is that now we can have states with different λ in the same multiplet as will see.

We now see how to find the states in a multiplet. We start with the algebra. Recall that

$$\{Q_\alpha, \bar{Q}_{\dot{\alpha}}\} = 2\sigma_{\alpha\dot{\alpha}}^\mu P_\mu \quad (4.25)$$

$$= 2(E\sigma^0 + E\sigma^3)_{\alpha\dot{\alpha}} \quad (4.26)$$

$$= 2E \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}_{\alpha\dot{\alpha}} \quad (4.27)$$

We can see from here that $\{Q_1, \bar{Q}_1\} = 4E$ but the rest of the anticommutators are zero. In particular,

$$\{Q_2, \bar{Q}_2\} = 0 \quad (4.28)$$

That means for any state in the representation

$$\langle P^\mu, \lambda | \bar{Q}_2 Q_2 | p^\mu \lambda \rangle = |Q_2 | P^\mu, \lambda \rangle|^2 = 0 \quad (4.29)$$

and hence $Q_2 = 0$ in this representation. Now consider Q_1 :

$$\{Q_1, Q_1\} = 4E \quad (4.30)$$

in this multiplet.

That means that we can define

$$a \equiv \frac{Q_1}{2\sqrt{E}} \quad , \quad a^\dagger \equiv \frac{Q_1}{2\sqrt{E}} \quad (4.31)$$

which gives

$$\{a, a^\dagger\} = 1 \quad , \quad (4.32)$$

$$\{a, a\} = \{a^\dagger, a^\dagger\} = 0 \quad (4.33)$$

This an algebra that we are already familiar with, the algebra of creation and annihilation operators. We can use these operators as we do in QM to build the representation by acting with these operators on the vacuum. We also know how they act on particles with spin.

We now consider the commutator of the annihilation operator with J^3 :

$$[a, J^3] = \frac{1}{\sqrt{2E}} \frac{1}{2} [Q_1, M^{12} - M^{21}] \quad (4.34)$$

$$= \frac{1}{\sqrt{2E}} \frac{1}{2} \left\{ (\sigma^{12})_1^\beta Q_\beta - (\sigma^{21})_1^\beta Q_\beta \right\} \quad (4.35)$$

$$= -\frac{1}{\sqrt{2E}} \frac{i}{4} (\sigma^1 \sigma^2 - \sigma^2 \sigma^1)_1^\beta Q_\beta \quad (4.36)$$

$$= \frac{1}{\sqrt{2E}} \frac{1}{2} (\sigma^3)_1^1 Q_1 \quad (4.37)$$

$$= \frac{1}{2} a \quad (4.38)$$

That means that if we take

$$J^3 (a |p, \lambda\rangle) = \left(aJ^3 - \frac{1}{2}(\sigma^3)_1^1 \right) a |p, \lambda\rangle \quad (4.39)$$

$$= (a\lambda - \frac{1}{2}a) |p, \lambda\rangle \quad (4.40)$$

$$= (\lambda - \frac{1}{2}) (a |p, \lambda\rangle) \quad (4.41)$$

So the state $a |a\lambda\rangle$ is an eigenstate of J^3 with helicity $\lambda - \frac{1}{2}$.

By the same argument one can show that $a^\dagger |p, \lambda\rangle$ has helicity $\lambda + \frac{1}{2}$. So to build a representation we start with a state that we call “the vacuum”. The quotes here are because this is not the real vacuum it is just the state of minimum helicity.

$$|\Omega\rangle \equiv |p, \lambda\rangle \quad (4.42)$$

By equation 4.41 we see that,

$$a^\dagger |\Omega\rangle = |p, \lambda + \frac{1}{2}\rangle \quad (4.43)$$

and since a^\dagger anticommutes with itself,

$$a^\dagger a^\dagger |\Omega\rangle = 0 \quad (4.44)$$

So the whole multiplet consists of

$$|p, \lambda\rangle \quad , \quad |p, \lambda + \frac{1}{2}\rangle \quad (4.45)$$

As usual we must add the CPT conjugates of this, so that means that we have

$$\left\{ |p, \pm\lambda\rangle \quad , \quad |p, \pm(\lambda + \frac{1}{2})\rangle \right\} \quad (4.46)$$

As an example consider $\lambda = 0$. Then we have a $\lambda = 0$ scalar and a $\lambda = \frac{1}{2}$ fermion. This is usually called the “Chiral Multiplet”. The name arises from having one chiral fermion and a scalar. Examples of such multiplets are

$\lambda = 0$ scalar	$\lambda = \frac{1}{2}$ fermion
squark	quark
slepton	lepton
Higgs	Higgsino

We can also have “Vector or Gauge multiplets” with $\lambda = \frac{1}{2}, 1$. Examples include

$\lambda = \frac{1}{2}$ fermion (“gauginos”)	$\lambda = 1$ boson
photino	photon
gluino	gluon
Wino	W
Zino	Z

For now we skip $\lambda = 1, \frac{1}{2}$ case but we discuss it more later. The last example is the $\lambda = \frac{3}{2}, 2$ multiplet,

$$\frac{\lambda = \frac{3}{2} \text{ fermion}}{\text{gravitino}} \quad \bigg| \quad \frac{\lambda = 2 \text{ boson}}{\text{graviton}}$$

We cannot have states with $\lambda > 2$. This is non-trivial but we will address this point later.

4.2 Massive Multiplets

We have,

$$P_\mu = (m, 0, 0, 0) \quad (4.47)$$

We start by considering the Casimirs,

$$P^2 = m^2 \quad (4.48)$$

and one can show that

$$\tilde{C}_2 = 2m^2 Y^i Y_i \quad (4.49)$$

where

$$Y_i \equiv J_i - \frac{1}{4m} (\bar{Q}\sigma Q)_i = \frac{B_i}{m} \quad (4.50)$$

These Y_i satisfy the $SU(2)$ algebra (Ex),

$$[Y_i, Y_j] = i\epsilon_{ijk} Y_k \quad (4.51)$$

The eigenvalues of Y^2 are $y(y+1)$. Thus we need to add two labels to our multiplets, one for each Casimir.

To find the rest of our labels we again consider the anticommutator,

$$\{Q_\alpha, Q_{\dot{\alpha}}\} = 2(\sigma_{\alpha\dot{\alpha}}^\mu) P_\mu \quad (4.52)$$

In the eigenspace of massive particles at rest this takes the form,

$$\{Q_\alpha, Q_{\dot{\alpha}}\} = 2m\delta_{\alpha\dot{\alpha}} \quad (4.53)$$

This is the same result we had for massless particles with a different scaling factors and a non-zero anticommutator for both Q_1 and Q_2 . Thus we need two sets of creation and annihilation operators,

$$a_{1,2} = \frac{1}{\sqrt{2m}} Q_{1,2} \quad (4.54)$$

$$a_{1,2}^\dagger = \frac{1}{\sqrt{2m}} \bar{Q}_{1,2} \quad (4.55)$$

and

$$\{a_p, a_q^\dagger\} = \delta_{pq} \quad (4.56)$$

We can define the “vacuum state”, $|\Omega\rangle$ such that

$$a_p |\Omega\rangle = 0 \quad (4.57)$$

With this we see that

$$Y_i |\Omega\rangle = J_i |\Omega\rangle - \frac{1}{4m} \bar{Q} \bar{\sigma}_i Q |\Omega\rangle = J_i |\Omega\rangle \quad (4.58)$$

So on acting on the vacuum the Y_i label is the same as the spin label, j .

The state is labelled by the mass, spin, momentum, and projected spin angular momentum,

$$|\Omega\rangle \equiv |m, y; P^\mu, j_3\rangle \quad (4.59)$$

For the rest of the mutliplet we can use the same trick as before

$$a_1 |j_3\rangle = |j_3 - \frac{1}{2}\rangle \quad (4.60)$$

$$a_1^\dagger |j_3\rangle = |j_3 + \frac{1}{2}\rangle \quad (4.61)$$

However for a_2 we have $\sigma_{22}^3 = -1$ which gives

$$a_2 |j_3\rangle = |j_3 + \frac{1}{2}\rangle \quad (4.62)$$

$$a_2^\dagger |j_3\rangle = |j_3 - \frac{1}{2}\rangle \quad (4.63)$$

That means that if we act on $|\Omega\rangle$ by a_1^\dagger ,

$$a_1^\dagger |\Omega\rangle = k_1 |m, j = y + \frac{1}{2}; P^\mu, j_3 + \frac{1}{2}\rangle + k_2 |m, j = y - \frac{1}{2}; P^\mu, j_3 + \frac{1}{2}\rangle \quad (4.64)$$

where k_1 and k_2 are Clebsh-Gordan coefficients. Similarly we have,

$$a_2^\dagger |\Omega\rangle = k_3 |m, j = y - \frac{1}{2}; P^\mu, j_3 - \frac{1}{2}\rangle + k_4 |m, j = y + \frac{1}{2}; P^\mu, j_3 - \frac{1}{2}\rangle \quad (4.65)$$

Acting on these states again with the same creation operator gives zero. So the only other thing we can do is

$$a_2^\dagger a_1^\dagger |\Omega\rangle = -a_1^\dagger a_2^\dagger |\Omega\rangle \quad (4.66)$$

This is a spin j object. In total we have,

$$\begin{aligned} 2 \text{ states } |m, j = y; P^\mu, j_3\rangle &\rightarrow 2(2y + 1) \\ 1 \text{ state } |m, j = y + \frac{1}{2}; P^\mu, j_3\rangle &\rightarrow 2(y + \frac{1}{2}) + 1 \\ 1 \text{ state } |m, j = y - \frac{1}{2}; P^\mu, j_3\rangle &\rightarrow 2(y - \frac{1}{2}) + 1 \end{aligned}$$

Above we assumed $y \neq 0$. This case is in fact a bit simpler. The state annihilated by a_α is,

$$|\Omega\rangle = |m, j = 0; p^\mu, j_3 = 0\rangle \quad (4.67)$$

then we have,

$$a_{1/2}^\dagger |\Omega\rangle = |m, j = 1/2; p^\mu, j = \pm 1/2\rangle \quad (4.68)$$

$$a_1^\dagger a_2^\dagger |\Omega\rangle = |m, j = 0; p^\mu, j_3 = 0\rangle \quad (4.69)$$

The last state appears to be the same as $|\Omega\rangle$ but it is not. $|\Omega\rangle$ is a scalar while $a_1^\dagger a_2^\dagger |\Omega\rangle$ is a pseudoscalar.

To see this recall that under parity we have, $(A, B) \leftrightarrow (B, A)$. So under parity we have $Q \leftrightarrow \bar{Q}$ up to a phase,

$$P^{-1} Q_\alpha P = \eta_P (\sigma^0)_{\alpha\dot{\beta}} \bar{Q}^{\dot{\beta}} = \eta_P \sigma_{\alpha\dot{\beta}}^0 \epsilon^{\dot{\beta}\gamma} \bar{Q}_\gamma \quad (4.70)$$

Similarly,

$$P^{-1} \bar{Q}_{\dot{\alpha}} P = \eta_P^* \bar{\sigma}_{\dot{\alpha}\beta}^0 \epsilon^{\beta\gamma} Q_\gamma \quad (4.71)$$

This is consistent with the algebra of the Q 's:

$$\{Q_\alpha, \bar{Q}_{\dot{\alpha}}\} = 2\sigma_{\alpha\dot{\alpha}}^\mu P_\mu \quad (4.72)$$

such that under parity, $P^\mu \rightarrow (P_0, -\mathbf{p})$. This is easy to show for a particular value of $\alpha, \dot{\alpha}$ as we show for $\alpha = 1, \dot{\alpha} = \dot{2}$.

$$P^{-1} Q_1 \bar{Q}_{\dot{2}} P = (\sigma^0)_{1\dot{\beta}} \epsilon^{\dot{\beta}\gamma} \bar{Q}_\gamma (\bar{\sigma}^0)_{\dot{2}\beta} \epsilon^{\beta\gamma} Q_\gamma \quad (4.73)$$

$$= -\bar{Q}_{\dot{2}} Q_1 \quad (4.74)$$

and so

$$P^{-1} \{Q_1, \bar{Q}_{\dot{2}}\} P = -\{\bar{Q}_{\dot{2}}, Q_1\} = -\{Q_1, \bar{Q}_{\dot{2}}\} \quad (4.75)$$

On the other hand the right hand side gives,

$$P^{-1} (2\sigma_{1\dot{2}}^\mu P_\mu) P = \begin{pmatrix} P^0 + P^3 & P^1 - iP^2 \\ P^1 + iP^2 & P^0 - P^3 \end{pmatrix}_{1\dot{2}} \quad (4.76)$$

$$= 2(P^1 - iP^2) \quad (4.77)$$

$$= -2\sigma_{1\dot{2}}^\mu P_\mu \quad (4.78)$$

This is consistent with the transformation of the anticommutator above as expected.

Also,

$$P^{-1} P^{-1} Q_\alpha P P = P^{-1} (\sigma^0)_{\alpha\dot{\beta}} \epsilon^{\dot{\beta}\dot{\alpha}} (\bar{\sigma}^0)_{\dot{\alpha}\beta} \epsilon^{\beta\gamma} Q_\gamma \quad (4.79)$$

$$= \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\}_\alpha^\gamma Q_\gamma \quad (4.80)$$

$$= -Q_\alpha \quad (4.81)$$

Now

$$|\Omega'\rangle \equiv a_1^\dagger a_2^\dagger |\Omega\rangle = |m, j = 0; P^\mu, 0\rangle \quad (4.82)$$

is not annihilated by the a 's but by the a^\dagger 's instead. These states are related by switching $a \leftrightarrow a^\dagger$. So under Parity, [\[Q 3: Show this\]](#)

$$|\Omega\rangle \leftrightarrow |\Omega'\rangle \quad (4.83)$$

We can then construct states that are Parity eigenstates,

$$|\pm\rangle = |\Omega\rangle \pm |\Omega'\rangle \quad (4.84)$$

which have Parity ± 1 ($|+\rangle$ is a scalar and $|-\rangle$ is a pseudoscalar).

For massive $N = 1$ multiplet $y = 0$,

$$|\Omega\rangle = |m, j = 0; p^\mu, 0\rangle \quad , \quad a_{1,2} |\Omega\rangle = 0 \quad (4.85)$$

Chapter 5

Extended Supersymmetry

We have seen the basics of supersymmetry with,

$$Q_\alpha^A \quad A = 1, \dots, \mathcal{N} \quad (5.1)$$

$$\bar{Q}_{\dot{\alpha}}^A \quad A = 1, \dots, \mathcal{N} \quad (5.2)$$

The algebra for this extended version is the same as the $\mathcal{N} = 1$ except for

$$\{Q_\alpha^A, \bar{Q}_{\dot{\alpha}}^B\} = 2\sigma_{\alpha\dot{\alpha}}^\mu P_\mu \delta^A_B \quad (5.3)$$

$$\{Q_\alpha^A, Q_\beta^B\} = \epsilon_{\alpha\beta} Z^{AB} \quad (5.4)$$

Z^{AB} is the new ingredient of extended supersymmetry and known as **central charges**. These central charges commute with all the operators,

$$[Z^{AB}, P^\mu] = [Z^{AB}, M^{\mu\nu}] = [Z^{AB}, Q_\alpha^A] = [Z^{AB}, Z^{CD}] = [Z^{AB}, T_a] = 0 \quad (5.5)$$

These charges provide what is called the Abelian invariant subalgebra of internal symmetries. For the rest of the internal symmetries we have the standard,

$$[T_a, T_b] = iC_{abc}T_c \quad (5.6)$$

Now recall earlier that we had an R-symmetry which was just a $U(1)$ phase under which the algebra was invariant. We define R - Symmetry to be elements of internal symmetry group \mathcal{G} ,

$$[Q_\alpha^A, T_a] = S_B^A Q_\alpha^B \quad (5.7)$$

If $Z^{AB} = 0$ then the R-symmetry group is just $U(N)$. You can see that if you multiply the Q 's by a unitary matrix U then the algebra is invariant. However if $Z^{AB} \neq 0$ then the R symmetry group is just a subgroup of $U(N)$ and you must study it case by case.

5.1 Representations of $\mathcal{N} > 1$ SUSY

5.1.1 Massless Particles

Choose the momentum to be $P_\mu = (E, 0, 0, E)$. Then we use,

$$\{Q_\alpha^A, \bar{Q}_{B,\dot{\alpha}}\} = 4E \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}_{\alpha\dot{\alpha}} \delta^A_B \quad (5.8)$$

As before we can choose

$$Q_2^A = 0 \quad (5.9)$$

and define

$$a^A \equiv \frac{Q_1^A}{2\sqrt{E}}, \quad a^{\dagger,A} = \frac{\bar{Q}_1^A}{2\sqrt{E}} \quad (5.10)$$

which satisfy,

$$\{a^A, a^{B,\dagger}\} = \delta^A_B \quad (5.11)$$

The states are

	States	helicity	# of States
“vacuum”	$ \Omega\rangle$	λ_{min}	1
	$a^{\dagger,A} \Omega\rangle$	$\lambda_{min} + 1/2$	N
	$a^{\dagger,A} a^{\dagger,B} \Omega\rangle$	$\lambda_{min} + 1$	$\frac{N(N+1)}{2} = \binom{N}{2}$
	$a^{\dagger,A} a^{\dagger,B} a^{\dagger,C} \Omega\rangle$	$\lambda_{min} + 3/2$	$\binom{N}{3}$
	\vdots	\vdots	\vdots
	$a^{\dagger,N} a^{\dagger,N-1} \dots a^{\dagger,1} \Omega\rangle$	$\lambda_{min} + N/2$	1

Total number of states within one multiplet is

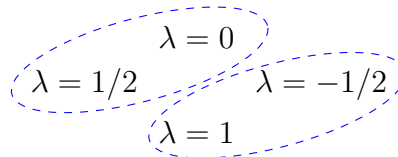
$$\sum_{n=0}^N \binom{N}{n} = 2^N \quad (5.12)$$

So the number of states in a multiplet can be very large.

We now consider some examples,

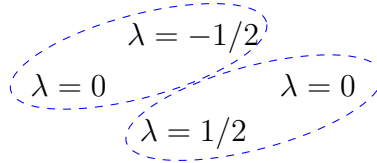
1. $N = 2$:

(a) $\lambda_{min} = 0$:



[Q 4: Don't we always need to include the $-\lambda$ states in each multiplet...?] We see the $N = 2$ multiplet is a combination of 2 $N = 1$ multiplets. This is a famous multiplet and is known as the $N = 2$ vector multiplet.

(b) $\lambda_{min} = 1/2$



This is called the $N = 2$ hypermultiplet.

$N = 4$

(a) $\lambda_{min} = -1$:

# of States	λ
1	-1
4	-1/2
6	0
4	1/2
1	1

2. $N = 8$. The particle content is:

# of States	λ
1	± 2
8	$\pm 3/2$
28	± 1
56	$\pm 1/2$
70	0

Below are a few remarks:

- In general $\lambda_{max} - \lambda_{min} = N/2$
- One can show that if we want a theory that is renormalizable you cannot have objects with helicity greater than 1. Thus we have a constraint on possible values of N . If we at most want $|\lambda| = 1$ then $\lambda_{max} - \lambda_{min}$ is at most 2 and thus we have the condition,

$$N \leq 4 \tag{5.13}$$

- “Strong Belief”: There are no massless particles with helicity greater than 2 since $|\lambda| > 1/2$ massless particles at low momentum must couple to conserved currents (e.g. $\partial^\mu j_\mu = 0$ for E&M where $\lambda = \pm 1$). For $\lambda = \pm 2$ the conserved current is $\partial^\mu T_{\mu\nu}$. The corresponding conserved charge is the 4-momentum. Beyond this there are no

more conserved currents to couple to. Since there are no more conserved currents there are no more conserved particles. This argument has also been extended to SUSY which allowed $\lambda = \pm 3/2$ spin particles. If this is true then that implies that

$$\lambda_{max} - \lambda_{min} \leq 4 \quad (5.14)$$

which implies that

$$N \leq 8 \quad (5.15)$$

This theory was a promising candidate for a grand unified theory as it unites all the possible states in one multiplet. A real *United States*.

- Another argument for seeing that $N = 8$ is a maximum is that if $N > 8$ then we have more than 1 “graviton”. This would be very strange since we would have a theory with different gravities.
- However, $N > 1$ SUSY are in general non-chiral. The SM is a chiral theory. However as we will argue now, for all the extended supersymmetry the multiplets are generically non-chiral.

The idea is that all these multiplets are $\lambda = \pm 1$ particles. Any gauge bosons transform in a particular representation of the corresponding group, the adjoint representation. The adjoint representation is a real representation and non-chiral. This implies that the corresponding particles within the same multiplet (the $\lambda = \pm 1/2$ particles) will transform under the same representation and hence will not be chiral. We know that the SM particles live in complex representations (the fundamental representations).

For the $N = 2$ hypermultiplet,

$$\begin{array}{ccc} & \lambda = -1/2 & \\ \lambda = 0 & & \lambda = 0 \\ & \lambda = 1/2 & \end{array}$$

the helicity $1/2$ comes together with the helicity $-1/2$. Hence they are innately non-chiral again. This kills all $N > 1$. We are only left with $N = 1$ and $N = 0$. For instance in $N = 1$,

$$\begin{pmatrix} 1/2 \\ 0 \end{pmatrix} \quad (5.16)$$

- However in $N = 1$ it has a prediction already: at least one extra particle for each SM particle with the same mass since its in the same multiplet. This prediction is clearly violated! The only way out is that supersymmetry is a broken symmetry (at low energies, $E \sim 10^2 \text{GeV}$).

Recall that for the massless case we found that the number of states in a multiplet is 2^N . To derive this relation we didn't need to use the central charges at all. This is because in the massless case, $Q_2 = 0$ which by

$$\{Q_\alpha^A, Q_\beta^B\} = \epsilon_{\alpha\beta} Z^{AB} \quad (5.17)$$

implies that $Z^{AB} = 0$.

We now address the massive case. We have,

$$P^\mu = (m, 0, 0, 0) \quad (5.18)$$

and we have

$$\{Q_\alpha^A, \bar{Q}_{B,\beta}\} = 2\sigma_{\alpha\dot{\beta}}^\mu P_\mu \delta^A_\beta = 2m \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \delta^A_\beta \quad (5.19)$$

Now the central charges do not need to be zero. There are two cases,

1. $Z^{AB} = 0$
2. $Z^{AB} \neq 0$

We consider each case separately. First consider $Z^{AB} = 0$:

As for $N = 1$ we now have,

$$a_\alpha^A \equiv \frac{Q_\alpha^A}{\sqrt{2m}} \quad (5.20)$$

We have $2N$ creation and annihilation operators and we have

$$2^{2N} \text{ states} \quad (5.21)$$

each of dimension $(2y + 1)^1$.

For example for $N = 2$,

$y = 0$	spin	# of states
$ \Omega\rangle$	0	1
$a_{\alpha}^{\dagger,A} \Omega\rangle$	1/2	4
$a^\dagger a^\dagger \Omega\rangle$	0(3states), 1(3states)	6
$a^\dagger a^\dagger a^\dagger$	1/2	4
$a^\dagger a^\dagger a^\dagger a^\dagger \Omega\rangle$	0	1

where we don't bother putting labels on the a^\dagger 's but just take them to be combinations of different values of A, α . The detailed structure of these multiplets is not particularly important for us but more to give a flavor of how the larger multiplets look like. Here we have $16 = 2^{2 \times 2}$ states as expected.

Now consider the second case, $Z^{AB} \neq 0$:

$$\mathcal{H} \equiv (\bar{\sigma}^0)^{\dot{\beta}\alpha} \{Q_\alpha^A - \Gamma_\alpha^A, \bar{Q}_{\dot{\beta},A} - \bar{\Gamma}_{\dot{\beta},A}\} \geq 0 \quad (5.22)$$

¹Recall that the degenerate states with spin j are $-j, -j + 1, \dots, j$.

where this number is greater than zero since we are multiplying a number by its complex conjugate. We define

$$\Gamma_\alpha^A = \epsilon_{\alpha\beta} U^{AB} \bar{Q}_{\dot{\gamma}\beta} (\bar{\sigma}^0)^{\dot{\gamma}\beta} \quad (5.23)$$

where U is an arbitrary unitary matrix. Using the SUSY algebra one can show

$$\mathcal{H} = 8mN - 2\text{Tr}(ZU^\dagger + UZ^\dagger) \geq 0 \quad (5.24)$$

[Q 5: Show.] Now any matrix can be written in the form,

$$Z \equiv HV \quad (5.25)$$

where $H = H^\dagger$ is positive and $VV^\dagger = 1$. Note we can isolate H ,

$$\sqrt{Z^\dagger Z} = \sqrt{HVV^\dagger H^\dagger} = H \quad (5.26)$$

This is known as the polar decomposition. Recall that we are free to choose the matrix U . We choose $U = V$. This sets

$$ZU^\dagger + UZ^\dagger = H + H = 2H \quad (5.27)$$

and so

$$\mathcal{H} = 8mN - 4\text{Tr}H = 8mN - 4\text{Tr}\sqrt{Z^\dagger Z} \geq 0 \quad (5.28)$$

where by square root of a $Z^\dagger Z$ we mean a matrix such that when multiplied by itself is equal to $Z^\dagger Z$. Thus we have,

$$m \geq \frac{\text{Tr}\sqrt{Z^\dagger Z}}{2N} \quad (5.29)$$

This is a bound on the mass of the particles and is usually known as the BPS bound.

The bound is saturated when

$$m = \frac{\text{Tr}\sqrt{Z^\dagger Z}}{2N} \quad (5.30)$$

These are known as BPS states. When this condition holds we have $\mathcal{H} = 0$ which implies that

$$Q_\alpha^A - \Gamma_\alpha^A = Q_\alpha^A - \epsilon_{\alpha\beta} V^{AB} \bar{Q}_{\dot{\gamma}\beta} (\bar{\sigma}^0)^{\dot{\gamma}\beta} \quad (5.31)$$

We have a combination of the Q 's and \bar{Q} 's that is equal to zero. Recall that for the massless case we had $Q_2 = 0$ which changed the number of states from $2^{2N} \rightarrow 2^N$. The BPS states have the same constraint and we have 2^N states. The corresponding multiplet is “shorter”. This is the main property that distinguishes these states.

As an example consider the case that $N = 2$. The central charge is antisymmetric so the most general matrix can be written,

$$Z^{AB} = \begin{pmatrix} 0 & q_1 \\ -q_1 & 0 \end{pmatrix} \quad (5.32)$$

²The matrix square root isn't unique but that need not bother us here.

where the degree of freedom of the central charge is called a charge and denoted by q_1 (the reason for subscript 1 will become clear soon). The BPS condition is

$$m \geq \frac{1}{4} 2q_1 = \frac{q_1}{2} \quad (5.33)$$

This can be generalized for any other N . Consider $N > 2$ with an even N . If N is even you can always bring a matrix to the form,

$$Z = \begin{pmatrix} 0 & q_1 & & & & \\ -q_1 & 0 & & & & \\ & & 0 & q_2 & & \\ & & -q_2 & 0 & \ddots & \\ & & & \ddots & \ddots & \\ & 0 & & & & 0 & q_N \\ & & & & & -q_N & 0 \end{pmatrix} \quad (5.34)$$

The same analysis we just did can be done for each block. The BPS condition will hold block by block and we have,

$$m \geq \frac{q_i}{2} \quad (5.35)$$

for all i . Some blocks may satisfy this condition and some may not.

If $k < N/2$ of the $q_i = 2m$ then we will have

$$2N - 2k \quad (5.36)$$

creation operators and so we will have $2^{2(N-k)}$ states.

- For $k = 0$ this implies we have 2^{2N} states (the standard long multiplet)
- For $0 < k < N/2$ we have $2^{2(N-k)}$ states and are usually called short multiplets
- For $k = N/2$ we have 2^N states and is known as an ultra-short multiplet

This is another use of superlatives in SUSY. We used super, hyper, and now ultra. We are competing with the comics which try to invent names for their heros.

Earlier considered the case with $k = 1$ and $N = 2$ so we had an ultra-short multiplet. Ultra-short multiplets play an important role due to the size of the multiplets. We now have a few comments

1. BPS states and bounds were started in soliton (e.g. monopoles) solutions of Yang-Mills systems. Here bound is an energy bound.
2. BPS states, (ultra) short states, tend to be stable. This is because the mass of these particles is small (its at the bottom of the lower bound). This stops particles from decaying.

3. For BPS states we have $mass \sim charge$. This condition is found in what's known as extremal black holes.
4. BPS states have been crucial for understanding strong-weak coupling duality (aka "s duality") in both field and string theory. In particular D-branes in string theory are BPS states.

5.A Important Commutators

[Q 6: Under Construction]

$$[a_\alpha, J^2] = \frac{1}{4} \epsilon_{ijk} \epsilon_{ilm} [M^{jk} M^{\ell m}] \quad (5.37)$$

$$= \frac{1}{4} \{ [a_\alpha, M^{jk} M^{jk}] - [a_\alpha, M^{jk} M^{kj}] \} \quad (5.38)$$

$$= \frac{1}{2} [a_\alpha, M^{jk} M^{jk}] \quad (5.39)$$

$$= \frac{1}{2} \left\{ (\sigma^{jk})_\alpha^\beta Q_\beta M^{jk} + M^{jk} a_\alpha M^{jk} - a_\alpha M^{jk} M^{jk} \right\} \quad (5.40)$$

$$= \frac{1}{2} \left\{ (\sigma^{jk})_\alpha^\beta a_\beta M^{jk} + M^{jk} (\sigma^{jk})_\alpha^\beta a_\beta \right\} \quad (5.41)$$

Now $(\sigma^{jk})_\alpha^\beta$ and M^{jk} live in different spaces and so we have,

$$[a_\alpha, J^2] = \frac{1}{2} (\sigma^{jk})_\alpha^\beta \{ a_\beta M^{jk} + M^{jk} a_\beta \} \quad (5.42)$$

$$= \frac{1}{2} (\sigma^{jk})_\alpha^\beta \left\{ 2M^{jk} a_\beta + (\sigma^{jk})_\beta^\gamma a_\gamma \right\} \quad (5.43)$$

Now we have,

$$\sigma^{jk} \sigma^{jk} = \frac{i^2}{4 \cdot 4} (\sigma^j \sigma^k - \sigma^k \sigma^j) (\sigma^j \sigma^k - \sigma^k \sigma^j) \quad (5.44)$$

$$= i^2 \cdot \frac{i^2}{4} \epsilon_{jkl} \epsilon_{jkm} \sigma^\ell \sigma^m \quad (5.45)$$

$$= \frac{1}{2} \sigma_\ell \sigma^\ell \quad (5.46)$$

$$= \frac{1}{2} (\sigma_1^2 + \sigma_2^2 + \sigma_3^2) \quad (5.47)$$

$$= \frac{3}{2} \quad (5.48)$$

Thus we can finally write,

$$[a_\alpha, J^2] = (\sigma^{jk})_\alpha^\beta M^{jk} a_\beta + \frac{3}{4} a_\alpha \quad (5.49)$$

$$= \frac{1}{2} \epsilon_{jkl} (\sigma^\ell)_\alpha^\beta M^{jk} a_\beta + \frac{3}{4} a_\alpha \quad (5.50)$$

$$= (\sigma^\ell)_\alpha^\beta J_\ell a_\beta + \frac{3}{4} a_\alpha \quad (5.51)$$

Taking the hermitian conjugate of this equation gives,

$$[J^2, a_\alpha^\dagger] = (\sigma^\ell)_{\dot{\alpha}}^{\dot{\beta}} a_{\dot{\beta}}^\dagger J_\ell + \frac{3}{4} a_{\dot{\alpha}}^\dagger \quad (5.52)$$

$$= (\sigma^\ell)_{\dot{\alpha}}^{\dot{\beta}} J_\ell a_{\dot{\beta}}^\dagger - \frac{3}{4} a_{\dot{\alpha}}^\dagger \quad (5.53)$$

These formula are quite general and holds for acting on states of any spin. Suppose for example we act on a spinless object, i.e. one that obyes $M^{jk} |j=0\rangle = 0$. In this case,

$$J^2 a_\alpha^\dagger |j=0\rangle = \left\{ \frac{3}{4} a_{\dot{\alpha}}^\dagger + a_{\dot{\alpha}}^\dagger J^2 \right\} |j=0\rangle \quad (5.54)$$

$$= \frac{3}{4} a_\alpha |j=0\rangle \quad (5.55)$$

So the state $a_{\dot{\alpha}}^\dagger |j=0\rangle$ is a $j = \frac{1}{2}$ object! $a_{\dot{\alpha}}^\dagger$ changes the j value of the state.

To better understand the results we now expand the J_ℓ term:

$$\sigma^\ell J_\ell = \sigma^1 J_1 + \sigma^2 J_2 + \sigma^3 J_3 \quad (5.56)$$

$$= -\frac{1}{2} (\sigma^1 - i\sigma^2) J_+ - \frac{1}{2} (\sigma^1 + i\sigma^2) J_- - \sigma^3 J^3 \quad (5.57)$$

$$= -\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} J_+ - \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} J_- - \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} J^3 \quad (5.58)$$

$$= -\begin{pmatrix} J^3 & J_- \\ J_+ & -J^3 \end{pmatrix} \quad (5.59)$$

and

$$\sigma^\ell a^\dagger J_\ell = \begin{pmatrix} a_1^\dagger J^3 & a_2^\dagger J_- \\ a_1^\dagger J_+ & -a_2^\dagger J^3 \end{pmatrix} \quad (5.60)$$

So,

$$\begin{pmatrix} (\sigma^\ell)_1^{\dot{\beta}} a_{\dot{\beta}}^\dagger J_\ell |j, j_3\rangle \\ (\sigma^\ell)_2^{\dot{\beta}} a_{\dot{\beta}}^\dagger J_\ell |j, j_3\rangle \end{pmatrix} = \begin{pmatrix} j_3 a_1^\dagger |j, j_3\rangle + A_- a_2^\dagger |j, j_3 - 1/2\rangle \\ A_+ a_1^\dagger |j, j_3 + 1/2\rangle - j_3 a_2^\dagger |j, j_3\rangle \end{pmatrix} \quad (5.61)$$

where

$$A_+ \equiv \sqrt{j(j+1) - j_3(j_3+1)} \quad (5.62)$$

$$A_- \equiv \sqrt{j(j+1) - j_3(j_3-1)} \quad (5.63)$$

Now consider,

$$\langle j, j_3 | a_\alpha^\dagger | j, j_3 \rangle = \frac{1}{j(j+1)} \langle j, j_3 | a_\alpha^\dagger J^2 | j, j_3 \rangle \quad (5.64)$$

$$= \frac{1}{j(j+1)} \langle j, j_3 | a_\beta^\dagger (\sigma^\ell)_\alpha^\beta J_\ell + \frac{3}{4} a_\alpha^\dagger + a_\alpha^\dagger J^2 | j, j_3 \rangle \quad (5.65)$$

$$\begin{pmatrix} \langle j, j_3 | a_\beta^\dagger | j, j_3 - 1/2 \rangle \\ \langle j, j_3 | a_\beta^\dagger | j, j_3 + 1/2 \rangle \end{pmatrix} = 0 \quad (5.66)$$

As a second example consider acting on a spinor. In this case $M^{jk} = \sigma^{jk}$ and,

$$(\sigma^{jk})_\alpha^\beta (\sigma^{jk})_\gamma^\delta = \frac{1}{4} \epsilon_{jkl} \epsilon_{jkm} (\sigma^\ell)_\alpha^\beta (\sigma^m)_\gamma^\delta \quad (5.67)$$

$$= \frac{1}{2} (\sigma^\ell)_\alpha^\beta (\sigma^\ell)_\gamma^\delta \quad (5.68)$$

$$= \delta_\alpha^\delta \delta_\gamma^\beta - \frac{1}{2} \delta_\alpha^\beta \delta_\gamma^\delta \quad (5.69)$$

and hence for the spinor representation,

$$(\sigma^{jk})_\alpha^\beta (M^{jk})_\gamma^\delta (a_\beta)_\delta^\chi = (a_\gamma)_\alpha^\chi - (a_\alpha)_\delta^\chi \quad (5.70)$$

and so we have,

$$[a_\alpha, J^2]_\gamma^\chi = (a_\gamma)_\alpha^\chi - (a_\alpha)_\gamma^\chi + \frac{3}{4} (a_\alpha)_\gamma^\chi \quad (5.71)$$

$$= (a_\gamma)_\alpha^\chi - \frac{1}{4} (a_\alpha)_\gamma^\chi \quad (5.72)$$

Acting on a spin 1/2 state doesn't return back an eigenstate. Instead it gives some linear combination of states.

Chapter 6

Superfields and Superspace

So far we have just dealt with supermultiplets in terms of one-particle states. In the case of field theory that is not enough. We need to talk about the interaction of the particles and have a field theory that is supersymmetric. *We have only the characters of the game but we need to know how to play.* For that we have to recall what we know from standard field theories. The particles are described by fields, $\phi(x^\mu)$ with x^μ as the four dimensional spacetime coordinates. The fields can be many types depending on how they transform under the Poincare group. What we want to do now is to make a supersymmetric generalization of $\phi(x^\mu)$. The fields will be a function of coordinates and a function of what we call superspace. We want:

$$\Phi(X) \tag{6.1}$$

with Φ transformaing under Super-Poincare and X are coordinates in superspace. We first need to define what superspace is. We use symmetries to define this space. We know that every continuous group defines a manifold. There is a mapping from a group \mathcal{G} to a manifold \mathcal{M} such that the elements of the group, $g = e^{i\alpha_a T^a}$ are labelled by the coordinates of the space, $\{\alpha_a\}$.

$$\mathcal{G} \rightarrow \mathcal{M}_{\mathcal{G}} \tag{6.2}$$

$$\{g = e^{i\alpha_a T^a}\} \rightarrow \{\alpha_a\} \tag{6.3}$$

We have as many parameters as generators of the group so the dimensionality of the space is the same as the dimension of the Lie group,

$$\dim\mathcal{G} = \dim\mathcal{M}_{\mathcal{G}} \tag{6.4}$$

We now consider some examples,

1. $\mathcal{G} = U(1)$: $g = e^{i\alpha Q}$ where $\alpha \in [0, 2\pi] \Rightarrow \mathcal{M}_{U(1)} = S^1$. The manifold is a circle.
2. $\mathcal{G} = SU(2)$: $g = \begin{pmatrix} p & q \\ -q^* & p^* \end{pmatrix}$ with the condition that $\det g = 1$ or $|p|^2 + |q|^2 = 1$.

If we parametrize p and q such that,

$$p = x_1 + ix_2 \tag{6.5}$$

$$q = x_3 + ix_4 \tag{6.6}$$

then the condition is just $\sum_{i=1}^4 x_i^2 = 1$ and hence $\mathcal{M}_{SU(2)} = S^3$. The manifold is a 3-sphere. This is also why $SU(2)$ is the covering group of $SO(3)$.

3. $\mathcal{G} = SL(2, \mathbb{C})$: $g = HV$; where $H = H^\dagger$ and $V = SU(2)$ by the polar decomposition. Since we have an $SL(2, \mathbb{C})$ matrix we also require $\det H = 1$. So you can see that the manifold corresponding the $SL(2, \mathbb{C})$ is the manifold corresponding to $SU(2)$ times the manifold corresponding to H , but H we can write,

$$H = x^\mu \sigma_\mu = \begin{pmatrix} x^0 + x^3 & x^1 + ix^2 \\ x^1 - ix^2 & x^0 - x^3 \end{pmatrix} \tag{6.7}$$

for some x^μ . [Q 7: Shouldn't we have a negative on all the x^i 's?]. $\det H = 1$ implies that

$$(x^0)^2 - \sum_i^3 (x^i)^2 = 1 \tag{6.8}$$

This is a hyperbole in 4 dimensions and is like \mathcal{R}^3 . So the corresponding manifold for $SL(2, \mathbb{C})$,

$$\mathcal{M}_{SL(2,\mathbb{C})} = \mathbb{R}^3 \times S^3 \tag{6.9}$$

since each of the two manifolds is simply connected we know that $SL(2, \mathbb{C})$ is simply connected.

More generally we can have coset. The coset of \mathcal{G}/\mathcal{H} is found by taking every element $g \in \mathcal{G}$ and identifying it with gh , where $h \in \mathcal{H}$. We again consider a few examples,

1. $\mathcal{G}/\mathcal{H} = \frac{U(1)_1 \times U(1)_2}{U(1)_1}$. In this case an element of \mathcal{G} and \mathcal{H} can be written as

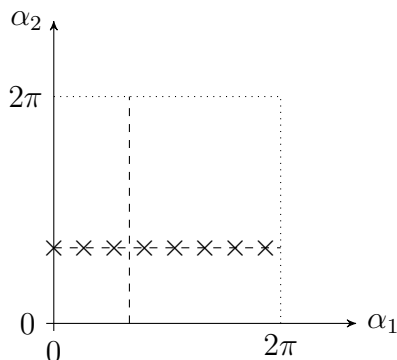
$$g = e^{i(\alpha_1 Q_1 + \alpha_2 Q_2)} \tag{6.10}$$

$$h = e^{i\beta Q_1} \tag{6.11}$$

then

$$gh = e^{i[(\alpha_1 + \beta)Q_1 + \alpha_2 Q_2]} = g \tag{6.12}$$

Pictorially we have,



We set all the values of α_1 on the line (marked by \times) to be equivalent to some value and the total group is just, $U(1)_2$.

2. $SU(2)/U(1) = S^2$

3. We can also define Minkowski space-time as a coset,

$$\frac{\text{Poincare}}{\text{Lorentz}} = \frac{\{\omega^{\mu\nu}, a^\mu\}}{\{\omega^{\mu\nu}\}} \rightarrow \{a^\mu = x^\mu\} = \text{Minkowski} \quad (6.13)$$

Note that since the Lorentz and the Poincare transformations do not commute this relation is not trivial but it can be shown to be true.

4. We now come to the most important example for us. $N = 1$ superspace is given by,

$$\frac{\text{Super Poincare}}{\text{Lorentz}} = \frac{\{\omega^{\mu\nu}, a^\mu, \theta_\alpha, \bar{\theta}_{\dot{\alpha}}\}}{\{\omega^{\mu\nu}\}} = \{a^\mu = x^\mu, \theta_\alpha, \bar{\theta}_{\dot{\alpha}}\} \quad (6.14)$$

An element of the Poincare group should be of the form,

$$g_{\text{Poincare}} = \exp \{i (\omega^{\mu\nu} M_{\mu\nu} + a^\mu P_\mu)\} \quad (6.15)$$

The elements of the Super Poincare group is then

$$g = \exp \{i (\omega^{\mu\nu} M_{\mu\nu} + a^\mu P_\mu + \theta^\alpha Q_\alpha + \bar{\theta}_{\dot{\alpha}} \bar{Q}^{\dot{\alpha}})\} \quad (6.16)$$

Note that normally have algebra in terms of anticommutators for the Q 's and not commutators. Using the Grassman numbers we can rewrite our algebra. Consider for example,

$$\{Q_\alpha, \bar{Q}_{\dot{\alpha}}\} = 2\sigma_{\alpha\dot{\alpha}}^\mu P_\mu \quad (6.17)$$

By multiplying from the left by θ^α and from the right by $\bar{\theta}^{\dot{\alpha}}$ this can be rewritten,

$$[\theta^\alpha Q_\alpha, \bar{Q}_{\dot{\beta}} \bar{\theta}^{\dot{\beta}}] = 2\theta^\alpha (\sigma_{\alpha\dot{\beta}}^\mu) \bar{\theta}^{\dot{\beta}} P_\mu \quad (6.18)$$

$$[\theta Q, \bar{\theta} \bar{Q}] = 2\theta \sigma^\mu \bar{\theta} P_\mu \quad (6.19)$$

To expand split the group elements we need to the BCH formula,

$$e^A e^B = e^{A+B+\frac{1}{2}[A,B]+\dots} \quad (6.20)$$

Returning back to our definition of superspace. We have x^μ which is a Minkowski variables and two anti-commuting numbers, θ and $\bar{\theta}$ which represent extra dimensions. We have 4 anticommuting dimensions.

6.1 Properties of $\theta^\alpha, \bar{\theta}_\alpha$ coordinates

Consider first one single θ parameter.

$$f(\theta) = f_0 + f_1\theta \quad (6.21)$$

The most general function of an anticommuting function is a linear function. *A Mathematicians paradise!*

We can take derivatives,

$$\frac{df}{d\theta} = f_1 \quad (6.22)$$

We define an integral such that

$$\int d\theta \frac{df}{d\theta} = 0 \quad (6.23)$$

In particular,

$$\int d\theta = 0 \quad (6.24)$$

Since the most general function will be linear, so the other integral we need to define,

$$\int \theta d\theta \equiv 1 \quad (6.25)$$

This normalization is done by convention. We impose this not to be zero to keep our discussion non-trivial. Then the Delta function in this space is

$$\delta(\theta) = \theta \quad (6.26)$$

since

$$\int d\theta \theta f(\theta) = f(0) \quad (= f_0) \quad (6.27)$$

and

$$\int d\theta f(\theta) = \int d\theta f_0 + f_1\theta = f_1 \quad (6.28)$$

So we see that

$$\int d\theta f(\theta) = \frac{df(\theta)}{d\theta} \quad (6.29)$$

So the integral and derivative are the same in this space; this is one of the curious properties of Grassman numbers.

So far we have focused on a single Grassman number. Now we move on to multiple θ 's. Recall that we defined

$$\theta\theta \equiv \theta^\alpha\theta_\alpha \quad (6.30)$$

$$\bar{\theta}\bar{\theta} \equiv \bar{\theta}_\alpha\bar{\theta}^{\dot{\alpha}} \quad (6.31)$$

According this notation we have,

$$\theta^\alpha \theta^\beta = -\frac{1}{2} \epsilon^{\alpha\beta} \theta\theta \quad (6.32)$$

$$\bar{\theta}^{\dot{\alpha}} \bar{\theta}^{\dot{\beta}} = \frac{1}{2} \epsilon^{\dot{\alpha}\dot{\beta}} \bar{\theta}\bar{\theta} \quad (6.33)$$

So we can take derivatives,

$$\frac{\partial \theta^\beta}{\partial \theta^\alpha} = \delta_\alpha^\beta \quad (6.34)$$

$$\frac{\partial \bar{\theta}^{\dot{\beta}}}{\partial \bar{\theta}^{\dot{\alpha}}} = \delta_{\dot{\alpha}}^{\dot{\beta}} \quad (6.35)$$

We can also take integrals. We know that

$$\int d\theta^1 d\theta^2 \theta^2 \theta^1 = 1 \quad (6.36)$$

Using equation 6.32 we can write $\theta^1 \theta^2 = \frac{1}{2} \theta\theta$ and so

$$\int \underbrace{\frac{1}{2} d\theta^1 d\theta^2}_{d^2\theta} \theta\theta = 1 \quad (6.37)$$

where we defined the differential for two dimensional Grassman variables above. Furthermore, you can write,

$$d^2\theta = -\frac{1}{4} d\theta^\alpha d\theta^\beta \epsilon_{\alpha\beta} \quad (6.38)$$

$$d^2\bar{\theta} = \frac{1}{4} d\bar{\theta}^{\dot{\alpha}} d\bar{\theta}^{\dot{\beta}} \epsilon_{\dot{\alpha}\dot{\beta}} \quad (6.39)$$

We can also write,

$$\int d^2\theta d^2\bar{\theta} (\theta\theta) (\bar{\theta}\bar{\theta}) = 1 \quad (6.40)$$

Extending the fact that the derivatives and integrals are the same we have,

$$\int d^2\theta = \frac{1}{4} \epsilon^{\alpha\beta} \frac{\partial}{\partial \theta^\alpha} \frac{\partial}{\partial \theta^\beta} \quad (6.41)$$

$$\int d^2\bar{\theta} = -\frac{1}{4} \epsilon^{\dot{\alpha}\dot{\beta}} \frac{\partial}{\partial \bar{\theta}^{\dot{\alpha}}} \frac{\partial}{\partial \bar{\theta}^{\dot{\beta}}} \quad (6.42)$$

6.2 Superfields

$N = 1$ superspace is given by

$$\{x^\mu, \theta^\alpha, \bar{\theta}_{\dot{\alpha}}\} \quad (6.43)$$

To define a superfield recall that a scalar field, $\phi(x^\mu)$ is a function of x^μ and it transforms under the Poincare group. For example under translations:

$$\phi \rightarrow e^{-ia^\mu P_\mu} \phi e^{ia^\mu P_\mu} \quad (6.44)$$

where P_μ are the generators of translations. But ϕ is also a function of x^μ . So we can also write with another representation of the momentum operator, \mathcal{P}_μ . Any function of x^μ can be transformed as,

$$\phi(x^\mu) \rightarrow e^{ia^\mu \mathcal{P}_\mu} \phi(x^\mu) = \phi(x^\mu + a^\mu) \quad (6.45)$$

$$(1 + ia^\nu \mathcal{P}_\nu) \phi(x^\mu) = \phi(x^\mu) + a^\nu \partial_\nu \phi(x^\mu) \quad (6.46)$$

$$\Rightarrow \mathcal{P}_\mu = -i\partial_\mu \quad (6.47)$$

We're doing something you've done since before you knew how to ride a bike.

We have two expressions for the same ϕ so we can find exactly how ϕ transforms:

$$(1 - ia^\mu P_\mu) \phi(x^\mu) (1 + ia^\mu P_\mu) = (1 + ia^\nu \mathcal{P}_\nu) \phi(x^\mu) \quad (6.48)$$

and so we have,

$$ia^\mu [\phi, P_\mu] = ia^\mu \mathcal{P}_\mu \phi \quad (6.49)$$

$$[\phi, a^\mu P_\mu] = -ia^\mu \partial_\mu \phi \quad (6.50)$$

We now follow these same steps but now for superfields.

For a general "scalar" superfield we write down every possible term using a exact Taylor expansion:¹ we have, *(take a deep breath and prepare for this)*

$$\begin{aligned} S(x, \theta, \bar{\theta}) = & \underbrace{\phi(x)}_{\text{scalar}} + \theta \underbrace{\psi(x)}_{\text{LSF}} + \bar{\theta} \underbrace{\bar{\chi}(x)}_{\text{RPF}} + \theta\theta M(x) + \bar{\theta}\bar{\theta} N(x) \\ & + (\theta\sigma^\mu\bar{\theta}) \underbrace{V_\mu(x)}_{\text{Vec. field}} + (\theta\theta) (\bar{\theta}\bar{\lambda}(x)) + (\bar{\theta}\bar{\theta}) (\theta\rho(x)) + (\theta\theta) (\bar{\theta}\bar{\theta}) D(x) \end{aligned} \quad (6.51)$$

where for convenience we write, LSF (RSF) as left spinor field (right spinor field). This is the most general representation of supersymmetry. As we will see later this representation is reducible, however its not easy to find out into what. This can be contrasted with the Poincare group where we had the irreducible representations, $\phi, \psi_\alpha, A_\mu, \dots$

Recall that when building a Poincare invariant Lagrangian we use products of representations of the Poincare group to build the most general Poincare invariant Lagrangian. Otherwise, it is really hard to make Poincare invariant terms! We want to go on to do

¹By scalar we do not demand that each constituents of the the terms making up the field be scalar but that each term is. In other words,

$$S\theta = \theta S \quad , \quad S\bar{\theta} = \bar{\theta} S$$

for any Grassman number θ .

the same thing here. We want to build the most general SUSY invariant Lagrangian. To do this we will use representations of supersymmetry.

Under Poincare transformations we have, $S \rightarrow e^{ia^\mu \mathcal{P}_\mu} S$. We now move on to how to transform our scalar superfield under superspace transformations (we already know how it transforms under Poincare transformations)

$$S(x, \theta, \bar{\theta}) \rightarrow e^{-i(\epsilon Q + \bar{\epsilon} \bar{Q})} S e^{i(\epsilon Q + \bar{\epsilon} \bar{Q})} \quad (6.52)$$

$$S(x, \theta, \bar{\theta}) \rightarrow e^{i(\epsilon \mathcal{Q} + \bar{\epsilon} \bar{\mathcal{Q}})} S \quad (6.53)$$

where ϵ is a infinitesimal 2 component Grassman object and we denote the differential form of the spinor charge operator \mathcal{Q} and the abstract form as Q (notice that the first and second equation above use different forms of the supercharge operator). Recall that for translations we knew that $e^{ia^\mu \mathcal{P}_\mu} \phi$ must transform to give $\phi(x^\mu + a^\mu)$. However, here it isn't quite obvious what the transformation will be. We allow the possibility that superspace transformations will also shift the real space coordinates. The most general vector quantity that's linear in the ϵ 's gives the transformation,

$$x^\mu \rightarrow x^\mu - (ic\epsilon\sigma^\mu\bar{\theta} + h.c.) \quad (6.54)$$

where c is a constant that we need to fix. We could not have a transformation that's a function of x^μ due to the assumed homogeneity of space-time. We have a general transformation,

$$S(x^\mu - ic\epsilon\sigma^\mu\bar{\theta}, \theta + \epsilon, \bar{\theta} + \bar{\epsilon}) \quad (6.55)$$

Doing a transformation in θ but keeping $\bar{\epsilon} = 0$ gives,

$$(1 + i\epsilon\mathcal{Q}) S(x^\mu, \theta, \bar{\theta}) = S(x^\mu, \theta, \bar{\theta}) - \left(ic\epsilon^\alpha \sigma^\mu_{\alpha\dot{\beta}} \bar{\theta}^{\dot{\beta}} \right) \partial_\mu S(x^\mu, \theta, \bar{\theta}) + \epsilon^\alpha \partial_\alpha S(x^\mu, \theta, \bar{\theta}) \quad (6.56)$$

$$i\epsilon^\alpha \mathcal{Q}_\alpha S(x^\mu, \theta, \bar{\theta}) = \epsilon^\alpha \left\{ -ic\sigma^\mu_{\alpha\dot{\beta}} \bar{\theta}^{\dot{\beta}} \partial_\mu + \partial_\alpha \right\} S(x^\mu, \theta, \bar{\theta}) \quad (6.57)$$

or

$$i\mathcal{Q}_\alpha = -ic\sigma^\mu_{\alpha\dot{\beta}} \bar{\theta}^{\dot{\beta}} \partial_\mu + \frac{\partial}{\partial\theta^\alpha} \quad (6.58)$$

$$(6.59)$$

Repeating the procedure only this time taking $\epsilon = 0$ and keeping nonzero $\bar{\epsilon}$ gives the differential forms of our generators:

$$\mathcal{Q}_\alpha = -i\frac{\partial}{\partial\theta^\alpha} - c\sigma^\mu_{\alpha\dot{\beta}} \bar{\theta}^{\dot{\beta}} \frac{\partial}{\partial x^\mu} \quad (6.60)$$

$$\bar{\mathcal{Q}}_{\dot{\alpha}} = i\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} + c^* \theta^\beta \sigma^\mu_{\beta\dot{\alpha}} \partial_\mu \quad (6.61)$$

$$\mathcal{P}_\mu = -i\partial_\mu \quad (6.62)$$

These objects are representations of the generators. To fix c we have to impose the commutation relation:

$$\{Q_\alpha, \bar{Q}_{\dot{\alpha}}\} = 2\sigma_{\alpha\dot{\alpha}}^\mu \mathcal{P}_\mu \quad (6.63)$$

It is a straightforward exercise to show that there are no restrictions on the imaginary part of c (which we set to zero) but just on the real part which must be 1^2 .

We now know how functions transform under SuperPoincare. We still have to find the analogue to the commutator in equation 6.50 which tells us how the fields transform. Comparing two transforms of S we have,

$$\exp(i(\epsilon Q + \bar{\epsilon}\bar{Q})) S \exp(-i(\epsilon Q + \bar{\epsilon}\bar{Q})) = \exp(i(\epsilon\mathcal{Q} + \bar{\epsilon}\bar{\mathcal{Q}})) S \quad (6.64)$$

$$\delta S = i [S, \epsilon Q + \bar{\epsilon}\bar{Q}] = i (\epsilon\mathcal{Q} + \bar{\epsilon}\bar{\mathcal{Q}}) S \quad (6.65)$$

To find how each of the components from equation 6.51 transform we just need to apply the differential form of the SUSY generators to the superfield. We write down the transformations of each component. We work through a few components explicitly.

Consider the action of \mathcal{Q} and $\bar{\mathcal{Q}}$ on the first part of S : the scalar field $\phi(x)$. We want to know what the contribution to the scalar field will be after transformation. Thus we apply the differential operator and then only keep scalar terms (the other terms will contribute to the transformation of other components). There is no hope of any term but the first three to contribute to the scalar field as they all have higher powers of θ or $\bar{\theta}$:³

$$i \left\{ \epsilon^\alpha \left(-i \frac{\partial}{\partial \theta^\alpha} - \sigma_{\alpha\dot{\beta}}^\mu \bar{\theta}^{\dot{\beta}} \partial_\mu \right) + \bar{\epsilon}^{\dot{\alpha}} \left(i \frac{\partial}{\partial \bar{\theta}^{\dot{\alpha}}} + \theta^\beta \sigma_{\beta\dot{\alpha}}^\mu \partial_\mu \right) \right\} \left\{ \phi + \theta\psi + \bar{\theta}\bar{\chi} + \dots \right\} \quad (6.66)$$

$$= i \left(-i\epsilon^\alpha \psi_\alpha + i\bar{\epsilon}^{\dot{\alpha}} \bar{\chi}_{\dot{\alpha}} \right) \quad (6.67)$$

$$= \epsilon\psi + \bar{\epsilon}\bar{\chi} \quad (6.68)$$

where we only kept terms without θ or $\bar{\theta}$ dependence and used $\epsilon^{\dot{\alpha}} \bar{\chi}_{\dot{\alpha}} = -\epsilon\bar{\chi}$. Thus we have,

$$\delta\phi = \epsilon\psi + \bar{\epsilon}\bar{\chi} \quad (6.69)$$

Notice that here we see that a scalar field, ϕ , goes to fermions under supersymmetry as expected.

²Note that to derive this relationship you must use the Grassman numbers product rule,

$$\frac{\partial}{\partial \theta} (fg) = \frac{\partial f}{\partial \theta} g - f \frac{\partial g}{\partial \theta}$$

³Here we need the relations:

$$\partial_\alpha \theta^\beta = \delta_\alpha^\beta, \quad \partial_\alpha \theta^{\dot{\beta}} = \delta_\alpha^{\dot{\beta}}, \quad \partial_\alpha \bar{\theta}^{\dot{\alpha}} = -\bar{\theta}^{\dot{\alpha}} \partial_\alpha$$

We now transform the fermion field. We have,

$$i \left\{ \epsilon^\alpha \left(-i \frac{\partial}{\partial \theta^\alpha} - \sigma_{\alpha\dot{\beta}}^\mu \bar{\theta}^{\dot{\beta}} \partial_\mu \right) + \bar{\epsilon}^{\dot{\alpha}} \left(i \frac{\partial}{\partial \bar{\theta}^{\dot{\alpha}}} + \theta^\beta \sigma_{\beta\dot{\alpha}}^\mu \partial_\mu \right) \right\} \left\{ \phi + \theta\theta M + \theta\sigma^\mu\bar{\theta}V_\mu + \dots \right\} \quad (6.70)$$

$$= i\bar{\epsilon}^{\dot{\alpha}}\theta^\beta (\sigma^\mu)_{\beta\dot{\alpha}} \partial_\mu \phi + 2\epsilon^\alpha \theta_\alpha M + \epsilon^{\dot{\alpha}} \sigma_{\beta\dot{\alpha}}^\mu \theta_\beta V_\mu \quad (6.71)$$

$$= \theta (i\sigma^\mu \bar{\epsilon} \partial_\mu \phi + 2\epsilon M + \sigma^\mu \bar{\epsilon} V_\mu) \quad (6.72)$$

so

$$\delta\psi = \sigma^\mu \bar{\epsilon} (i\partial_\mu \phi + V_\mu) + 2\epsilon M \quad (6.73)$$

As a last example we explore the transformations of the D term as it will turn out to be particularly important in supersymmetry. We have,

$$i \left\{ \epsilon^\alpha \left(-i \frac{\partial}{\partial \theta^\alpha} - \sigma_{\alpha\dot{\beta}}^\mu \bar{\theta}^{\dot{\beta}} \partial_\mu \right) + \bar{\epsilon}^{\dot{\alpha}} \left(i \frac{\partial}{\partial \bar{\theta}^{\dot{\alpha}}} + \theta^\beta \sigma_{\beta\dot{\alpha}}^\mu \partial_\mu \right) \right\} \left\{ \bar{\theta}^2 \theta \rho + \theta^2 \bar{\theta} \bar{\lambda} + \dots \right\} \quad (6.74)$$

$$= -i\epsilon^\alpha \sigma_{\alpha\dot{\beta}}^\mu \theta^2 \bar{\theta}^{\dot{\beta}} \bar{\theta}_{\dot{\alpha}} \partial_\mu \lambda^{\dot{\alpha}} - i\bar{\epsilon}_{\dot{\alpha}} (\sigma^\mu)^{\beta\dot{\alpha}} \bar{\theta}^2 \theta_\beta \theta^\alpha \partial_\mu \rho \quad (6.75)$$

$$= i\theta^2 \bar{\theta}^2 \partial_\mu \left(\frac{1}{2} \epsilon^\alpha \sigma_{\alpha\dot{\beta}}^\mu \lambda^{\dot{\beta}} - \frac{1}{2} \rho_\beta (\sigma^\mu)^{\beta\dot{\alpha}} \bar{\epsilon}_{\dot{\alpha}} \right) \quad (6.76)$$

We leave the rest of the transformations as an exercise. We list them all below,

$$\delta\phi = \epsilon\psi + \bar{\psi}\bar{\chi} \quad (6.77)$$

$$\delta\psi = 2\epsilon M + \sigma^\mu \bar{\epsilon} (i\partial_\mu \phi + V_\mu) \quad (6.78)$$

$$\delta\bar{\chi} = 2\bar{\epsilon}N - \epsilon\sigma^\mu (i\partial_\mu \phi - V_\mu) \quad (6.79)$$

$$\delta M = \bar{\epsilon}\bar{\lambda} - \frac{i}{2} \partial_\mu \psi \sigma^\mu \bar{\epsilon} \quad (6.80)$$

$$\delta N = \epsilon\rho + \frac{i}{2} \epsilon\sigma^\mu \partial_\mu \bar{\chi} \quad (6.81)$$

$$\delta V_\mu = \epsilon\sigma_\mu \bar{\lambda} + \rho\sigma_\mu \bar{\epsilon} + \frac{i}{2} \partial^\nu \psi \sigma_\mu \bar{\sigma}_\nu \epsilon - \bar{\epsilon} \bar{\sigma}_\nu \sigma_\mu \partial^\nu \bar{\chi} \quad (6.82)$$

$$\delta \bar{\lambda} = 2\bar{\epsilon}D + \frac{i}{2} (\bar{\sigma}^\nu \sigma^\mu \bar{\epsilon}) \partial_\mu V_\nu + i (\bar{\sigma}^\mu \epsilon) \partial_\mu M \quad (6.83)$$

$$\delta \rho = 2\epsilon D - \frac{i}{2} (\sigma^\nu \bar{\sigma}^\mu \epsilon) \partial_\mu V_\nu + i\sigma^\mu \bar{\epsilon} \partial_\mu N \quad (6.84)$$

$$\delta D = \frac{i}{2} \partial_\mu (\epsilon\sigma^\mu \bar{\lambda} - \rho\sigma^\mu \bar{\epsilon}) \quad (6.85)$$

Note the transformation of D is by a total derivative. We will use this property soon.

To find supersymmetry invariant terms we want to find the type of operations we can apply to a superfield (the representation of supersymmetry) such that it remains a superfield:

- If S_1 and S_2 are superfields then $S_1 S_2$ is also a superfield. **Proof:** Suppose $S = S_1 S_2$

then

$$\delta S = i [S_1 S_2, \epsilon Q + \bar{\epsilon} \bar{Q}] \quad (6.86)$$

$$\begin{aligned} &= i \left(S_1 S_2 (\epsilon Q + \bar{\epsilon} \bar{Q}) - S_1 (\epsilon Q + \bar{\epsilon} \bar{Q}) S_2 + S_1 (\epsilon Q + \bar{\epsilon} \bar{Q}) S_2 \right. \\ &\quad \left. - (\epsilon Q + \bar{\epsilon} \bar{Q}) S_1 S_2 \right) \end{aligned} \quad (6.87)$$

$$= i (S_1 [S_2, \epsilon Q + \bar{\epsilon} \bar{Q}] + [S_1, \epsilon Q + \bar{\epsilon} \bar{Q}] S_2) \quad (6.88)$$

$$= S_1 i (\epsilon Q + \bar{\epsilon} \bar{Q}) S_2 + i (\epsilon Q + \bar{\epsilon} \bar{Q}) S_1 S_2 \quad (6.89)$$

$$= i (\epsilon Q + \bar{\epsilon} \bar{Q}) (S_1 S_2) \quad (6.90)$$

where in the last step we used the fact that Q and \bar{Q} are differential operators and obey the product rule.

- Linear combinations of superfields are superfields since,

$$[\alpha S_1 + \beta S_2, \epsilon Q + \bar{\epsilon} \bar{Q}] = \alpha (\epsilon Q + \bar{\epsilon} \bar{Q}) S_1 + \beta (\epsilon Q + \bar{\epsilon} \bar{Q}) S_2 \quad (6.91)$$

$$= (\epsilon Q + \bar{\epsilon} \bar{Q}) (\alpha S_1 + \beta S_2) \quad (6.92)$$

- $\partial_\mu S$ is a superfield since,

$$[\partial_\mu S, \epsilon Q + \bar{\epsilon} \bar{Q}] = \partial_\mu [S, \epsilon Q + \bar{\epsilon} \bar{Q}] \quad (6.93)$$

$$= \partial_\mu (\epsilon Q + \bar{\epsilon} \bar{Q}) S \quad (6.94)$$

$$= (\epsilon Q + \bar{\epsilon} \bar{Q}) \partial_\mu S \quad (6.95)$$

- However, $\partial_\alpha S$ is **not** a superfield since,

$$\delta (\partial_\alpha S) = i [\partial_\alpha S, \epsilon Q + \bar{\epsilon} \bar{Q}] \quad (6.96)$$

$$= i \partial_\alpha [S, \epsilon Q + \bar{\epsilon} \bar{Q}] \quad (6.97)$$

$$= i \partial_\alpha (\epsilon Q + \bar{\epsilon} \bar{Q}) S \quad (6.98)$$

$$\neq i (\epsilon Q + \bar{\epsilon} \bar{Q}) \partial_\alpha S \quad (6.99)$$

Note the subtle point here. We were able to pull out the ∂_α in the second line since Q and \bar{Q} are the abstract objects. However, afterwards the same cannot be done since then we are in a particular representation of the Q 's namely, the differential representation.

We have seen such a situation in the past where the derivative didn't transform in the way that we want it too - in building a gauge invariant Lagrangian. We will have a similar situation here. We define covariant derivatives,

$$\mathcal{D}_\alpha \equiv \partial_\alpha + i \sigma_{\alpha\dot{\beta}}^\mu \bar{\theta}^{\dot{\beta}} \partial_\mu \quad (6.100)$$

$$\bar{\mathcal{D}}_{\dot{\alpha}} \equiv -\bar{\partial}_{\dot{\alpha}} - i \theta^\beta \sigma_{\beta\dot{\alpha}}^\mu \partial_\mu \quad (6.101)$$

These satisfy,

$$\{\mathcal{D}_\alpha, \mathcal{Q}_\beta\} = \{\mathcal{D}_\alpha, \bar{\mathcal{Q}}_{\dot{\beta}}\} = \{\bar{\mathcal{D}}_{\dot{\alpha}}, \mathcal{Q}_\beta\} = \{\bar{\mathcal{D}}_{\dot{\alpha}}, \bar{\mathcal{Q}}_{\dot{\beta}}\} = 0 \quad (6.102)$$

We prove these relations in appendix 6.A

Since these derivatives anticommute with the \mathcal{Q} 's they will commute with the $\epsilon\mathcal{Q}$'s. That means that will obey

$$[\mathcal{D}_\alpha, \epsilon\mathcal{Q} + \bar{\epsilon}\bar{\mathcal{Q}}] = 0 \quad (6.103)$$

and so $\mathcal{D}_\alpha S$ is a superfield.

Also one can show that

$$\{\mathcal{D}_\alpha, \bar{\mathcal{D}}_{\dot{\alpha}}\} = 2i\sigma_{\alpha\dot{\alpha}}^\mu \partial_\mu \quad (6.104)$$

They have very similar properties to the \mathcal{Q} 's however they have the very special property that $\mathcal{D}_\alpha S$ is a superfield. Continuing with the remarks

- $S = f(x)$ is a superfield only if $f = \text{const}$. However, $S = \phi(x)$ for example is not a superfield (since $\delta\phi = \epsilon\psi + \bar{\epsilon}\bar{\chi}$, a term with θ 's).
- However this doesn't mean that by truncating the superfield you can never get a superfield. There are some ways to eliminate some components are still have a superfield. This is equivalent to saying that S is a reducible representation of SUSY. In general we can impose consistent constraints on S .

We now consider some examples of truncated superfields.

1. Chiral Superfield: Denoted by Φ it is defined such that when acted upon by the superderivative we get zero,

$$\bar{\mathcal{D}}_{\dot{\alpha}}\Phi = 0 \quad (6.105)$$

Such an object will transform to itself under supersymmetry since \mathcal{D} acting on a field is still a superfield.

2. Antichiral Superfield: Denoted by $\bar{\Phi}$ and we impose the constraint,

$$\mathcal{D}_\alpha\bar{\Phi} = 0 \quad (6.106)$$

3. Vector Superfield: $V = V^\dagger$. We note that while we call this a vector superfield it is still scalar in nature in the sense that all its terms are bosonic. The origin of this name will become clearer when we discuss this below.

4. Linear Superfield: Denoted by L and obeys the constraints,

$$\mathcal{D}\mathcal{D}L = 0 \quad (6.107)$$

$$L = L^\dagger \quad (6.108)$$

The Chiral superfield will include the quarks and leptons and the vector superfield will include the gauge fields.

6.3 Chiral Superfields

We have the constraint,

$$\bar{\mathcal{D}}_{\dot{\alpha}}\Phi = 0 \quad (6.109)$$

To find the components of Φ we will use a trick. We define a new coordinate,

$$y^\mu \equiv x^\mu + i\theta\sigma^\mu\bar{\theta} \quad (6.110)$$

If we consider now $\Phi(y, \theta, \bar{\theta})$ then the constraint for Φ takes the form,

$$\bar{\mathcal{D}}_{\dot{\alpha}}\Phi = -\partial_{\dot{\alpha}}\Phi(x, \theta, \bar{\theta}) - i\theta^\beta\sigma_{\beta\dot{\alpha}}^\mu\frac{\partial}{\partial x^\mu}\Phi(x, \theta, \bar{\theta}) \quad (6.111)$$

$$= -\bar{\partial}_{\dot{\alpha}}\Phi(y, \theta, \bar{\theta}) - \frac{\partial\Phi(y, \theta, \bar{\theta})}{\partial y^\mu}\frac{\partial y^\mu}{\partial\bar{\theta}^{\dot{\alpha}}} - i\theta^\beta\sigma_{\beta\dot{\alpha}}^\mu\frac{\partial}{\partial y^\mu}\Phi(y, \theta, \bar{\theta}) \quad (6.112)$$

$$= -\bar{\partial}_{\dot{\alpha}}\Phi - \partial_\mu\Phi(-i\theta\sigma^\mu)_{\dot{\alpha}} - i(\theta\sigma^\mu)_{\dot{\alpha}}\partial_\mu\Phi \quad (6.113)$$

$$= -\bar{\partial}_{\dot{\alpha}}\Phi \quad (6.114)$$

where we have used,

$$\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}}\left(\theta_\beta(\sigma^\mu)^{\beta\dot{\beta}}\theta_{\dot{\beta}}\right) = \theta_\beta(\sigma^\mu)^{\beta\dot{\beta}}\epsilon_{\dot{\beta}\dot{\alpha}} \quad (6.115)$$

$$= -(\theta\sigma^\mu)_{\dot{\alpha}} \quad (6.116)$$

Combining Eq 6.114 with the definition of a superfield we have,

$$\bar{\partial}_{\dot{\alpha}}\Phi(y, \theta, \bar{\theta}) = 0 \quad (6.117)$$

and so the field doesn't depend on $\bar{\theta}$! This gives a chiral field a very simple expansion. So a chiral field is just

$$\Phi(\overbrace{y^\mu, \theta}^{\text{no } \bar{\theta}}) = \phi(y^\mu) + \overbrace{\sqrt{2}\theta\psi(y^\mu) + \theta\theta F(y^\mu)}^{\text{convention}} \quad (6.118)$$

The $\phi(y^\mu)$ is a scalar object and will describe the squarks, sleptons, and the Higgs for instance. The $\psi(y^\mu)$ will describe the quarks, leptons, and Higgsinos. The $F(y^\mu)$ is a new object that will turn out not to be physical and known as an auxiliary field.

ϕ is complex so it has two components. ψ is complex but it has an index giving a total of 4 components. F is complex and has two components. F is needed in this regard to ensure that we have an equal number of bosonic and fermionic components.

Now we have a simple expression for Φ in terms of its independent components. However, y^μ is not a physically interesting variable. We can go back and rewrite our expression in terms of x^μ . To do this we expand each component in x^μ :

$$\begin{aligned} \Phi(y, \theta) &= \phi(x) + i\theta\sigma^\mu\bar{\theta}\frac{\partial}{\partial x^\mu}\phi(x) - \frac{1}{2}\theta\sigma^\mu\bar{\theta}\theta\sigma^\nu\bar{\theta}\frac{\partial^2}{\partial x^\mu\partial x^\nu}\phi(x) \\ &+ \sqrt{2}\theta\left(\psi(x) + i\theta\sigma^\mu\bar{\theta}\frac{\partial}{\partial x^\mu}\psi(x)\right) + \theta^2 F(x) \end{aligned} \quad (6.119)$$

To simplify this expression note that

$$\theta\sigma^\mu\bar{\theta}\theta\sigma^\nu\bar{\theta} = \theta^\alpha\sigma^\mu_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\theta^\beta\sigma^\nu_{\beta\dot{\beta}}\bar{\theta}^{\dot{\beta}} \quad (6.120)$$

$$= \frac{1}{4}\epsilon^{\alpha\beta}\epsilon^{\dot{\alpha}\dot{\beta}}\theta^2\bar{\theta}^2\sigma^\mu_{\alpha\dot{\alpha}}\sigma^\nu_{\beta\dot{\beta}} \quad (6.121)$$

$$= \frac{1}{4}\theta^2\bar{\theta}^2\sigma^\nu_{\beta\dot{\beta}}(\bar{\sigma}^\mu)^{\dot{\beta}\beta} \quad (6.122)$$

$$= \frac{1}{2}\theta^2\bar{\theta}^2\eta^{\mu\nu} \quad (6.123)$$

and

$$\theta^\beta\theta^\alpha\sigma^\mu_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial_\mu\psi_\beta = -\frac{1}{2}\epsilon^{\beta\alpha}\sigma^\mu_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial_\mu\psi_\beta \quad (6.124)$$

$$= \frac{1}{2}\sigma^\mu_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial_\mu\psi^\alpha \quad (6.125)$$

$$= -\frac{1}{2}\partial_\mu\psi\sigma^\mu\bar{\theta} \quad (6.126)$$

So we have,

$$\begin{aligned} \Phi(x, \theta, \bar{\theta}) &= \phi(x) + \sqrt{2}\theta\psi(x) + \theta\theta F(x) + i\theta\sigma^\mu\bar{\theta}\partial_\mu\phi - \frac{i}{\sqrt{2}}(\theta\theta)\partial_\mu\psi\sigma^\mu\bar{\theta} \\ &\quad - \frac{1}{4}(\theta\theta)(\bar{\theta}\bar{\theta})\square\phi \end{aligned} \quad (6.127)$$

By switching variables we get back the older components in the general superfield but with fewer variables. We only have ϕ , ψ , and F as variables in a chiral superfield. To find the transformations of the variables of a chiral superfield we can just compare with the general superfield transformations we found earlier. This gives,

$$\delta\phi = \sqrt{2}\epsilon\psi \quad (6.128)$$

$$\delta\psi = i\sqrt{2}\sigma^\mu\bar{\epsilon}\partial_\mu\phi + \sqrt{2}\epsilon F \quad (6.129)$$

$$\delta F = \sqrt{2}i\partial_\mu(\bar{\epsilon}\bar{\sigma}^\mu\psi) \quad (6.130)$$

Again we see that fermions go to bosons and bosons go to fermions under SUSY transformations. Notice

- δF is just a total derivative. We will need to keep this in mind when we discuss Lagrangians.
- Products of chiral superfields are also chiral superfields since,

$$\mathcal{D}_{\dot{\alpha}}(\Phi_1\Phi_2) = (\mathcal{D}_{\dot{\alpha}}\Phi_1)\Phi_2 + \Phi_1(\mathcal{D}_{\dot{\alpha}}\Phi_2) = 0 \quad (6.131)$$

In general any (holomorphic) function of a chiral superfield, $f(\Phi)$, will be a chiral superfield since,

$$\mathcal{D}_{\dot{\alpha}}f(\Phi) = \frac{\partial f(\Phi)}{\partial\Phi} \cdot \mathcal{D}_{\dot{\alpha}}\Phi = 0 \quad (6.132)$$

- If Φ is chiral, then $\bar{\Phi} = \Phi^\dagger$ is anti-chiral (which obeys $\mathcal{D}_\alpha \bar{\Phi} = 0$)
- $\Phi\Phi^\dagger, \Phi + \Phi^\dagger$ are superfields but don't have any particular chirality.

6.4 Vector Superfields

For vector superfields we impose the condition,

$$V(x, \theta, \bar{\theta}) = V^\dagger(x, \theta, \bar{\theta}) \quad (6.133)$$

We now write the general vector superfield,

$$\begin{aligned} V(x, \theta, \bar{\theta}) = & C(x) + i\theta\chi(x) - i\bar{\theta}\bar{\chi} + \frac{i}{2}\theta\theta(M(x) + iN(x)) \\ & - \frac{i}{2}\bar{\theta}\bar{\theta}(M(x) - iN(x)) + \theta\sigma^\mu\bar{\theta}V_\mu(x) + i\theta\theta\bar{\theta}(\bar{\lambda}(x) + \frac{i}{2}\bar{\sigma}^\mu\partial_\mu\chi(x)) \\ & - i\bar{\theta}\bar{\theta}\theta\left(\lambda(x) + \frac{i}{2}\sigma^\mu\partial_\mu\bar{\chi}(x)\right) + \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}\left(D - \frac{1}{2}\square C\right) \end{aligned} \quad (6.134)$$

The important thing about this superfield is it has components C, M, N, D, V_μ real bosonic components which gives $1 + 1 + 1 + 1 + 4$ bosonic components. You have $\chi_\alpha, \lambda_\alpha$ complex fermionic components which gives $4 + 4$ components. The field is called a vector superfield since it includes the V_μ component. This is the relevant part.

Consider some chiral superfield Λ . The combination $i(\Lambda - \Lambda^\dagger)$ is a vector superfield since $i(\Lambda - \Lambda^\dagger)$ is real. Comparing components with the general form of the vector superfield, (Exercise)

$$C = i(\phi - \phi^\dagger) \quad (6.135)$$

$$\chi = \sqrt{2}\psi \quad (6.136)$$

$$\frac{1}{2}(M + iN) = F \quad (6.137)$$

$$V_\mu = -\partial_\mu(\phi + \phi^\dagger) \quad (6.138)$$

$$\lambda = D = 0 \quad (6.139)$$

We can use this as a way to define a transformation on vector fields. This will let us generalize the standard gauge transformations. Recall that under a gauge transformation a vector transforms as,

$$A_\mu(x) \xrightarrow{\text{gauge}} A_\mu(x) - \partial_\mu\alpha(x) \quad (6.140)$$

The generalized gauge transformation is

$$V(x, \theta, \bar{\theta}) \rightarrow V(x, \theta, \bar{\theta}) + i(\Lambda(x, \theta, \bar{\theta}) - \Lambda^\dagger(x, \theta, \bar{\theta})) \quad (6.141)$$

From expression 6.138 we have that under the transformation above,

$$V_\mu \rightarrow V_\mu - \partial_\mu \left(\frac{\phi + \phi^\dagger}{2} \right) \quad (6.142)$$

$$= V_\mu - \partial_\mu \underbrace{\text{Re}(\phi)}_\alpha \quad (6.143)$$

This induces a standard gauge transformation for the vector component of V . Λ isn't a physical quantity; it just a convenient superfield. So we can fix the gauge by choosing a particular value for ϕ, ψ , and F within Λ to gauge away some of the components of V . This is what is usually called the Wess-Zumino gauge.

6.4.1 Wess-Zumino Gauge

In the Wess-Zumino gauge we have,

$$V_{WZ} = (\theta\sigma^\mu\bar{\theta}) V_\mu(x) + i\theta\theta\bar{\theta}\bar{\lambda}(x) - i\bar{\theta}\bar{\theta}\theta\lambda(x) + \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}D(x) \quad (6.144)$$

We started with a standard vector superfield with all the different components. We then shifted our field which gave the reduced version of V . We now see the prominence of the vector component, which is now the first component of the vector superfield. $V_\mu(x)$ will contain all the gauge particles (photons, W^\pm , Z , gluons). $\lambda(x)$ will contain the gauginos (photino, Wino, Zino, gluino). $D(x)$ is an extra component or auxillary field.

We will want to find the square of this superfield as well. Due to the anticommuting property of θ and $\bar{\theta}$ we only need to keep the first term,

$$V_{WZ}^2 = \theta_\alpha (\sigma^\mu)^{\alpha\dot{\beta}} \bar{\theta}_{\dot{\beta}} \theta_\beta (\sigma^\nu)^{\beta\dot{\alpha}} \bar{\theta}_{\dot{\alpha}} V_\mu V_\nu \quad (6.145)$$

$$= \frac{1}{4} (\theta\theta) (\bar{\theta}\bar{\theta}) (\sigma^\mu)_\beta^{\dot{\beta}} (\sigma^\nu)^{\beta\dot{\beta}} V_\mu V_\nu \quad (6.146)$$

$$= \frac{1}{2} (\theta\theta) (\bar{\theta}\bar{\theta}) V_\mu V^\mu \quad (6.147)$$

and clearly,

$$V_{WZ}^n = 0 \quad \forall n > 2 \quad (6.148)$$

This is a nice property of using the Wess-Zumino gauge.

Even though the Wess-Zumino gauge has some nice properties its important to note the WZ gauge is not supersymmetric. That means that if we take V_{WZ} and transform it under supersymmetry it is no longer in the Wess Zumino gauge:

$$V_{WZ} \xrightarrow{SUSY} V_{\mathcal{WZ}} \quad (6.149)$$

However, under a combination of SUSY and generalized gauge transformation we can always do the following:

$$V_{WZ} \xrightarrow{SUSY} V_{\mathcal{WZ}} \xrightarrow{gen. gauge} V_{\mathcal{WZ}}'' \quad (6.150)$$

Under a standard gauge transformation we know how matter fields and vector fields transform. For example for a $U(1)$ gauge the fields get a phase. We now try to see how superfields transform under gauge transformations.

6.5 Transformations of Superfields

6.5.1 Chiral Superfields

Lets first consider non-SUSY transformations. We can have a $U(1)$ gauge:

$$\phi(x) \rightarrow e^{iq\alpha}\phi(x) \quad (6.151)$$

$$V_\mu(x) \rightarrow V_\mu + \partial_\mu\alpha \quad (6.152)$$

where $\alpha(x)$ is a parameter and q is the charge.

We know how vector superfields transform. We hypothesize a form for the chiral superfield:

$$V \rightarrow V - \frac{i}{2}(\Lambda - \Lambda^\dagger) \quad (6.153)$$

$$\Phi \rightarrow e^{iq\Lambda}\Phi \quad (6.154)$$

Since Λ is a chiral superfield the transformed Φ is also a chiral superfield.

We now want to define the field strength superfield; the analogue to (for $U(1)$ gauge) $F_{\mu\nu} \equiv \partial_\mu V_\nu - \partial_\nu V_\mu$. We define a new object

$$W_\alpha \equiv -\frac{1}{4}\bar{\mathcal{D}}^2\mathcal{D}_\alpha V \quad (6.155)$$

We now consider the properties of this object,

- This is the first superfield we've considered so far that carries an index.
- W_α is chiral
- W_α is invariant under generalized gauge transformation

These two properties are easy to show after writing W_α in components,

$$W_\alpha(y, \theta) = \lambda_\alpha(y) + \theta_\alpha D(y) + (\sigma^{\mu\nu}\theta)_\alpha F_{\mu\nu} - i\theta\theta\sigma^\mu_{\alpha\dot{\beta}}\partial_\mu\bar{\lambda}^{\dot{\beta}} \quad (6.156)$$

where $F_{\mu\nu} \equiv \partial_\mu V_\nu - \partial_\nu V_\mu$. This is called the field strength superfield since one of its components includes the field strength tensor from the SM.

6.A Derivatives and spinor charges

Above we made the following claim:

$$\{\mathcal{D}_\alpha, \mathcal{Q}_\beta\} = \{\mathcal{D}_\alpha, \bar{\mathcal{Q}}_\beta\} = \{\bar{\mathcal{D}}_{\dot{\alpha}}, \mathcal{Q}_\beta\} = \{\bar{\mathcal{D}}_{\dot{\alpha}}, \bar{\mathcal{Q}}_\beta\} = 0 \quad (6.157)$$

We prove these properties below:

$$\{\mathcal{D}_\alpha, \mathcal{Q}_\beta\} = \left\{ \partial_\alpha + i\sigma_{\alpha\dot{\beta}}^\mu \bar{\theta}^{\dot{\beta}} \partial_\mu, -i\partial_\beta - \sigma_{\beta\dot{\gamma}}^\nu \bar{\theta}^{\dot{\gamma}} \partial_\nu \right\} \quad (6.158)$$

The derivative term trivially anticommute. The cross-terms anticommute as well since $\{\partial_\alpha, \bar{\theta}^{\dot{\beta}}\} = 0$ and the rest of the factors just pull right out. The same occurs with the the final term due to the anticommutator, $\{\bar{\theta}^{\dot{\beta}}, \bar{\theta}^{\dot{\gamma}}\} = 0$.

We now consider a less trivial example,

$$\{\mathcal{D}_\alpha, \bar{\mathcal{Q}}_{\dot{\alpha}}\} = \left\{ \partial_\alpha + i\sigma_{\alpha\dot{\beta}}^\mu \bar{\theta}^{\dot{\beta}} \partial_\mu, i\partial_{\dot{\alpha}} + \theta^\beta \sigma_{\beta\dot{\alpha}}^\mu \partial_\mu \right\} \quad (6.159)$$

The two derivative and $\theta, \bar{\theta}$ terms again trivially anticommute giving zero. The possible concern here are the cross terms. We have,

$$\left\{ \sigma_{\alpha\dot{\beta}}^\mu \bar{\theta}^{\dot{\beta}} \partial_\mu, \partial_{\dot{\alpha}} \right\} = \sigma_{\alpha\dot{\beta}}^\mu \bar{\theta}^{\dot{\beta}} \partial_\mu \partial_{\dot{\alpha}} + \sigma_{\alpha\dot{\beta}}^\mu \delta_{\dot{\alpha}}^{\dot{\beta}} \partial_\mu - \sigma_{\alpha\dot{\beta}}^\mu \bar{\theta}^{\dot{\beta}} \partial_\mu \partial_{\dot{\alpha}} = \sigma_{\alpha\dot{\alpha}}^\mu \partial_\mu \quad (6.160)$$

$$\left\{ \partial_\alpha, \theta^\beta \sigma_{\beta\dot{\alpha}}^\mu \partial_\mu \right\} = \sigma_{\alpha\dot{\alpha}}^\mu \partial_\mu \quad (6.161)$$

But these two terms have a relative minus sign in the anticommutator hence,

$$\{\mathcal{D}_\alpha, \bar{\mathcal{Q}}_{\dot{\alpha}}\} = 0 \quad (6.162)$$

By symmetry the other two anticommutators are also zero.

6.B Useful Relations

$$\psi \sigma^\mu \bar{\chi} = \psi^\alpha \sigma_{\alpha\dot{\alpha}}^\mu \bar{\chi}^{\dot{\alpha}} \quad (6.163)$$

$$= (\psi \sigma^\mu)_{\dot{\alpha}} \bar{\chi}^{\dot{\alpha}} \quad (6.164)$$

$$= -(\psi \sigma^\mu)^{\dot{\alpha}} \bar{\chi}_{\dot{\alpha}} \quad (6.165)$$

$$= \bar{\chi}_{\dot{\alpha}} (\bar{\sigma}^\mu)^{\dot{\alpha}\beta} \psi_\beta \quad (6.166)$$

$$\theta^\alpha \psi^\beta = -\frac{1}{2} \epsilon^{\alpha\beta} \theta \psi \quad (6.167)$$

$$= \frac{1}{2} \epsilon^{\beta\alpha} \psi \theta \quad (6.168)$$

$$= -\psi^\beta \theta^\alpha \quad (6.169)$$

So two contravariant spinors anticommute.

$$\theta\sigma^\mu\bar{\theta}\theta\sigma^\nu\bar{\theta} = \frac{1}{2}\theta^2\bar{\theta}^2\eta^{\mu\nu} \quad (6.170)$$

Consider the transformation $y^\mu = x^\mu + i\theta\sigma^\mu\bar{\theta}$,

$$\frac{\partial}{\partial\theta^\alpha} = \frac{\partial y^\mu}{\partial\theta^\alpha} \frac{\partial}{\partial y^\mu} + \frac{\partial}{\partial\theta^\alpha} = i(\sigma^\mu\bar{\theta})_\alpha \frac{\partial}{\partial y^\mu} + \frac{\partial}{\partial\theta^\alpha} \quad (6.171)$$

$$\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} = \frac{\partial y^\mu}{\partial\bar{\theta}^{\dot{\alpha}}} \frac{\partial}{\partial y^\mu} + \frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} = -i(\theta\sigma^\mu)_{\dot{\alpha}} \frac{\partial}{\partial y^\mu} + \frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} \quad (6.172)$$

$$\frac{\partial}{\partial x^\mu} = \frac{\partial}{\partial y^\mu} \quad (6.173)$$

Using these relationships we can find the covariant derivatives in these coordinates,

$$\mathcal{D}_\alpha^y = \partial_\alpha + 2i(\sigma^\mu)_{\alpha\dot{\beta}}\bar{\theta}^{\dot{\beta}}\partial_\mu \quad (6.174)$$

$$\mathcal{D}_{\dot{\alpha}}^y = -\partial_{\dot{\alpha}} \quad (6.175)$$

Chapter 7

Four Dimensional Supersymmetric Lagrangians

7.1 $\mathcal{N} = 1$ Global Supersymmetry

We want to find couplings among superfields (Φ, V, W_α) which include particles of the SM. For this to work we need a way to build Lagrangians that are invariant up to a total derivative under a SUSY transformation.

7.1.1 Chiral Superfields

To find an object $\mathcal{L}(\Phi)$ such that $\delta\mathcal{L}$ is a total derivative under SUSY transformations we use the fact that

- For a general superfield, $S = \dots + (\theta\theta)(\bar{\theta}\bar{\theta})D(x)$, the D term transforms as,

$$\delta D = \frac{i}{2} \partial_\mu (\epsilon \sigma^\mu \bar{\lambda} - \rho \sigma^\mu \bar{\epsilon}) \quad (7.1)$$

- For a chiral superfield, $\Phi = \dots + (\theta\theta)F(x)$, the F term transforms as,

$$\delta F = \partial_\mu (i\sqrt{2}\bar{\epsilon}\sigma^\mu\psi) \quad (7.2)$$

Now recall that any holomorphic function of chiral superfield is also a chiral superfield. So we can write the most general Lagrangian with only chiral superfields as,

$$\mathcal{L} = K(\Phi, \Phi^\dagger) \Big|_D + \left(W(\Phi) \Big|_F + h.c. \right) \quad (7.3)$$

where $\Big|_D$ refers to the pulling out only the D term of the corresponding superfield (without $\theta, \bar{\theta}$) and $\Big|_F$ refers to pulling out only the F term of the corresponding general superfield

(without $\theta, \bar{\theta}$). The function, K is known as the Kähler potential and is a real function of Φ and Φ^\dagger . The chiral superfield, W , is known as the superpotential.

As an example we consider a renormalizable Lagrangian for the chiral superfield. The dimension of a superfield, Φ is the same as the dimension of its scalar component, ϕ . We want to get back to Lagrangians that we recognize from QFT. We take the of the fermionic component and scalar component to be the same as for the Klein Gordon and Dirac equations,

$$[\Phi] = [\phi] = 1 \quad , \quad [\psi] = 3/2 \quad (7.4)$$

From the expansion of the chiral superfield,

$$\Phi = \phi + \sqrt{2}\theta\psi + \theta\theta F + \dots \quad (7.5)$$

we have,

$$[\theta] = -\frac{1}{2} \quad , \quad [F] = 2 \quad (7.6)$$

This already hints that F isn't a standard bosonic field. Renormalizable Lagrangian have terms such as,

$$\partial_\mu \phi^\dagger \partial^\mu \phi \quad , \quad m^2 \underbrace{\phi^\dagger \phi}_{\text{dim}=2} \quad , \quad g \underbrace{(\phi^\dagger \phi)^2}_{\text{dim}=4} \quad (7.7)$$

Operators with dimension greater than 4 are not renormalizable. In the Kähler potential the D and F terms appear in the form,

$$K = \dots + \theta^2 \bar{\theta}^2 K_D \quad (7.8)$$

$$W = \dots + (\theta\theta) W_F \quad (7.9)$$

In this notation $K \Big|_D = K_D$). However, in this discussion we are allowed to have dimensionful coefficients in front of our operators so we must have,

$$[K_D] \leq 4 \quad (7.10)$$

$$[W_F] \leq 4 \quad (7.11)$$

This requires

$$[K] \leq 2 \quad (7.12)$$

$$[W] \leq 3 \quad (7.13)$$

Thus we cannot for example have terms of the form $K = \Phi^\dagger \Phi \Phi^\dagger \Phi$. Since \mathcal{K} must be real the only other form it can be is $\Phi^\dagger + \Phi$, however this leads to a total derivative in the Lagrangian and so does not contribute. So in general we have,

$$K = \Phi^\dagger \Phi \quad , \quad W = \alpha + \lambda \Phi + \frac{m^2}{2} \Phi^2 + \frac{g}{3} \Phi^3 \quad (7.14)$$

The corresponding Lagrangian is known as the Wess Zumino Model,

$$\mathcal{L} = \Phi^\dagger \Phi \Big|_D + \left(\left(\alpha + \lambda \Phi + \frac{m^2}{2} \Phi^2 + \frac{g}{3} \Phi^3 \right) \Big|_F + h.c. \right) \quad (7.15)$$

We now expand the result in components. Recall the expression for the chiral superfield,

$$\Phi = \phi + \sqrt{2} \theta \psi + \theta \theta F + i \theta \sigma^\mu \bar{\theta} \partial_\mu \phi - \frac{i}{\sqrt{2}} (\theta \theta) \partial_\mu \psi \sigma^\mu \bar{\theta} - \frac{1}{4} (\theta \theta) (\bar{\theta} \bar{\theta}) \partial_\mu \partial^\mu \phi \quad (7.16)$$

$$\Phi^\dagger = \phi^* + \sqrt{2} \bar{\psi} \bar{\theta} + \bar{\theta} \bar{\theta} F^* - i \theta \sigma^\mu \bar{\theta} \partial_\mu \phi^* + \frac{i}{\sqrt{2}} (\bar{\theta} \bar{\theta}) \theta \sigma^\mu \partial_\mu \bar{\psi} - \frac{1}{4} (\theta \theta) (\bar{\theta} \bar{\theta}) \partial_\mu \partial^\mu \phi^* \quad (7.17)$$

We need to multiply the two superfields together and we keep only terms with a $\theta^2 \bar{\theta}^2$ coefficient (keeping in mind that $\theta^3 = 0$). This procedure gives,

$$K \Big|_D = -\frac{1}{4} (\phi^* \partial_\mu \partial^\mu \phi + \phi \partial_\mu \partial^\mu \phi^* - 2 \partial^\mu \phi \partial_\mu \phi^*) + F F^* - \frac{i}{2} (-\psi \sigma^\mu \partial_\mu \bar{\psi} + \partial_\mu \psi \sigma^\mu \bar{\psi}) \quad (7.18)$$

$$= \partial_\mu \phi^* \partial^\mu \phi + F F^* + i (\bar{\psi} \bar{\sigma}^\mu \partial_\mu \psi) \quad (7.19)$$

where we used $\sigma_{\alpha\dot{\alpha}}^\mu = \bar{\sigma}_{\dot{\alpha}\alpha}^\mu$ and $\theta^\alpha \theta^\beta = -\frac{1}{2} \epsilon^{\alpha\beta} \theta^2$. We now need to calculate the contribution due to the superpotential.

$$\alpha \Big|_F = 0 \quad (7.20)$$

$$\lambda \Phi \Big|_F = \lambda F \quad (7.21)$$

$$\frac{m}{2} \Phi^2 \Big|_F = m \left(-\frac{1}{2} \psi^2 + F \phi \right) \quad (7.22)$$

$$\frac{g}{3} \Phi^3 \Big|_F = g (-\psi^2 \phi + F \phi^2) \quad (7.23)$$

We have,

$$W \Big|_F = (\lambda + m \phi + g \phi^2) F - \frac{1}{2} (m + 2g \phi) \psi^2 \quad (7.24)$$

Above we calculated the contributions to the Lagrangian explicitly. Alternatively we can do an exact Taylor expansion due to the anticommuting nature of the Grassman variables,

$$W(\Phi) = W(\phi) + \underbrace{(\Phi - \phi)}_{\dots + \sqrt{2} \theta \psi + \theta \theta F + \dots} \frac{\partial W}{\partial \phi} + \frac{1}{2} \underbrace{(\Phi - \phi)^2}_{\dots + (\theta \psi)(\theta \psi) + \dots} \frac{\partial^2 W}{\partial \phi^2} \quad (7.25)$$

Calculating the derivatives of W with respect the superfield evaluated at $\theta = \bar{\theta} = 0$ (we expanded around $\Phi = \phi$):

$$\frac{\partial W}{\partial \phi} = \lambda + m \phi + g \phi^2 \quad (7.26)$$

$$\frac{\partial^2 W}{\partial \phi^2} = m + 2g \phi \quad (7.27)$$

where we have defined

$$\frac{\partial W}{\partial \phi} \equiv \frac{\partial W}{\partial \Phi} \Big|_{\Phi=\phi} \quad (7.28)$$

Thus we can write the full Lagrangian as,

$$\mathcal{L} = \partial^\mu \phi^* \partial_\mu \phi + i\bar{\psi} \bar{\sigma}^\mu \partial_\mu \psi + FF^* + \left(\frac{\partial W}{\partial \phi} F + h.c. \right) \Big|_F - \frac{1}{2} \left(\frac{\partial^2 W}{\partial \phi^2} \psi \psi + h.c. \right) \Big|_F \quad (7.29)$$

The first and second terms are the standard scalar field and fermion kinetic terms respectively!

The part of the Lagrangian that depends on the auxiliary field F takes the form,

$$\mathcal{L}_{(F)} = FF^* + \frac{\partial W}{\partial \phi} F + \frac{\partial W^*}{\partial \phi^*} F^* \quad (7.30)$$

Notice that this is quadratic and without any space-time derivatives. This means that the field F does not propagate. So we can easily eliminate F using the field equations

$$\frac{\partial \mathcal{L}_{(F)}}{\partial F} = F^* + \frac{\partial W}{\partial \phi} = 0 \quad (7.31)$$

$$\frac{\partial \mathcal{L}_{(F)}}{\partial F^*} = F + \frac{\partial W^*}{\partial \phi^*} = 0 \quad (7.32)$$

which implies that

$$F = -\frac{\partial W^*}{\partial \phi^*} \quad (7.33)$$

Substituting the result back in the Lagrangian gives

$$\mathcal{L}_{(F)} = \left| \frac{\partial W}{\partial \phi} \right|^2 - 2 \left| \frac{\partial W}{\partial \phi} \right|^2 = - \left| \frac{\partial W}{\partial \phi} \right|^2 \equiv -V_{(F)}(\phi) \quad (7.34)$$

This defines the scalar potential since it is the part in the Lagrangian that only depends on the scalar field, ϕ . From this expression we can easily see that it is a positive definite scalar potential,

$$V(\phi) \geq 0 \quad (7.35)$$

Starting from the more general Lagrangian,

$$\int d^2\theta \lambda_{ijk} \Phi_i \Phi_j \Phi_k + y_{ij} \Phi_i \Phi_j + h.c. + \int d^4\theta \Phi_i^\dagger \Phi_i \quad (7.36)$$

it is straight forward to show that the F term contribution to the scalar potential is,

$$V_{(F)}(\phi_i, \phi_i^*) = F_i^\dagger F_i = \left(\frac{\partial W}{\partial \phi_i} \right)^\dagger \left(\frac{\partial W}{\partial \phi_i} \right) \quad (7.37)$$

In summary we have,

$$\mathcal{L} = \partial^\mu \phi^* \partial_\mu \phi + i\bar{\psi} \bar{\sigma}^\mu \partial_\mu \psi - \frac{1}{2} \left(\frac{\partial^2 W}{\partial \phi^2} \psi \psi + h.c. \right) - V(\phi, \phi^*) \quad (7.38)$$

Remarks:

- This Lagrangian with $\mathcal{N} = 1$ is a particular case of a standard $\mathcal{N} = 0$ Lagrangian.
- This is a theory of a scalar field and a Majorana fermion since the fermion kinetic terms are $i\bar{\psi}\bar{\sigma}^\mu\partial\psi$, and mass terms are $m^2(\psi\psi + \bar{\psi}\bar{\psi})$.
- Suppose you have a W of the form,

$$W = \frac{m}{2}\Phi^2 + \frac{g}{3}\Phi^3 \tag{7.39}$$

The scalar field mass term is

$$\left. \frac{\partial W}{\partial \Phi} \right|_{\Phi=\phi} = m^2 |\phi|^2 \tag{7.40}$$

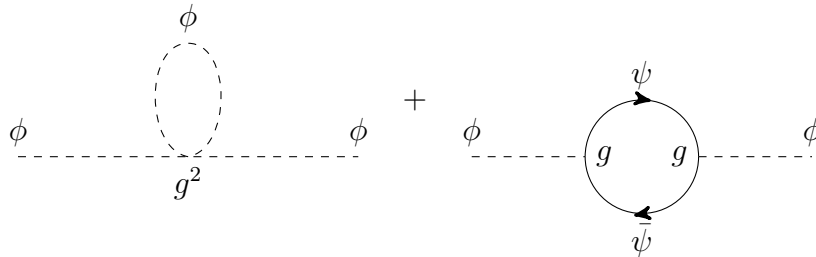
The fermionic mass term is

$$\frac{1}{2} \frac{\partial^2 W}{\partial \phi^2} \psi\psi + h.c. = \frac{1}{2} m\psi\psi + h.c. \tag{7.41}$$

(we will go through this minimization process more carefully soon) so the mass of the fermionic and bosonic fields are determined by the same parameter m .

- The Yukawa coupling ($\phi\psi\psi$) is equal to g . However, g is also the scalar self-coupling ($\sim g^2 |\phi|^4$).

The Yukawa coupling equally the self-coupling is the source of what is called the “miraculous cancellations in supersymmetry perturbation theory. Diagrams that look completely different in non-supersymmetry theories have the same coupling in SUSY. For example we have,



If the couplings were arbitrary then you would get completely different contributions from each diagram and they each diverge. If the couplings match up then instead they add up to give you a finite contribution. This is the origin to the idea that supersymmetric theories have much better quantum behavior than standard theories. If this field is the Higgs field then this explains how SUSY can help solve the hierarchy problem.

In general we don't need to stick to Kähler potentials of the type that we considered. You can have many scalar fields or a non-renormalizable potential. We consider the former generalization,

$$K(\Phi^i, \Phi^{j\dagger}) \quad , \quad W(\Phi^i) \tag{7.42}$$

As we did for W we can expand K around $\Phi^i = \phi^i$. This gives in components (Ex),

$$K(\Phi^i, \Phi^{j,\dagger}) \Big|_D = \frac{\partial^2 K}{\partial \phi_i \partial \phi_j} \partial_\mu \phi^i \partial^\mu \phi^{*j} \quad (7.43)$$

The kinetic terms are no longer the standard kinetic terms. This is usually written as

$$K_{ij^*} \partial_\mu \phi^i \partial^\mu \phi^{*j} \quad (7.44)$$

K_{ij^*} is a metric in the space with coordinates ϕ^i which is a complex Kähler-manifold,

$$g_{ij^*} = K_{ij^*} \quad (7.45)$$

This is often called a non-linear sigma model for historical reasons. A Kähler manifold is that the metric is given in terms of a potential. Not any metric can be written as a second derivative of an object. This is a property of Kähler manifolds.

7.1.2 Miraculous Calculation in Detail

Shifting the superfields we can reduce the most general superpotential to a form of,

$$\frac{m}{2} \Phi^2 + \frac{g}{3} \Phi^3 \quad (7.46)$$

To do a calculation we want to write out the Lagrangian in detail in terms of only the real fields. We define,

$$\phi \equiv \frac{\phi_1 + i\phi_2}{\sqrt{2}} \quad (7.47)$$

Explicitly we have,

$$V(\phi) = (m\phi + g\phi^2)(m\phi^* + g\phi^{*2}) \quad (7.48)$$

$$= m^2 |\phi|^2 + mg(\phi\phi^{*2} + \phi^*\phi^2) + g^2 |\phi|^4 \quad (7.49)$$

$$= \frac{m^2}{2} (\phi_1^2 + \phi_2^2) + \frac{mg}{\sqrt{2}} (\phi_1^3 + \phi_1\phi_2^2) + \frac{g^2}{4} (\phi_1^4 + \phi_2^4 + 2\phi_1^2\phi_2^2) \quad (7.50)$$

The fermion mass/Yukawa term is,

$$\frac{1}{2} \left(\frac{\partial^2 W}{\partial \phi^2} \Big|_D + h.c. \right) = \frac{1}{2} \{ (m + 2g\phi) \psi^2 + (m + 2g\phi^*) \bar{\psi}^2 \} \quad (7.51)$$

$$= \frac{1}{2} \{ m(\psi^2 + \bar{\psi}^2) + 2g(\phi_1 + i\phi_2) \psi^2 + 2g(\phi_1 - i\phi_2) \bar{\psi}^2 \} \quad (7.52)$$

Now since we have a Majorana fermion we can make the definition,

$$\Psi \equiv \begin{pmatrix} \psi \\ \bar{\psi} \end{pmatrix} \quad (7.53)$$

which gives

$$\frac{1}{2} \left(\frac{\partial^2 W}{\partial \phi^2} \Big|_D + h.c. \right) = \frac{1}{2} \{ m \bar{\Psi} \Psi + 2g \bar{\Psi} (\phi_1 + i\gamma^5 \phi_2) \Psi \} \quad (7.54)$$

and the kinetic term for the fermion is

$$\mathcal{L}_{kin}^{(f)} = i \bar{\psi} \sigma^\mu \partial_\mu \psi \quad (7.55)$$

$$= \frac{i}{2} (\bar{\psi} \sigma^\mu \partial_\mu \psi - h.c.) \quad (7.56)$$

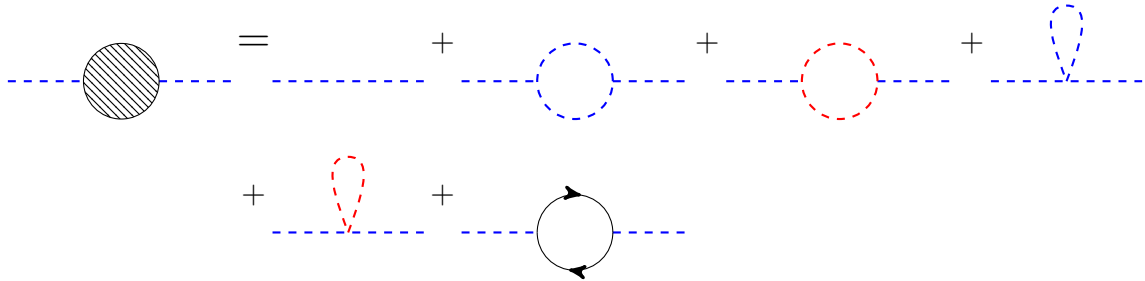
$$= \frac{i}{2} (\bar{\psi} \bar{\sigma}^\mu \partial_\mu \psi + x s p \sigma^\mu \partial_\mu \bar{\psi}) \quad (7.57)$$

$$= \frac{i}{2} \bar{\Psi} \not{\partial} \Psi \quad (7.58)$$

So we can write the full Lagrangian as,

$$\begin{aligned} \mathcal{L} = & \frac{1}{2} \partial^\mu \phi_i \partial_\mu \phi_i - \frac{m^2}{2} \phi_i \phi_i + \frac{1}{2} \bar{\Psi} (i\not{\partial} - m) \Psi - \frac{mg}{\sqrt{2}} (\phi_1^3 + \phi_1 \phi_2^2) \\ & - \frac{g^2}{4} (\phi_1^4 + \phi_2^4 + 2\phi_1^2 \phi_2^2) + g \bar{\Psi} (\phi_1 - i\gamma^5 \phi_2) \Psi \end{aligned} \quad (7.59)$$

Our goal is now to calculate the ϕ_1 propagator to second order. We have (we denote ϕ_1 with a blue dashed line and ϕ_2 with a red dashed line),



The first two loop diagrams are logarithmically divergent and give natural contributions to the propagator. The third loop has an amplitude of,

$$i\mathcal{M} = \int \frac{d^4 \ell}{(2\pi)^4} \frac{1}{\mathcal{S}} \frac{i(-ig^2)}{\ell^2 - m^2 + i\epsilon} \quad (7.60)$$

where $\mathcal{S} = 1/3$ is the symmetry factor.

The fourth loop gives the same amplitude and with the same sign and a symmetry factor of 1.

The final fermionic loop has an amplitude of

$$\begin{aligned} & \text{fermionic loop} \\ i\mathcal{M} = & \overset{\uparrow}{-2} \int \frac{d^4 \ell}{(2\pi)^4} \frac{1}{\mathcal{S}} (ig)^2 \text{Tr} \frac{i(\not{\ell} + m)}{\ell^2 - m^2 + i\epsilon} \frac{i(\not{p} - \not{\ell} + m)}{(\ell - p)^2 - m^2 + i\epsilon} \end{aligned} \quad (7.61)$$

with a symmetry factor of $\mathcal{S} = 2$ and the factor of $1/2$ in front is to have a proper normalization (we have an extra $1/2$ in the Lagrangian in front of the Dirac Lagrangian).

The trace is

$$\text{Tr} [i(\ell + m)i(\ell - \not{p} + m)] = -4\ell \cdot p + 4\ell^2 + 4m^2 \quad (7.62)$$

$$= 2((\ell - p)^2 - \ell^2 - m^2) + 4\ell^2 + 4m^2 \quad (7.63)$$

$$= 2(p - \ell)^2 + 2\ell^2 + 2m^2 \quad (7.64)$$

$$= 2((p - \ell)^2 - m^2) + 2(\ell^2 - m^2) + 6m^2 \quad (7.65)$$

and so we can write the amplitude as,

$$i\mathcal{M} = -2g^2 \int \frac{d^4\ell}{(2\pi)^4} \left\{ \frac{1}{\ell^2 - m^2 + i\epsilon} + \frac{1}{(\ell - p)^2 - m^2 + i\epsilon} + \frac{3m^2}{((\ell - p)^2 - m^2 + i\epsilon)(\ell^2 - m^2 + i\epsilon)} \right\} \quad (7.66)$$

Adding up the quadratically divergent one loop contributions we have an amplitude of,

$$g^2 \int \frac{d^4\ell}{(2\pi)^4} \left\{ \frac{1}{\ell^2 - m^2 + i\epsilon} - \frac{1}{(\ell - p)^2 - m^2 + i\epsilon} - \frac{3m^2}{((\ell - p)^2 - m^2 + i\epsilon)(\ell^2 - m^2 + i\epsilon)} \right\} \quad (7.67)$$

Notice the the first two contributions cancel! We have lost the quadratic divergence. The diagrams scale logarithmically instead giving natural corrections to the Higgs's mass.

7.1.3 Vector Superfields

In our discussion above we did not mention gauge invariance when we constructed the superpotential. In fact if the chiral superfield we presented above was charged under a gauge group then we didn't write a gauge invariant superpotential. To do so you have to write fields whose charges add to zero, just like in non-supersymmetric QFT.

Thus far our discussion was for chiral scalar fields. We now move onto vector (scalar) superfields. We want to relate the couplings of Φ^i, V, W_α . We first recall what we do in non-SUSY models. We have a kinetic term for instance of the form, $\partial_\mu\phi\partial^\mu\phi^\dagger$ and there was a symmetry, $\phi \rightarrow e^{i\alpha q}\phi$. If $\alpha = \alpha(x)$ then the kinetic term is no longer invariant. We have steps,

(a) We introduce the covariant derivative,

$$D_\mu\phi = \partial_\mu\phi - iqA_\mu\phi \quad (7.68)$$

where A_μ is a field that transforms as

$$A_\mu \rightarrow A_\mu + \partial_\mu\alpha \quad (7.69)$$

Our Lagrangian now takes the form,

$$\mathcal{L} = D_\mu\phi D^\mu\phi^* + \dots \quad (7.70)$$

and this defines how the gauge fields couple to the scalar for instance.

(b) We then add a kinetic term for A_μ . For a $U(1)$ gauge boson,

$$-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \quad (7.71)$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

We now follow these steps for supersymmetry only now instead of promoting global transformations to local transformations we force supersymmetry invariance.

(a) The kinetic terms arise from the Kähler potential, $K = \Phi^\dagger \Phi$, and we want to see how it transforms. This term will not be invariant under the transformation (Λ is some chiral superfield),

$$\Phi \rightarrow e^{iq\Lambda} \Phi \quad (7.72)$$

because

$$\Phi^\dagger \Phi \rightarrow \Phi^\dagger e^{iq(\Lambda - \Lambda^\dagger)} \Phi \quad (7.73)$$

so in general if Λ is a superfield this will not be invariant. The solution is to introduce a gauge field such that the Kähler term is invariant. We introduce V such that

$$K = \Phi^\dagger e^{2qV} \Phi \quad (7.74)$$

where V transforms as,

$$V \rightarrow V - \frac{i}{2}(\Lambda - \Lambda^\dagger) \quad (7.75)$$

With this K is invariant since,

$$K \rightarrow \Phi^\dagger e^{-iq\Lambda^\dagger} e^{2q(V - \frac{i}{2}(\Lambda - \Lambda^\dagger))} e^{iq\Lambda} \Phi = \Phi^\dagger e^{2qV} \Phi \quad (7.76)$$

This is a generalized gauge transformation.

The derivatives in non-SUSY theories tell us how ordinary fields couple to gauge fields. In the same way this prescription will tell us how chiral superfields couple to vector superfields.

(b) We now need to add a kinetic term for V . Recall that W_α is a chiral superfield (not the completely unrelated superpotential!) given by,

$$W_\alpha(y, \theta) = \lambda_\alpha(y) + \theta_\alpha D(y) + (\sigma^{\mu\nu} \theta)_\alpha F_{\mu\nu} - i\theta\theta\sigma_{\alpha\dot{\beta}}^\mu \partial_\mu \bar{\lambda}^{\dot{\beta}} \quad (7.77)$$

$$\mathcal{L}_{kin} = \overset{\text{coupling}}{\uparrow} \tau W^\alpha W_\alpha \Big|_F \quad (7.78)$$

Since W_α is a chiral superfield so is $W^\alpha W_\alpha$. Thus to make this into a Lagrangian we were forced to take the F term of it. We chose this particular term because as

we saw in an earlier lecture W_α has an $F_{\mu\nu}$ term embedded in it. $W^\alpha W_\alpha$ is going to generalize the $F_{\mu\nu}F^{\mu\nu}$ that we had earlier.

If we want this term to be renormalizable then τ needs to be a constant. In a more general case we can have it as a function of the chiral fields,

$$\mathcal{L}_{kin} = [f(\Phi)W^\alpha W_\alpha] \Big|_F + h.c. \quad (7.79)$$

where we have chosen the coefficient to be a function only of a chiral field Φ since we want the Lagrangian to be able to take the F term of the object in square brackets which will only hold if the term is a chiral field¹. $f(\Phi)$ is known as the gauge kinetic function.

- (c) We followed the same steps as we did for non-supersymmetric case. Introduced a gauge field to enforce gauge invariance and then add a kinetic term. However, in the supersymmetric case there is something else that we can add that is allowed by all the symmetries of the theory. This term is known as the Fayet Iliopoulos term:

$$\mathcal{L}_{FI} = \xi V \Big|_D \quad (7.80)$$

where ξ is a constant. This term is only present for $U(1)$ gauge theories since for these theories the gauge field is not charged under $U(1)$ (the photon is chargeless). [Q 8: Doesn't V transform as above?] For non-abelian gauge theories the gauge fields (and their corresponding D terms) would transform under the gauge group and therefore have to be forbidden.

These are all the ingredients of supersymmetric-QED. The renormalizable Lagrangian will be,

$$\mathcal{L} = (\Phi^\dagger e^{2qV} \Phi) \Big|_D + \left(W(\Phi) \Big|_F + h.c. \right) + \left(\tau W^\alpha W_\alpha \Big|_F + h.c. \right) + \xi V \Big|_D \quad (7.81)$$

The superpotential is quite restricted since it must be invariant under supersymmetric and gauge transformations. If we have a single superfield Φ charged under $U(1)$ then we are forced to have $W = 0$ since otherwise it wouldn't be gauge invariant ($W(\phi)$ doesn't involve Φ^\dagger , only powers of Φ).

As we did for the case of chiral superfields we now write this out in terms of components. We do this piece by piece

¹Recall that a function of a chiral superfield is a chiral superfield.

1.

$$\begin{aligned} \Phi^\dagger e^{2qV} \Phi \Big|_D &= \theta^2 \bar{\theta}^2 \frac{1}{4} V_\mu V_\mu (\theta^* \phi) + \left(i \theta^2 \bar{\theta} \bar{\lambda} \sqrt{2} \bar{\theta} (\bar{\psi} \phi) + h.c. \right) \\ &+ \frac{1}{2} \theta^2 \bar{\theta}^2 D (\theta^* \phi) + \bar{\theta}^2 \theta^2 F^* F + 2 (\theta \sigma \bar{\theta}) \bar{\theta} \bar{\psi} \theta \psi V_\mu \Big|_D \end{aligned} \quad (7.82)$$

$$\begin{aligned} &= FF^* + \partial_\mu \phi \partial^\mu \phi^* + i \bar{\psi} \bar{\sigma}^\mu \partial_\mu \psi \\ &+ q V^\mu \left(\frac{1}{2} \bar{\psi} \bar{\sigma}_\mu \psi + \frac{i}{2} \phi^* \partial_\mu \phi - \frac{i}{2} \phi \partial_\mu \phi^* \right) \\ &+ \frac{i}{\sqrt{2}} q (\phi \bar{\lambda} \bar{\psi} - \phi^* \lambda \psi) + \frac{q}{2} \left(D + \frac{1}{2} V_\mu V^\mu \right) |\phi|^2 \end{aligned} \quad (7.83)$$

where to derive this expression we have used the Wess Zumino gauge (though the full expression is gauge invariant so this is a general result). In this gauge V^2 is simple and $V^3 = 0$ so the exponential is cut off at second order and is somewhat manageable. Notice that this Lagrangian is precisely what we expected for QED. We could have written it in a shorter way but we want to be explicit. It has the standard kinetic terms for a scalar and fermion field. This is a gauge theory so it must come out in terms of covariant derivatives. Compare this with,

$$D_\mu \phi D^\mu \phi^* = \left(\partial_\mu - \frac{i}{2} q V_\mu \right) \phi \left(\partial^\mu + \frac{i}{2} q V^\mu \right) \phi^* \quad (7.84)$$

$$= \partial_\mu \phi \partial^\mu \phi^* - i \frac{q}{2} V_\mu (\phi \partial^\mu \phi^* - \partial^\mu \phi \phi^*) + \frac{1}{4} q^2 V_\mu V^\mu |\phi|^2 \quad (7.85)$$

and

$$i \bar{\psi} \bar{\sigma}^\mu D_\mu \psi = i \bar{\psi} \bar{\sigma}^\mu \left(\partial_\mu - \frac{i}{2} q V_\mu \right) \psi \quad (7.86)$$

$$= i \bar{\psi} \bar{\sigma}^\mu \partial_\mu \psi + \frac{q}{2} \bar{\psi} \bar{\sigma}^\mu V_\mu \psi \quad (7.87)$$

so we have,

$$\Phi^\dagger e^{2qV} \Phi \Big|_D = D_\mu \Phi D^\mu \Phi^* + i \bar{\psi} \bar{\sigma}^\mu D_\mu \psi + \overbrace{\frac{i}{\sqrt{2}} q (\phi \bar{\lambda} \bar{\psi} - \phi^* \lambda \psi)}^{\text{New interactions due to SUSY}} + FF^* + \frac{q}{2} D |\phi|^2 \quad (7.88)$$

It is no coincidence that we recovered our old interactions from non-supersymmetry SUSY. We imposed the Lagrangian to be gauge invariant. By doing this we are restricting ourselves to a very small number of possible Lagrangian terms.

2. We considered the special case of one chiral superfield, Φ , and hence $W(\Phi) = 0$. In any case we won't see any new interactions in this case over before since it doesn't involve any gauge field interactions.

3. Next we consider the term index superfield term,

$$W^\alpha W_\alpha \Big|_F + h.c. = \frac{1}{2} D^2 - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + i \lambda \sigma^\mu \partial_\mu \bar{\lambda} - \frac{i}{8} F_{\mu\nu} \tilde{F}^{\mu\nu} \quad (7.89)$$

where,

$$\tilde{F}_{\mu\nu} = \epsilon_{\mu\nu\rho\sigma} F^{\rho\sigma} \quad (7.90)$$

- As we hoped we have a kinetic term for the gauge boson, $\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$.
- We have a Dirac kinetic term for the λ field. We associate these fields with the gauginos.
- The D field only appears on its own and doesn't propagate in spacetime. It is just an auxillary field (analogous to F for the chiral field).
- The last term $\propto F_{\mu\nu} \tilde{F}^{\mu\nu}$ is equal to a total derivative in QED,

$$F_{\mu\nu} \tilde{F}^{\mu\nu} = \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} \quad (7.91)$$

$$= \epsilon^{\mu\nu\rho\sigma} [(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial_\rho A_\sigma - \partial_\sigma A_\rho)] \quad (7.92)$$

$$= \epsilon^{\mu\nu\rho\sigma} [-(\partial_\rho \partial_\mu A_\nu) A_\sigma - (\partial_\nu \partial_\sigma A_\mu) A_\rho + (\partial_\nu \partial_\rho A_\mu) A_\sigma - (\partial_\nu \partial_\sigma A_\mu) A_\rho] \quad (7.93)$$

$$= 0 \quad (7.94)$$

(since $\partial_\alpha \partial_\beta A_\gamma$ is symmetric under $\alpha \leftrightarrow \beta$). We should point out though that while this is true in abelian gauge theory, this doesn't hold in non-abelian gauge theory and this term becomes important in QCD.

4. The last term we need to consider is

$$\xi V \Big|_D = \xi D \quad (7.95)$$

As in the case of the chiral superfield we eliminate the auxillary field. The D dependent part of \mathcal{L} is

$$\mathcal{L}_{(D)} = \frac{q}{2} D |\phi|^2 + \frac{1}{2} D^2 + \xi D \quad (7.96)$$

So the field equation gives,

$$D = -\frac{q}{2} |\phi|^2 - \xi \quad (7.97)$$

We can now substitute this back into the D part of the Lagrangian,

$$\mathcal{L}_{(D)} = \left(\frac{q}{2} |\phi|^2 + \xi \right) D + \frac{1}{2} D^2 \quad (7.98)$$

$$= -\frac{1}{2} \left(\frac{q}{2} |\phi|^2 + \xi \right)^2 \quad (7.99)$$

$$= -\mathcal{V}_{(D)} \quad (7.100)$$

(we denote scalar potential by \mathcal{V} to avoid confusion with the vector superfield) This is a part of the potential of the field ϕ as it has no spacetime derivatives. In general the total scalar potential is

$$\mathcal{V} = \mathcal{V}_{(F)} + \mathcal{V}_{(D)} \quad (7.101)$$

$$= \left| \frac{\partial W}{\partial \phi} \right|^2 + \frac{1}{2} \left(\xi + \frac{1}{2} q |\phi|^2 \right)^2 \quad (7.102)$$

$$\geq 0 \quad (7.103)$$

7.2 Action as a Superspace Integral

The Lagrangian we have written can alternatively be written as an integral over superspace. Recall that without supersymmetry we have,

$$S = \int d^4x \mathcal{L} \quad (7.104)$$

Recall that

$$\int d^2\theta \theta = 1 \quad (7.105)$$

$$\int d^4\theta \theta^2 \bar{\theta}^2 = 1 \quad (7.106)$$

We now want to put the action in a manifestly supersymmetry invariant form. To do that consider the Kähler potential with an integral over superspace,

$$\begin{aligned} \int d^4\theta \Phi^\dagger \Phi &= \left(\phi^\dagger + i\theta\sigma^\mu\bar{\theta}\partial_\mu\phi^\dagger - \frac{1}{4}\bar{\theta}^2\theta^2\partial^2\phi^* + \sqrt{2}\bar{\theta}\bar{\psi} - \frac{i}{\sqrt{2}}\theta\sigma^\mu\partial_\mu\bar{\psi}\bar{\theta}^2 + \bar{\theta}^2 F^* \right) \\ &\times \left(\phi - i\theta\sigma^\mu\partial_\mu\phi - \frac{1}{4}\theta^2\bar{\theta}^2\partial^2\phi + \sqrt{2}\theta\psi + \frac{i}{\sqrt{2}}\theta^2\partial_\mu\psi\sigma^\mu\bar{\theta} + \theta^2 F \right) \end{aligned} \quad (7.107)$$

$$= (F^\dagger F + \partial^\mu\phi^\dagger\partial_\mu\phi + i\bar{\psi}\bar{\sigma}^\mu\partial_\mu\psi) - \frac{1}{4}\partial^2(\phi^\dagger\phi) + \frac{i}{2}\partial_\mu(\bar{\psi}\bar{\sigma}^\mu\psi) \quad (7.108)$$

$$\int d^4x d^4\theta \Phi^\dagger \Phi = \int d^4x K \Big|_D \quad (7.109)$$

where we have thrown away the surface terms. In other words the taking the D term is equivalent to integrating over superspace.

Now consider the superpotential. The superpotential is holomorphic. Here we integrate only over half of superspace,

$$\int d^2\theta W(\Phi) = \int d^2\theta W + \theta^\alpha\partial_\alpha W + \frac{1}{2}\theta^\alpha\theta^\beta\partial_\alpha\partial_\beta W \Big|_{\theta=0} = \int d^2\theta\theta^2\partial^\alpha\partial_\alpha W \Big|_{\theta=0} \quad (7.110)$$

since W is a chiral superfield its form is fixed,

$$\int d^2\theta \partial^\alpha \partial_\alpha W(\Phi) = F_W + \partial^2 \left(-\frac{1}{4} \bar{\theta}^2 \phi_W^* \right) + \partial_\mu \left(\frac{i}{\sqrt{2}} \psi_W \sigma^\mu \bar{\theta} \right) \quad (7.111)$$

$$\int d^4x d^2\theta \partial^\alpha \partial_\alpha W(\Phi) = F_W \quad (7.112)$$

$$= \int d^4x W \Big|_F \quad (7.113)$$

where ϕ_W, ψ_W, F_W are the scalar, fermionic, and F terms of W (which are some combination of the components Φ).

We previously wrote the Lagrangian as

$$\mathcal{L} = K \Big|_D + \left(W \Big|_F + h.c. \right) + \left(W^\alpha W_\alpha \Big|_F + h.c. \right) \quad (7.114)$$

Notice that the D term is precisely the term with $\theta^2 \bar{\theta}^2$ and the F term has θ^2 . So we can write this as

$$\mathcal{L} = \int d^4\theta K + \left(\int d^2\theta W + h.c. \right) + \left(\int d^2\theta W^\alpha W_\alpha + h.c. \right) \quad (7.115)$$

with the understanding that we set $\bar{\theta} = 0$ in the superpotential.

There is a big difference between the Kähler potential and the other two terms since it has an integral over all of superspace while the other two terms have integrals of half of superspace.

7.3 Non-abelian generalization

Thus far we have only discussed abelian gauge theories. We briefly discuss the non-abelian case. In this case V is replaced by $V_a T^a$, where T^a are the group generators in the representation of the chiral fields. This messes up our earlier construction which had,

$$e^{2qV} \rightarrow e^{iq\Lambda^\dagger} e^{2qV} e^{-iq\Lambda} \quad (7.116)$$

which worked well because we had,

$$\Phi \rightarrow e^{iq\Lambda} \Phi \quad (7.117)$$

The natural extension to these ideas to nonabelian gauge theories is $V_a T^a \rightarrow V_a T^a - \frac{i}{2} (\Lambda_a T^a - \Lambda_a^\dagger T^a)$ but these leads to non-gauge invariant Lagrangian since $[T^a, T^b] \neq 0$. There are two things we can do to overcome this: we could either change the form of the transformation of V or we need to change the form $e^{2qV} \rightarrow e^{iq\Lambda^\dagger} e^{2qV} e^{-iq\Lambda}$. Its easier to change the way V transforms. Consider,

$$e^{2qV'} = e^{iq\Lambda^\dagger} e^{2qV} e^{-iq\Lambda} \quad (7.118)$$

$$V' = V - \frac{i}{2} (\Lambda - \Lambda^\dagger) - \frac{iq}{2} [V, \Lambda - \Lambda^\dagger] \quad (7.119)$$

Before we had,

$$W_\alpha = -\frac{1}{q}(\bar{D}\bar{D})\mathcal{D}_\alpha V \quad (7.120)$$

This is extended to

$$W_\alpha^a = -\frac{1}{q}(\bar{D}\bar{D})\mathcal{D}_\alpha (e^{2qV} V e^{-2qV}) \quad (7.121)$$

which gives,

$$W_\alpha^a W_a^\alpha \quad (7.122)$$

for the kinetic term.

The most general action is:

$$\begin{aligned} S & \left[K \left(\Phi_i^\dagger e^{qV} \Phi_i \right), W(\Phi), f(\Phi_i), \xi \right] \\ & = \int d^4x d^4\theta (K + \xi V) + \int d^4x d^2\theta [W + f(\Phi_i) W^\alpha W_\alpha + h.c.] \end{aligned} \quad (7.123)$$

where ξ is only non-zero in $U(1)$. Given K, W, f , and ξ our supersymmetric theory is determined.

7.3.1 General Results for Renormalizable Gauge Theories

We now calculate some general results for gauge theories. To simplify our discussion we set to Fayet-Iliopoulos term to zero and work with a renormalizable supersymmetric field theory. We denote the generators of the gauge symmetry by t_a . The action is given by

$$S = \int d^4x d^4\theta K + \int d^2\theta W + \tau W^\alpha W_\alpha + h.c. \quad (7.124)$$

The Kähler potential is given by,

$$K = \sum_i \Phi_i^\dagger e^{2gt^a V^a} \Phi_i \quad (7.125)$$

where we Φ_i has flavor i and is a vector in terms of its non-abelian charge and t^a are the generators of the adjoint representation. In principle one could have terms with different flavor, $\Phi_i^\dagger e^{2gt^a V^a} \Phi_j$ as long as they have the same quantum numbers, but we conventionally begin with diagonalized the fields such that the Kähler term in diagonal (the same is done in regular QFT where we begin with diagonal kinetic terms). To calculate K explicitly we work in the shifted spatial coordinate y . This gives,

$$\begin{aligned} K & = \left(\phi_i + \sqrt{2}\theta\psi_i + \theta^2 F_i \right) \\ & \quad \left[1 + 2(\theta\bar{\sigma}\bar{\theta})t^a V_\mu^a + 2(i\theta^2\bar{\theta}\bar{\lambda}^a - i\bar{\theta}\theta\lambda^a)t^a + \theta^2\bar{\theta}^2 D^a t^a + \theta^2\bar{\theta}^2 V_\mu^a V^{b,\mu} t^a t^b \right] \\ & \quad \left(\phi_i + \sqrt{2}\theta\psi_i + \theta^2 F_i \right) \end{aligned} \quad (7.126)$$

$$\begin{aligned} & = \theta^2\bar{\theta}^2 V_\mu^a V^{b,\mu} \left[\phi_i^\dagger t^a t^b \phi_i \right] + \left(2i\theta^2\bar{\theta}\bar{\lambda}^a \sqrt{2}\bar{\theta} (\bar{\psi}_i^T t^a \phi_i) + h.c. \right) \\ & \quad \theta^2\bar{\theta}^2 D^a \left(\phi_i^\dagger t^a \phi_i \right) + \bar{\theta}^2\theta^2 \left(F_i^\dagger F_i \right) + 4(\theta\sigma^a\bar{\theta}) \bar{\theta}\bar{\psi}_i^T t^a \theta\psi_i V_\mu^a \end{aligned} \quad (7.127)$$

Using the relations,

$$\bar{\theta}\lambda\bar{\theta}\bar{\psi} = \frac{1}{2}\bar{\theta}^2\lambda\bar{\psi}, \quad \theta\sigma^\sigma\bar{\theta}\bar{\psi}_i\theta\psi_i = \frac{1}{2}\theta^2\bar{\theta}^2\bar{\psi}_i\bar{\sigma}^\mu\psi_i \quad (7.128)$$

we have,

$$\int d^4\theta K = V_\mu^a V^{b,\mu} \left(\phi_i^\dagger t^a t^b \phi_i \right) + \left(\sqrt{2}i\bar{\lambda}^a \bar{\psi}_i^T t^a \phi_i + h.c. \right) + D^a \phi_i^\dagger t^a \phi_i \\ F_i^\dagger F_i + 2\bar{\psi}_i^T \bar{\sigma}^\mu t^a \psi_i V_\mu^a \quad (7.129)$$

When deriving this relationship its important to keep track of all the different indices:

$$\begin{aligned} \alpha &\rightarrow \text{Grassman} \\ \mu &\rightarrow \text{Lorenz} \\ i &\rightarrow \text{flavor} \\ a, b &\rightarrow \text{group indices} \\ \text{none} &\rightarrow \begin{array}{l} \text{non-abelian charge} \\ \text{(suppressed)} \end{array} \end{aligned}$$

In particular its important to keep in mind that each term has a contraction of vectors with color indices. For example,

$$\bar{\psi}_i^\dagger \bar{\sigma}^\mu t^a \psi_i V_\mu^a = \bar{\psi}_{i,m}^* \bar{\sigma}^\mu t_{m,n}^a \psi_{j,n} V_\mu^a \quad (7.130)$$

We now move onto the vector kinetic terms,

$$W^{a,\alpha} W_\alpha^a = \left(-i\lambda^{a,\alpha} + \theta^\alpha D^a - \frac{i}{2} (\sigma^\mu \bar{\sigma}^\nu \theta)^\alpha F_{\mu\nu}^a + \theta^2 \sigma^{\mu,\alpha}{}_\beta \partial_\mu \bar{\lambda}^{a,\beta} \right) \quad (7.131)$$

$$\left(-i\lambda_\alpha^a + \theta_\alpha D^a - \frac{i}{2} (\sigma^\rho \bar{\sigma}^\sigma \theta)_\alpha F_{\rho\sigma}^a + \theta^2 \sigma_{\alpha\dot{\gamma}}^\rho \partial_\rho \bar{\lambda}^{a,\dot{\gamma}} \right) \quad (7.132)$$

$$= -i\theta^2 \lambda^a \sigma^\rho \partial_\rho \bar{\lambda}^a + \theta^2 D^a D^a + i\theta^2 \lambda_\alpha^a \sigma^{\mu\alpha}{}_\beta \partial_\mu \bar{\lambda}^{a,\beta} \\ - \frac{1}{4} (\sigma^\mu \bar{\sigma}^\nu \theta)^\alpha (\sigma^\rho \bar{\sigma}^\sigma \theta)_\alpha F_{\mu\nu}^a F_{\rho\sigma}^a \quad (7.133)$$

This can simplified using,

$$(\sigma^\mu \bar{\sigma}^\nu \theta)^\alpha (\sigma^\rho \bar{\sigma}^\sigma \theta)_\alpha = \theta^2 (g^{\mu\nu} g^{\sigma\rho} - g^{\mu\sigma} g^{\nu\rho} + g^{\mu\rho} g^{\nu\sigma} + i\epsilon^{\mu\nu\sigma\rho}) \quad (7.134)$$

Thus we have,

$$\int d^2\theta W^{a,\alpha} W_\alpha^a = -2i\lambda^a \not{\partial} \bar{\lambda}^a + D^a D^a - \frac{1}{2} F_{\mu\nu}^a F^{a,\mu\nu} + \frac{i}{4} \tilde{F}_{a,\mu\nu} F^{a,\mu\nu} \quad (7.135)$$

where it is understood that $\not{\partial}$ has a Pauli matrix with the correct index structure.

Adding in the coupling and the hermitian conjugate we have,

$$\frac{1}{4} \int d^4\theta W^{a,\alpha} W_\alpha^a + h.c. = -\lambda^a i \not{\partial} \bar{\lambda}^a + \frac{1}{2} D^a D^a - \frac{1}{4} F_{\mu\nu}^a F^{a,\mu\nu} \quad (7.136)$$

where we have dropped $\tilde{F}F$ since its a total derivative ² and used the fact the the rest of the Lagrangian is real (we will soon see explitley that D is real).

We are now in a position to find the contribution to the scalar potential from the Kähler and vector kinetic terms (also known as the D terms). We have,

$$V_D \equiv -\mathcal{L}_D = -D^a \phi_i^\dagger t^a \phi_i - \frac{1}{2} D^a D^a \quad (7.137)$$

These are the only terms in which D appears. Thus we can go ahead and integrate it out,

$$\frac{\partial \mathcal{L}}{\partial D^a} = \phi_i^\dagger t^a \phi_i + D^a = 0 \quad (7.138)$$

$$\Rightarrow D^a = -\phi_i^\dagger t^a \phi_i \quad (7.139)$$

which is real since the generators in the adjoint representation are real.

From the above we see that

$$V_D = \frac{1}{2} D^a D^a \quad (7.140)$$

The renormalizable superpotential has more freedom then the Kähler potential so its slightly more difficult to find it's associated scalar potential. We can write the most general superpotential as,

$$W[\Phi_i] = \sum_j g_j \prod_i \Phi_i^{n_{j,i}} = \sum_j g_j \Phi_1^{n_{j,1}} \Phi^{n_{j,2}} \dots \quad (7.141)$$

where $n_{j,i}$ denotes how many powers of field i are in term j .

We now find the scalar potential,

$$\int d^2\theta W + h.c. = \sum_j g_j \Phi_1^{n_{j,1}} \Phi_2^{n_{j,2}} \Phi_3^{n_{j,3}} \dots + h.c. \quad (7.142)$$

$$= \sum_j g_j \left\{ F_1 n_{j,1} \phi_1^{n_{j,1}-1} (\phi_2^{n_{j,2}} \phi_3^{n_{j,3}} \dots) \right. \\ \left. + F_2 n_{j,2} \phi_2^{n_{j,2}-1} (\phi_1^{n_{j,1}} \phi_3^{n_{j,3}} \dots) + \dots \right\} + h.c. \quad (7.143)$$

$$= \sum_j g_j \sum_i F_i n_{j,i} \phi_i^{n_{j,i}-1} \prod_{k \neq i} \phi_k^{n_{j,k}} + h.c. \quad (7.144)$$

²There are subtleties associated with this as there can be important instanton effects. This point should be revisited in the future.

The only other part which contributes to the scalar potential terms arises from the Kähler term. The scalar potential is then given by,

$$V_F = -\mathcal{L}_F = -\sum_i F_i^\dagger F_i + g_j \sum_i F_i n_{j,i} \phi_i^{n_{j,i}-1} \prod_{k \neq i} \phi_k^{n_{j,k}} + h.c. \quad (7.145)$$

These are the only places where F appears. Its now easy to integrate it out:

$$F_i^\dagger = -\sum_j g_j n_{j,i} \phi_i^{n_{j,i}-1} \prod_{k \neq i} \phi_k^{n_{j,k}} \quad (7.146)$$

Explicitly taking the derivative of Eq.7.141 gives,

$$\frac{\partial W}{\partial \phi_i} = \sum_j g_j n_{j,i} \phi_i^{n_{j,i}-1} \prod_{k \neq i} \phi_k^{n_{j,k}} \quad (7.147)$$

Therefore we can write,

$$F_i^\dagger = -\frac{\partial W}{\partial \phi_i} \quad (7.148)$$

which gives a scalar potential,

$$V_F = \sum_i F_i^\dagger F_i = \sum_i \left| \frac{\partial W}{\partial \phi_i} \right|^2 \quad (7.149)$$

7.4 R Symmetry

We now further investigate a topic that was mentioned in passing earlier. A Lagrangian can have many different symmetries, but one particularly interesting one in the context of SUSY is R symmetry. Recall that this is the name we give for any internal symmetry that does not commute with supersymmetry, but can coexist with SUSY due to phase invariance of the SUSY commutations relations,

$$\{Q_\alpha, Q_\beta\} = \{\bar{Q}_{\dot{\alpha}}, \bar{Q}_{\dot{\beta}}\} = 0 \quad (7.150)$$

$$\{Q_\alpha, \bar{Q}_{\dot{\beta}}\} = 2P_\mu \sigma_{\alpha\dot{\beta}}^\mu \quad (7.151)$$

$$[Q_\alpha, R] \neq 0 \quad (7.152)$$

This means that you can have a SUSY invariant theory but still have an additional symmetry which differentiates between bosons and fermions (since it doesn't need to commute with Q_α). We call this phase symmetry, $U(1)_R$.

To see this explicitly consider the effect of an R symmetry on Q_α :

$$e^{iR\phi} Q_\alpha e^{-iR\phi} = e^{-i\phi} Q_\alpha \quad (7.153)$$

$$(1 + iR\phi - \dots) Q_\alpha (1 - iR\phi - \dots) = -i\phi Q_\alpha \quad (7.154)$$

$$i [Q_\alpha, R] \phi = i\phi Q_\alpha \quad (7.155)$$

$$[Q_\alpha, R] = Q_\alpha \quad (7.156)$$

Thus this phase shift symmetry implies that the commutation between the R symmetry generator and Q_α is nontrivial and hence bosons and fermions can have a different R charge.

This symmetry acts on superspace and has a peculiar property that the Grassman variables are charged under the symmetry. To see this consider the commutator acting on a bosonic state (we denote the $U(1)_R$ charge of particle i by r_i):

$$[Q_\epsilon, R] |\phi\rangle = r_\phi Q_\epsilon |\phi\rangle - R\epsilon |\psi\rangle \quad (7.157)$$

$$= (r_\phi - r_\psi) Q_\epsilon |\phi\rangle \quad (7.158)$$

For this to equal to $+Q_\epsilon |\phi\rangle$ we must have

$$r_\psi = r_\phi - 1 \quad (7.159)$$

So the $U(1)_R$ charge of a fermion is 1 less than the bosonic $U(1)_R$ charge. In order for a superfield,

$$S(x, \theta, \bar{\theta}) = \phi(x) + \sqrt{2}\theta\psi(x) + \dots \quad (7.160)$$

to have a well defined R charge we then must have $U(1)_R$ charge of θ of 1 and -1 for $\bar{\theta}$.

7.5 Non-Renormalization Theorems

We know that the whole Lagrangian will depend on Kähler potential, super potential, gauge kinetic function and Fayet Iliopoulos term (K, W, f, ξ). We'd like to see how they behave under quantum corrections. The claim is that

- $K(\Phi, \Phi^\dagger)$ gets corrections order by order in perturbation theory
- $f(\Phi)$ only gets corrections are one loop order
- $W(\Phi)$ is nonrenormalizable at any order in perturbation theory. This doesn't mean that it doesn't get corrected. There are corrections but they are non perturbative.
- ξ also doesn't get corrected (though this is not as peculiar since ξ has no field dependence)

This was first proved in 1977 by Grisaru, Rocek, Seigel using what's known as "supergraphs". These are generalization of Feynman diagrams to superfields. The result that they found was that quantum corrections only come as

$$\int d^4x d^4\theta \{...\} \quad (7.161)$$

except the one loop corrections of f (remember that except for the Kähler potential every term only has integrals over half of superspace). In 1993 Seiberg (based on string theory

arguments from Witten) was able to prove these results using symmetry arguments and holomorphicity. We use the second technique to prove the results³.

We can write the action explicitly by,

$$S = \int d^4x d^4\theta [K(\Phi, \Phi^\dagger, e^V) + \xi V_{U(1)}] + \int d^4x d^2\theta [W(\Phi) + f(\Phi) W^\alpha W_\alpha + h.c.] \quad (7.162)$$

where we write $V_{U(1)}$ to remember that we only include this extra term in the special case of a $U(1)$ symmetry.

We introduce two chiral superfields fields called “spurion” fields, X and Y each with components,

$$X = (x, \psi_x, F_x) \quad , \quad Y = (y, \psi_y, F_y) \quad (7.163)$$

We consider these fields to be very heavy and hence don't propagate. The only effect they have is through their VEVs. Having a theory with these “background” fields should be completely equivalent to having explicit parameters in the Lagrangian at low enough energies.

This mimics a phenomena that's seen in string theory, where there are usually no arbitrary parameters and all parameters arise as expectation values of fields.

With this in mind we can write the action,

$$S = \int d^4x d^4\theta (K + \xi V_{U(1)}) + \int d^4x \int d^2\theta (Y W(\Phi) + X W^\alpha W_\alpha + h.c.) \quad (7.164)$$

X is going to mimic the gauge coupling and Y is substituting the Yukawa couplings and self-couplings of the matter fields by self-couplings of Y . If we set $Y = 0$ then we are killing all couplings of the matter fields. Note that X and Y don't change the terms in the Lagrangian. W and $W^\alpha W_\alpha$ are still unspecified, no more constrained functions. They just have a heavy field in front. For our actual theory the VEV's of X and Y are specified but we consider the more general theory in which we let them be variables.

To prove the non-renormalization theorems we use

- (i) Symmetries
- (ii) Holomorphicity
- (iii) Limits of $\langle \phi_X \rangle \rightarrow \infty$ and $\langle \phi_Y \rangle \rightarrow 0$.

Symmetries

In our Lagrangian we have a few symmetries:

1. Gauge symmetries
2. Supersymmetry

³This argument follows Wienberg 27.8.

3. R Symmetry

In our case, we assign the charges for the fields as shown below:

Field	Φ	V	X	Y	θ	$\bar{\theta}$	W^α
$U(1)_R$	0	0	0	-2	1	-1	1

The reasoning as follows. Suppose Y has charge -2 (we assign this arbitrarily). Then $W[\Phi]$ trivially has an R charge of 0 ($d^2\theta$ has R charge of 2 since integrals in Grassman space take derivatives of θ 's).

W^α is not an independent superfield but comes from V (recall that $W^\alpha \sim \bar{\mathcal{D}}^2 \mathcal{D}^\alpha V$). V must have an R charge of 0 since otherwise the Kähler potential couldn't exist. Each \mathcal{D} behaves like a θ . So the charge for W^α is $2 - 1 + 0 = 1$.

Note that the Kähler and Fayet-Iliopoulos terms are trivially invariant,

$$\int d^4x d^4\theta [K + \xi V_{U(1)}] \quad (7.165)$$

is invariant under $U(1)_R$ transformations.

W^α has charge 1. So the charge of $d^2\theta X W^\alpha W_\alpha$ is $-2 + 1 + 1 = 0$. Thus this is a symmetry of our Lagrangian.

This symmetry did not exist before. However, by adding the spurion fields X and Y we added these symmetries. At energies well below the masses of the introducing spurions (which we take to infinity) the physics of this “fictitious theory” and the physical theory will be identical.

4. We also have another symmetry which is called Peccei-Quinn(PQ) symmetry. Every field is a singlet under this symmetry except X which transforms as,

$$X \rightarrow X + i\alpha \quad (7.166)$$

where $\alpha \in \mathbb{R}$. To see that this is a symmetry consider the vector kinetic term,

$$XW^\alpha W_\alpha = \left(\phi_X + \sqrt{2}\theta\psi + \dots \right) \left(-i\lambda + \theta D - \frac{i}{2}(\sigma^\mu \bar{\sigma}^\nu \theta) F_{\mu\nu} + \theta^2(\sigma^\mu \partial_\mu \bar{\lambda}) \right)^\alpha \quad (7.167)$$

$$\times \left(-i\lambda + \theta D - \frac{i}{2}(\sigma^\mu \bar{\sigma}^\nu \theta) F_{\mu\nu} + \theta^2(\sigma^\nu \partial_\nu \bar{\lambda}) \right)_\alpha + h.c. \quad (7.168)$$

$$= \phi_X \left(-2i\lambda \not{\partial} \bar{\lambda} + D^2 - \int d^2\theta i\theta \sigma^\mu \sigma^\nu \theta F_{\mu\nu} D - \frac{1}{4} \theta \theta \sigma^\mu \bar{\sigma}^\nu \sigma^\alpha \bar{\sigma}^\beta \theta F_{\mu\nu} F_{\alpha\beta} \right) + \dots \quad (7.169)$$

We focus on the terms proportional to ϕ_X . Taking the $d^2\theta$ integral its easy to see that the $F_{\mu\nu} D$ cross-terms vanish. Furthermore the $\lambda i \partial_\mu \sigma^\mu \lambda$ term is real (its a kinetic term) so its contribution is of the form,

$$\phi_X 2\lambda i \partial_\mu \sigma^\mu \lambda + h.c. = \text{Re}(\phi_X) 2\lambda i \partial_\mu \sigma^\mu \lambda \quad (7.170)$$

This is independent of the imaginary part of ϕ_X and so is invariant under PQ.

The D^2 term has the same property since the only other D term contribution comes from,

$$\Phi^\dagger e^{qV} \Phi = (\phi^* + \dots) (\theta^2 \bar{\theta}^2 D + \dots) (\phi + \dots) = D |\phi|^2 \quad (7.171)$$

which gives a D term equation of motion,

$$\frac{\partial \mathcal{L}}{\partial D} = |\phi|^2 + 2\phi_X D + h.c. = 0 \quad (7.172)$$

Since only the real part of ϕ_X gets a VEV,

$$D = \frac{|\phi|^2}{4 \langle \phi_X \rangle} \in \mathbb{R} \quad (7.173)$$

Since D is real the term $\phi_X D + h.c.$ is independent of the imaginary part of ϕ_X .

There is only one more term that could potentially transform under PQ,

$$\Delta \mathcal{L} = \int d^2\theta \phi_X (\sigma^\mu \bar{\sigma}^\nu \theta) (\sigma^\omega \bar{\sigma}^\tau \theta) F_{\mu\nu} F_{\omega\tau} + h.c. \quad (7.174)$$

$$= \phi_X \text{Tr} [\sigma^\mu \bar{\sigma}^\nu \sigma^\omega \bar{\sigma}^\tau] F_{\mu\nu} F_{\omega\tau} + h.c. \quad (7.175)$$

$$= \phi_X (g^{\mu\nu} g^{\omega\tau} + g^{\mu\omega} g^{\nu\tau} + g^{\mu\tau} g^{\omega\nu} + i\epsilon^{\mu\nu\omega\tau}) F_{\mu\nu} F_{\omega\tau} + h.c. \quad (7.176)$$

$$\sim \phi_X \left(F_{\mu\nu} F^{\mu\nu} + i\tilde{F}_{\mu\nu} F^{\mu\nu} \right) + h.c. \quad (7.177)$$

$$\sim \text{Re}(\phi_X) F^2 + \text{Im}(\phi_X) \tilde{F}F \quad (7.178)$$

where $\tilde{F}_{\mu\nu} \equiv \epsilon_{\mu\nu\rho\tau} F^{\rho\tau}$. But for $U(1)$ $\tilde{F}F$ is a total derivative. So even though it does depend on the imaginary part of ϕ_X , it doesn't contribute to the Lagrangian. ϕ_X is called an axion field.

Holomorphicity

Consider the quantum-correction action (Wilsonian action). The action associated with integrating all momentum greater than some cutoff in the path integral, such that,

$$\exp(S_\lambda) = \int_{|p|>\lambda} \int \mathcal{D}\phi \exp(iS) \quad (7.179)$$

We have,

$$\begin{aligned} S_\lambda = & \int d^4x \int d^4\theta [J(\Phi, \Phi^\dagger, e^V, X, Y, \mathcal{D}\Phi, \dots) + \xi(X, X^\dagger, Y, Y^\dagger) V_{U(1)}] \\ & + \int d^4x \int d^2\theta [H(\Phi, X, Y, W^\alpha)] + h.c. \end{aligned} \quad (7.180)$$

where J and Ξ , are arbitrary functions, and H is holomorphic. The $U(1)_R$ invariance will help us constrain terms in the Lagrangian.

For $J(\Phi, \Phi^\dagger, e^V, X, Y, \mathcal{D}, \dots)$ the only thing we know is that Y appears together with Y^\dagger to any possible power to preserve $U(1)_R$. Its almost completely arbitrary function.

The situation is quite different for H . The only objects that transform under R -symmetry are Y (R charge of 2) and W^α (R charge of 1). The charge of H must be -2 (since $d^2\theta$ has charge 2). This restricts H to be of the form,

$$H = Yh(X, \Phi) + g(X, \Phi)W^\alpha W_\alpha \tag{7.181}$$

Under PQ symmetry only X transforms and it transforms as a shift in the imaginary component. This means that any power of X greater than 2 will not be invariant under this symmetry! We must have $h(X, \Phi) = h(\Phi)$. Terms linear in X are allowed as the shift only contributes to a total derivative. Thus we have $g(X, \Phi) = g(\Phi) + \alpha X$ where α is a constant (it must be a constant to not spoil the PQ symmetry [Q 9: check this]). We have,

$$H = Yh(\Phi) + (\alpha X + g(\Phi))W^\alpha W_\alpha \tag{7.182}$$

Notice that we have significantly reduced the possible dependencies. This is starting to look very similar to the Lagrangian we started with.

Limits

Now we can use the last tool in our repertoire which is the limits, $\langle\phi_X\rangle \rightarrow \infty$ and $\langle\phi_Y\rangle \rightarrow 0$. In the limit that $\langle\phi_Y\rangle \rightarrow 0$ the quantum corrected superpotential must be equal to the tree level superpotential up to corrections of $1/M_{X,Y}$ (any loops will have more couplings). This sets,

$$YW(\Phi) = Yh(\Phi) \Rightarrow W(\Phi) = h(\Phi) \tag{7.183}$$

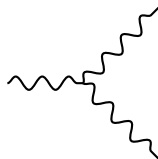
for large $\langle\phi_Y\rangle$. But $h(\Phi)$ is independent of Y ! So it must be equal to $W(\Phi)$ for all Y (including the one which corresponds to our theory $\langle\phi_Y\rangle = 1$). This means that W is never renormalized by higher order corrections. The superpotential at tree level is exact! Any loop diagrams will always cancel at higher orders.

Similar observations can be done for X . Due to holomorphy, as for W the gauge coupling can only be of the form,

$$f(\Phi) = \alpha X + g(\Phi) \tag{7.184}$$

Note that the gauge kinetic terms are $\sim \langle\phi_X\rangle F_{\mu\nu}F^{\mu\nu} \sim \langle\phi_X\rangle \partial^{[\mu}A^{\nu]}\partial_{[\mu}A_{\nu]}$. Each ∂_μ gets Fourier transformed into a $1/\langle\phi_X\rangle$ and thus the propagator is proportional to $1/\langle\phi_X\rangle$.

Gauge self couplings (which arise from the same term),



contribute an $\langle \phi_X \rangle$. Every other term doesn't have any $\langle \phi_X \rangle$ contribution.

We could count the number of $\langle \phi_X \rangle$ powers in any diagram with. It is given by

$$N_{\langle \phi_X \rangle} = V_W - I_W \tag{7.185}$$

where I_W is the number of X propagators. We also know that the number of loops, L , in a gauge theory is

$$L = I_W - V_W + 1 \tag{7.186}$$

Combining these results we have

$$N_{\langle \phi_X \rangle} = 1 - L \tag{7.187}$$

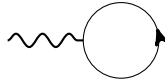
The amplitude can't diverge in the limit that $\langle \phi_X \rangle \rightarrow 0$ since this is unphysical.

For $L = 0$ (tree level), $N_{\langle \phi_X \rangle} = 1$. The power of X is equal to one and hence αX is the tree level result ($\alpha = 1$). For $L = 1$ we have $N_{\langle \phi_X \rangle} = 0$ which gives the term $g(\Phi)$. In the renormalizable case we only have Φ and no Φ^\dagger . which one can show leads to $g(\Phi)$ being a constant.

The Kähler potential, being non-holomorphic, is corrected to all orders. In general we also have a FI-term,

$$(\xi (X, X^\dagger, Y, Y^\dagger) V_{U(1)}) \Big|_D \tag{7.188}$$

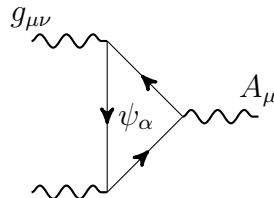
Gauge invariance: $V \rightarrow \frac{i}{2} (\Lambda - \Lambda^\dagger)$ implies that ξ must be a constant. Any dependence on the chiral superfields will destroy this invariance. However, this is not necessarily the same constant that we started with. ξ can only get contributions from diagrams known as tadpole diagrams,



But these diagrams happen to be proportional to all the charges,

$$\sum_i q_i = \text{Tr} (Q_{U(1)}) \tag{7.189}$$

But if $\text{Tr} Q \neq 0$ the theory is inconsistent in the sense that it has what is called “gravitational anomalies”. These occur when you have couplings of the type,



This diagram is proportional to $\text{Tr} (Q)$. If this is different from zero then it tells you that you are breaking gauge invariance. If there are no gravitational anomalies then the FI term is not corrected.

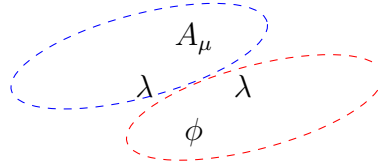
7.6 $\mathcal{N} = 2, 4$ Global SUSY

Recall that for $\mathcal{N} = 1$ the action is dependent on three arbitrary functions and one parameter,

$$S = S[K, W, f; \xi] \quad (7.190)$$

The most general Lagrangian has $\mathcal{N} = 0$ as in the SM. $\mathcal{N} = 1$ is just a special case of $\mathcal{N} = 0$. On the same note extended SUSY theories ($\mathcal{N} > 1$) are special cases of $\mathcal{N} = 1$ models. Thus have a larger \mathcal{N} puts restrictions on the dependencies above.

For $\mathcal{N} = 2$ vector multiplet,



where the states in blue are a $\mathcal{N} = 1$ vector multiplet (so it comes from an $\mathcal{N} = 1$ vector superfield, V) and the states in red come from a chiral $\mathcal{N} = 1$ chiral superfield, Φ .

It so happens that this particular case is the simplest. You can write the $\mathcal{N} = 2$ action in terms of $\mathcal{N} = 1$ actions. It turns out that $W = 0$, and K, f can be written in terms of a single holomorphic function, $\mathcal{F}(\Phi)$. This function is called the prepotential. This is good news since in $\mathcal{N} = 1$ we had no information about K . However, now K will be derived from a holomorphic function, \mathcal{F} . In particular,

$$f(\Phi) = \frac{d^2 \mathcal{F}}{d\Phi^2} \quad (7.191)$$

$$K(\Phi, \Phi^\dagger) = \frac{1}{2i} \left[\Phi^\dagger e^{2V} \frac{d\mathcal{F}}{d\Phi} - h.c. \right] \quad (7.192)$$

Full perturbative action is just given in one loop,

$$\mathcal{F} = \begin{cases} \Phi^2 \\ \Phi^2 \log \frac{\Phi^2}{\Lambda^2} \end{cases} \quad (7.193)$$

where Λ denotes some cutoff. However, we could have some non-perturbative corrections. So in general we have,

$$\mathcal{F}(\Phi) = \mathcal{F}_{1 \text{ loop}} + \mathcal{F}_{\text{non-pert}} \quad (7.194)$$

Perturbative means that $\mathcal{F}_{1 \text{ loop}}$ will be a series in the powers of the couplings $\sum_n g^n a_n$. Non-perturbative means that something cannot be written in this form. For example $\sum A_n e^{-a_n/g^2}$ since we know this function and all its derivatives vanish at zero. You cannot get any information about \mathcal{F} from $\mathcal{F}_{\text{non-pert}}$ since the Taylor expansion vanishes at zero. Non-perturbative effects can be important and in general need to be understood.

There are also vector and hypermultiplets. These are in general more complicated.

We now consider $\mathcal{N} = 4$. We can write the vector multiplet as,

$$\underbrace{\begin{pmatrix} & A_\mu & \\ xl & & \psi_1 \\ & \phi_1 & \end{pmatrix}}_{\mathcal{N}=2 \text{ vector}} + \begin{pmatrix} & \phi_2 & \\ \psi_3 & & \psi_2 \\ & \phi_3 & \end{pmatrix}_{\mathcal{N}=2 \text{ hyper}}$$

The effective action does not depend on any arbitrary functions. There is only one free parameter,

$$f = \tau = \frac{\Theta}{2\pi} + \frac{4\pi i}{g^2} \quad (7.195)$$

This means that $\text{Re}\tau$ multiplies $F_{\mu\nu}\tilde{F}^{\mu\nu}$ and $\text{Im}\tau$ is the coefficient of $F^{\mu\nu}F_{\mu\nu}$. This is the only free parameter in a $\mathcal{N} = 4$ theory.

Notice,

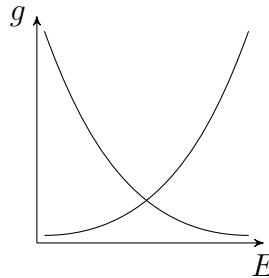
- $\mathcal{N} = r$ is not only renormalizable but finite
- The β function follows,

$$\beta(g) = M^2 \frac{dg}{M^2} \quad (7.196)$$

where M^2 is the renormalization scale. For $\mathcal{N} = 1$ the β function is

$$bg^3 \quad (7.197)$$

Depending on the value of b the coupling increases or decreases,



For $\mathcal{N} = 4$ $\beta = 0$ and the coupling does not change with energy. This means that we have what's called a conformal field theory. This theory is very well behaved.

- There is something called S-duality which essentially takes

$$\tau \rightarrow \frac{a\tau + b}{c\tau + d} \quad (7.198)$$

where $a, b, c, d \in \mathbb{Z}$ form what is called a $SL(2, \mathbb{Z})$ group. For particular choices of the coefficients,

$$\tau \rightarrow \frac{1}{\tau} \quad (7.199)$$

Remember that the imaginary part of τ is the gauge coupling so your changing the coupling from weak to strong. This is one of the beautiful properties of these theories; you can control the strong coupling from the strong coupling.

- $\mathcal{N} = 4$ has an AdS/CFG duality to a string theory in 5 dimensions.

7.7 Supergravity

We know that in gauge theories that if we set a field ϕ such that

$$\phi \rightarrow e^{i\alpha} \phi \quad (7.200)$$

and α goes from a constant to $\alpha(x)$ then we get a gauge theory. In SUSY we know that

$$\delta S = i (\epsilon \mathcal{Q} + \bar{\epsilon} \bar{\mathcal{Q}}) S \quad (7.201)$$

What happens if we make ϵ a function of spacetime coordinates? SUSY is a part of a whole spacetime symmetry. If we make ϵ a function of x then we also change the structure of spacetime. If we make Poincare local we get gravity. If we make SUSY local then we get supergravity.

Gravity is determined by a corresponding metric which can be written in terms of tetrad fields, e_μ^a which are connected to the metric through

$$g_{\mu\nu} = e_\mu^a e_\nu^b \eta_{ab} \quad (7.202)$$

In supergravity we have a field, ψ_μ , called the gravitino (one gravity for each supersymmetry). In this sense gravitino is the gauge field of supergravity.

The supergravity action is

$$S_{SG} = \int d^4x \sqrt{g} \{ R + (\psi_\mu \sigma_\nu D_\rho \bar{\psi}_\sigma - \bar{\psi}_\mu \sigma_\nu D_\rho \psi_\sigma) \epsilon^{\mu\nu\rho\sigma} + \dots \} \quad (7.203)$$

where R is the Einstein-Hilbert term and the second term is called a Rarita-Schwinger term.

The total action will be

$$S = S_{SG} + S[K, W, f; \xi] \quad (7.204)$$

This is similar as global supersymmetry except there are extra symmetries:

- Kähler symmetry:

$$K \rightarrow K(\phi, \phi^\dagger) + h(\phi) + h^*(\phi^*) \quad (7.205)$$

$$W \rightarrow e^{h(\phi)} W \quad (7.206)$$

S depends on the determinate of the metric,

$$g = K - \log |W|^2 \quad (7.207)$$

- The F-term potential is:

$$V_{F-term} = e^{M/M_p^2} \left\{ K_{ij^*}^{-1} D_i W D_{j^*} W^* - 3 \frac{|W|^2}{M_p^4} \right\} \quad (7.208)$$

where

$$D_i W = \partial_i W + (\partial_i K) \frac{W}{M_p^2} \quad (7.209)$$

This is called the Kähler covariant derivative.

Notice that when we send $M_p \rightarrow \infty$ we reduce to the standard supersymmetric case. Further notice that V_{F-term} is not longer positive as it was in supersymmetry.

Chapter 8

Supersymmetry Breaking

8.1 Basics

Recall that when we talk about symmetry breaking in gauge theories we have a field ϕ that transforms as,

$$\phi \rightarrow (e^{i\alpha^a T^a})_i^j \phi_j \quad (8.1)$$

such that

$$\delta\phi_i = i\alpha^a (T^a)_i^j \phi_j \quad (8.2)$$

The symmetry is broken if $\delta\phi_i \neq 0$ for the vacuum state. In other words in the vacuum the field is not invariant under the transformation. Another way to say it is

$$(\alpha^a T^a)_i^j \phi_{j(vac)} \neq 0 \quad (8.3)$$

The typical example for the $U(1)$ case is for

$$\phi = \rho e^{i\theta} \quad (8.4)$$

then under a transformation $e^{i\alpha}$,

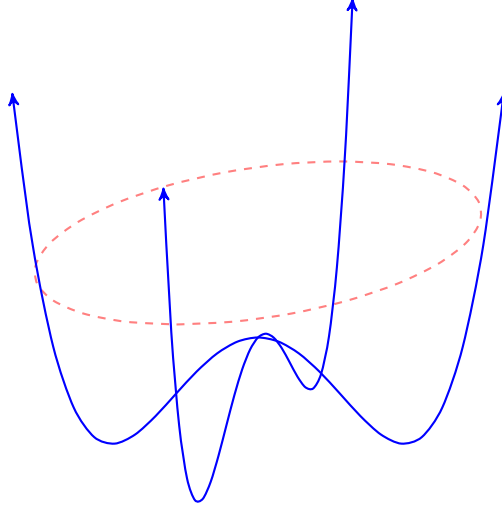
$$\phi' = \rho e^{i\alpha} e^{i\theta} \quad (8.5)$$

which gives

$$\delta\rho = 0 \quad (8.6)$$

$$\delta\theta = \alpha \quad (8.7)$$

Now suppose you have a potential of the form,



Any transformation changes the angle θ but leaves the modulus invariant. Taking the state from vacua to vacua. The field α is the Goldstone boson.

We now move onto supersymmetry. Supersymmetry is broken if the vacuum state ($|\text{vac}\rangle$) is transformed under supersymmetry, i.e.

$$Q_\alpha |\text{vac}\rangle \neq 0 \quad (8.8)$$

Let's consider,

$$\{Q_\alpha, \bar{Q}_{\dot{\beta}}\} = 2(\sigma^\mu)_{\alpha\dot{\beta}} P_\mu \quad (8.9)$$

Suppose we multiply both sides by $(\bar{\sigma}^\nu)^{\dot{\beta}\alpha}$:

$$(\bar{\sigma}^\nu)^{\dot{\beta}\alpha} \{Q_\alpha, \bar{Q}_{\dot{\beta}}\} = 2(\bar{\sigma}^\nu)^{\dot{\beta}\alpha} (\sigma^\mu)_{\alpha\dot{\beta}} P_\mu \quad (8.10)$$

$$= 4P^\nu \quad (8.11)$$

Now consider $\nu = 0$. This implies that we have,

$$\{Q^\alpha, Q^\dagger_\alpha\} = 4P^0 \rightarrow 4E \quad (8.12)$$

where we have defined $Q^{\alpha,\dagger} \equiv (\bar{\sigma}^0)^{\alpha\dot{\beta}} \bar{Q}_{\dot{\beta}}$.

Note,

1. The left hand side of our expression is positive definite. Acting with this operator on any state says that that the energy of that state must be positive. This is a result we also saw earlier where we found the scalar potential was positive.
2. We can sandwich this expression between the vacuum state:

$$\langle \text{vac} | Q^\alpha Q^\dagger_\alpha + Q^{\dagger,\alpha} Q_\alpha | \text{vac} \rangle \geq 0 \quad (8.13)$$

If supersymmetry is broken then we have $Q_\alpha |\text{vac}\rangle \neq 0$ and the energy is zero. Thus SUSY being broken is equivalent to having energy of the vacuum greater than zero,

$$\text{SUSY} \Leftrightarrow E > 0 \quad (8.14)$$

The energy is the order parameter of supersymmetry breaking in the same way that the vev was order parameter of gauge symmetry breaking.

8.2 F and D Breaking

8.2.1 F -Term

Suppose we have a chiral superfield. We know that the transformation of the supersymmetry of each of the components is of the form,

$$\delta\phi = \sqrt{2}\epsilon\psi \quad (8.15)$$

$$\delta\psi = \sqrt{2}\epsilon F + \sqrt{2}i\sigma^\mu\bar{\epsilon}\partial_\mu\phi \quad (8.16)$$

$$\delta F = i\sqrt{2}\bar{\epsilon}\bar{\sigma}^\mu\partial_\mu\psi \quad (8.17)$$

We want to look on the right hand side if any terms are different from zero for the vacuum then we know we have supersymmetry breaking. As we know a scalar can get a vacuum expectation value. However, a fermion could not get a VEV as then under Lorentz transforms the VEV itself would also transform, breaking Lorentz invariance. So you cannot have ψ getting a VEV. We cannot have $\delta\phi \neq 0$ on the vacuum as this would require,

$$\langle vac|\psi|vac\rangle \neq 0 \quad (8.18)$$

Similarly we cannot have $\delta F \neq 0$ as this would require a directional VEV. This only leaves $\delta\psi$ which can be non-zero. However, $\partial_\mu\phi$ is also a vector and cannot attain a VEV by Lorentz invariance. The only term that can get a VEV without breaking Lorentz invariance is the F term in $\delta\psi$. So we can have SUSY breaking if

$$\langle vac|F|vac\rangle \neq 0 \quad (8.19)$$

which will give,

$$\langle vac|\delta\phi|vac\rangle = \langle vac|\delta F|vac\rangle = 0 \quad (8.20)$$

$$\langle vac|\delta\psi|vac\rangle = \sqrt{2}\epsilon \langle vac|F|vac\rangle \neq 0 \quad (8.21)$$

Recall that in Gauge symmetry breaking ρ was invariant and only θ transformed and $\delta\theta$ corresponded to a Goldstone boson (a scalar). We have an analogous situation here. Carrying this analogy we call ψ as a Goldstone-fermion or a “goldstino”. Note that the goldstino is not the supersymmetric partner of a Goldstone boson. It is just the fermion that transforms non-trivially under supersymmetry.

Remember that we were constructing the corresponding Lagrangian we have found that the F term of the scalar potential was equal to

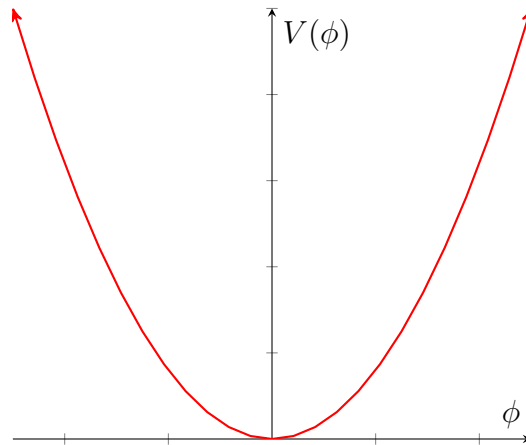
$$V_{F-Term} = K_{ij}^{-1} \frac{\partial W}{\partial\phi_i} \frac{\partial W^*}{\partial\phi_j^*} \propto FF^* \quad (8.22)$$

It is interesting that in the vacuum F will be different then zero. Furthermore, since the potential is positive definite we have

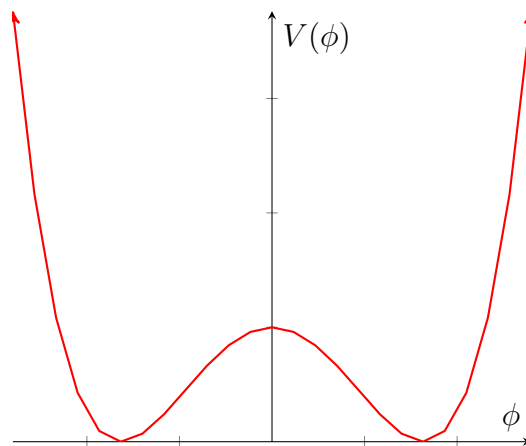
$$V_F > 0 \Leftrightarrow \langle vac|F|vac\rangle \neq 0 \quad (8.23)$$

So as we mentioned earlier SUSY breaking corresponds to energy of the vacuum being greater than zero.

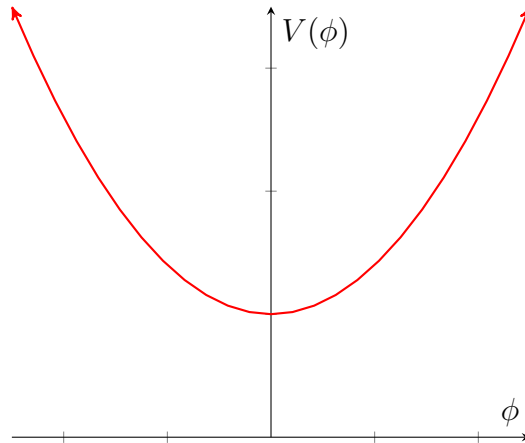
We now consider some forms for the potential.



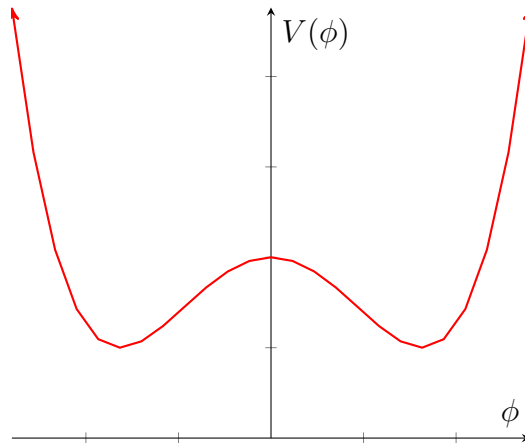
Here we have $\langle \phi \rangle = 0$ and $\langle V \rangle = 0$. So the potential is both gauge symmetric and supersymmetric. A potential of the form,



This potential is supersymmetric as we have $V = 0$ on the vacuum but not gauge invariant. A potential of the form,



is gauge symmetric and non-supersymmetric (as is the case in the SM) and the last possibility is,



which is non-supersymmetric and not gauge invariant.

O'Raifeartaigh Model

We now discuss a toy model with a concrete superpotential with an expectation value of F to be greater than zero. For that you need three superfields,

$$\Phi_1, \Phi_2, \Phi_3 \quad (8.24)$$

The Kähler potential is taken to be the canonical one but a superpotential of the form,

$$W = g\Phi_1 (\Phi_3^2 - m^2) + M\Phi_2\Phi_3 \quad (8.25)$$

and we assume $M \gg m$. Recall that

$$F_i^* = -\frac{\partial W}{\partial \phi_i} = -\frac{\partial W}{\partial \Phi_i} \Big|_{\Phi_i = \phi_i} \quad (8.26)$$

so we have,

$$F_1^* = -g(\phi_3^2 - m^2) \quad (8.27)$$

$$F_2^* = M\phi_3 \quad (8.28)$$

$$F_3^* = 2g\phi_1\phi_3 + M\phi_2 \quad (8.29)$$

If we have any of these terms are different from zero on the vacuum then we break supersymmetry. You have to cook a good superpotential to be able to do this; you need to find a set of three equations that cannot be solved simultaneously.

To see that these are inconsistent suppose you set the first equation equal to zero. This implies $\langle\phi_3\rangle = \pm m$ but then you can't have $\langle F_2\rangle = 0$ unless $m = 0$. This was a particular example but can be shown more generally. Since you cannot have

$$\langle F_1\rangle = \langle F_2\rangle = \langle F_3\rangle = 0 \quad (8.30)$$

it implies that you have supersymmetry breaking.

We now want to find the spectrum of the model (i.e. the masses of all the particles). The potential in this case is just,

$$V = \left| \frac{\partial W}{\partial \phi_i} \right|^2 \quad (8.31)$$

since the second derivative of the Kähler potential is just the identity matrix in this case. This is equal to,

$$V = g^2 |\phi_3^2 - m^2|^2 + M^2 |\phi_3|^2 + |2g\phi_1\phi_3 + M\phi_2|^2 \quad (8.32)$$

To minimize this potential one needs to treat the real and imaginary parts of ϕ_3 independently but can use the modulus of ϕ_1 and ϕ_2 as a parameter. If $m^2 < \frac{M^2}{2g^2}$ then this procedure gives a minimum at,

$$\langle\phi_2\rangle = \langle\phi_3\rangle = 0, \quad \langle\phi_1\rangle \text{ is arbitrary} \quad (8.33)$$

At this point we have,

$$\langle V\rangle = g^2 m^4 > 0 \quad (8.34)$$

The potential has one flat direction which is called a ‘‘modulus’’. Since this direction is completely flat it implies that the mass of ϕ_1 is zero. We now compute the mass of the other two. $\langle\phi_1\rangle$ can be any value but we compute the masses of ϕ_2 and ϕ_3 at the point where its zero for simplicity.

We start with the fermions. Recall that the fermion masses are given in terms of the Yukawa couplings. Recall that the mass term is,

$$\left\langle \frac{\partial^2 W}{\partial \phi_i \partial \phi_j} \right\rangle \psi_i \psi_j \quad (8.35)$$

So the mass matrix for the fermions is

$$m_{ij}^f \equiv \left\langle \frac{\partial^2 W}{\partial \phi_i \partial \phi_j} \right\rangle \quad (8.36)$$

At $\langle\phi_2\rangle = \langle\phi_3\rangle = 0$ the mass matrix is very simple. We only have two nonzero terms,

$$m_{ij}^f = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & M \\ 0 & M & 0 \end{pmatrix} \quad (8.37)$$

The first row equalling zero implies that $m_{\psi_1} = 0$. For ψ_2 and ψ_3 you need to diagonalize the 2×2 matrix. The eigenvalues are $\pm M$. However for the fermion you can always redefine the field using $\psi \rightarrow e^{i\pi\gamma_5/2}\psi$. This leaves the kinetic term invariant but switches the sign of the mass term. Thus we have,

$$m_{\psi_2} = m_{\psi_3} = M \quad (8.38)$$

Recall that

$$\delta\psi_1 \sim \langle F_1 \rangle \neq 0 \quad (8.39)$$

so ψ_1 is our Goldstino. It is massless as we predicted earlier.

We now consider the scalars. To find their masses we need to take the second derivatives of the potential. The quadratic piece of the potential is given by,

$$V_{quad} = -mg^2(\phi_3^2 + \phi_3^{*2}) + M^2|\phi_3|^2 + M^2|\phi_2|^2 \quad (8.40)$$

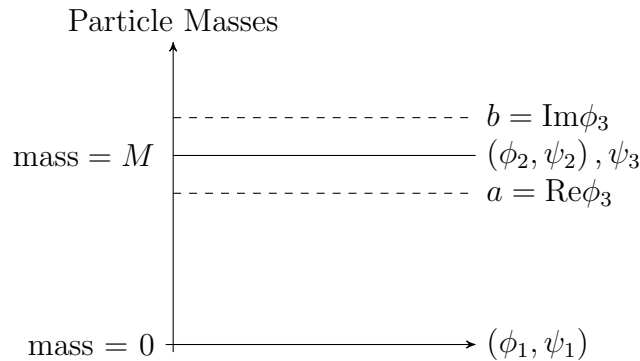
Its easy to take derivatives with respect to ϕ_2 since only its modulus appears here, however ϕ_3 is more complicated as you have to consider its real and imaginary components separately, $\phi_3 = m_a + ib$. Taking the second derivatives gives a diagonal matrix with the masses,

$$m_{\phi_2} = M \quad (8.41)$$

$$m_a = \sqrt{M^2 - 2m^2g^2} \quad (8.42)$$

$$m_b = \sqrt{M^2 + 2m^2g^2} \quad (8.43)$$

So the spectrum is as follows,



So now ψ_3 doesn't come with a bosonic particle with the same mass but with two particles with different masses. Recall that F_1 is the quantity that breaks supersymmetry by a quantity gm^2 . This is what end up gives us the splitting. This is an explicit example of how SUSY is broken. This splitting which we expect to be in the real world and should be of the order of a TeV energy. We expect to find scalar partners to the fermions on the order of a TeV scale away from the fermions.

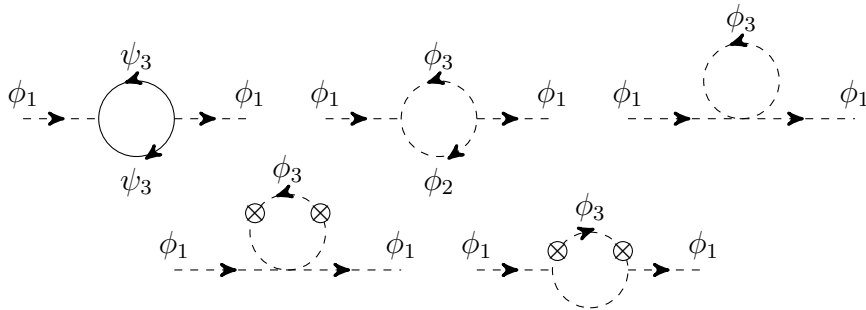
In this model we have one scalar that is heavier than the fermion but there is also another one that is lighter than the fermion. This is the general behavior that appears in tree level supersymmetry breaking and is a phenomenological problem since if we had such light scalars we should have seen them by now.

Because of the non-renormalization theorems we proved earlier the superpotential at tree level is exact to all orders in perturbation theory. Having SUSY unbroken at tree level implies that it is unbroken to all orders in perturbation theory. Thus we must break SUSY using nonperturbative effects. We discuss this more later.

Above we saw that at tree level the partner to the goldstino, ϕ_1 , was massless. This was an unexpected consequence of the SUSY breaking model. Since we have a SUSY breaking theory, this doesn't need to hold to all orders and indeed it doesn't. We now calculate the first order correction to its mass. The Lagrangian is given by,

$$\begin{aligned} \Delta\mathcal{L} = \sum_i (i\bar{\psi}_i \not{\partial}\psi_i + \partial_\mu\phi_i^\dagger\partial^\mu\phi_i) - M(\psi_2\psi_3 + h.c.) - g(\phi_1\psi_3\psi_3 + h.c.) - 2g(\phi_3\psi_1\psi_3 + h.c.) \\ - M^2\phi_3^*\phi_3 - M^2\phi_2^*\phi_2 + g^2m^2(\phi_3^2 + h.c.) - 2gM(\phi_1\phi_2\phi_3 + h.c.) - 4g^2\phi_1\phi_1^*\phi_3\phi_3^* \end{aligned} \quad (8.44)$$

We work in two component notation implementing the techniques of appendix A.. We treat g as a small parameter and the fields and their conjugates as independent degrees of freedom. The first order mass correction for ϕ_1 is given by,



where the crosses indicate insertions of $-2igm^2$. We want to study the mass correction which is a momenta independent quantity. Thus we set the incoming momenta to zero.

The first three diagrams give, [Q 10: These need fixing...]

$$i\mathcal{M}_1 = (-ig)^2 \int \frac{d^4\ell}{(2\pi)^4} (-1) i^2 \text{Tr} \frac{(\ell \cdot \bar{\sigma} + M)(\ell \cdot \sigma + M)}{(\ell^2 - M^2)^2} \quad (8.45)$$

$$= -2g^2 \int \frac{d^4\ell}{(2\pi)^4} \frac{\ell^2 + M^2}{(\ell^2 - M^2)^2} \quad (8.46)$$

$$= -2g^2 \int \frac{d^4\ell}{(2\pi)^4} \frac{1}{\ell^2 - M^2} + \frac{2M^2}{(\ell^2 - M^2)^2} \quad (8.47)$$

$$i\mathcal{M}_2 = (-2igM)^2 \frac{1}{2} \int \frac{d^4\ell}{(2\pi)^4} \frac{i^2}{(\ell^2 - M^2)^2} \quad (8.48)$$

$$= 2g^2 \int \frac{d^4\ell}{(2\pi)^4} \frac{M^2}{(\ell^2 - M^2)^2} \quad (8.49)$$

$$i\mathcal{M}_3 = -i(4g^2) \int \frac{d^4\ell}{(2\pi)^4} \frac{i}{\ell^2 - M^2} \quad (8.50)$$

$$= 4g^2 \int \frac{d^4\ell}{(2\pi)^4} \frac{1}{\ell^2 - M^2} \quad (8.51)$$

where to get this result we need to

1. include the correct symmetry factors
2. use a negative sign for the fermion loop
3. take the trace over the fermion loop

These diagrams delicately cancel,

$$\mathcal{M}_1 + \mathcal{M}_2 + \mathcal{M}_3 = 0 \quad (8.52)$$

The final two contributions are, [Q 11: check symmetry factors]

$$i\mathcal{M}_4 = -i(4g^2)(2ig^2m^2)^2 \int \frac{d^4\ell}{(2\pi)^4} \frac{i^3}{(\ell^2 - M^2)^3} \quad (8.53)$$

$$i\mathcal{M}_5 = (-2igM)^2(2ig^2m^2)^2 \frac{1}{2} \int \frac{d^4\ell}{(2\pi)^4} \frac{i^4}{(\ell^2 - M^2)^4} \quad (8.54)$$

[Q 12: Finish this section]

8.2.2 General O’Raifeartaigh Models

Supersymmetry is conserved if

$$\left. \frac{\partial W}{\partial \phi_n} \right|_{\phi=\phi_0} = 0 \quad \forall n = 1, \dots, N \quad (8.55)$$

There are N ϕ fields and N equations, so there should generically exist some point $\vec{\phi} = \vec{\phi}_0$ such that this holds true.

One way around this is to suppose that W is a linear combination of superfields,

$$\vec{Y} = (Y_1, \dots, Y_{N_Y}), \quad \vec{X} = (X_1, \dots, X_{N_X}) \quad (8.56)$$

where Y_i have R charges of 2 and X_n don't carry an R charge. Then superpotential has to take the form,

$$W = \sum_i Y_i f_i(X_1, \dots, X_{N_X}) \quad (8.57)$$

which gives the SUSY conserving conditions,

$$f_i(\vec{X}) = 0 \text{ for } i = 1, \dots, N_Y \quad (8.58)$$

$$\sum_i Y_i \frac{\partial f_i(\vec{X})}{\partial X_n} = 0 \text{ for } n = 1, \dots, N_X \quad (8.59)$$

The first set of equations is made up of N_X unknowns and N_Y equations, thus if $N_Y > N_X$ it can't be generically solved. He goes on to define

$$\mathcal{V}_{n,i} \equiv \left. \frac{\partial f_i}{\partial x_n} \right|_{x=x_0} \quad (8.60)$$

which gives the potential,

$$V(x, y) = \sum_i \left| f_i(\vec{X}) \right|^2 + \sum_n \left| \sum_i y_i \mathcal{V}_{n,i} \right|^2 \quad (8.61)$$

Lets now consider the minima. The obvious minima is at,

$$(\vec{x}, \vec{y}) = (\vec{x}_0, 0) \quad (8.62)$$

where the \vec{x}_0 is the set of fields that minimizes f_i . However, there is actually a continuous set of minima as well.

To see this we fix x at its minima value and we consider $V(x_0, y)$ which is a function on the space of \vec{y} of dimension N_Y . The first term in the potential doesn't depend on \vec{y} so its irrelevant. We want to find the directions that the second term still vanishes. Therefore, we want the directions of \vec{y} such that

$$\underbrace{\mathcal{V}}_{N_Y \times N_X \text{ matrix}} \vec{y} = 0 \quad (8.63)$$

This has $N_Y - N_X$ linearly independent solutions. Therefore, there are $N_Y - N_X$ flat directions (i.e. $N_Y - N_X$ Goldstinos). In the simple case of $N_Y = 2, N_X = 1$ above there was a single Goldstino as required.

8.2.3 D Term

Suppose you have a vector superfield,

$$V = V(\lambda, A_\mu, D) \quad (8.64)$$

One can show that [Q 13: check] the only way to have supersymmetry breaking is to use the gaugino,

$$\delta\lambda \sim \epsilon D \quad (8.65)$$

with

$$\langle D \rangle \neq 0 \quad (8.66)$$

So the gaugino is the goldstino. [Q 14: Do exercises!]

8.3 SUSY breaking in $\mathcal{N} = 1$ supergravity

- Supergravity multiplet adds new auxillary fields which we denote F_g . Supersymmetry breaking requires nonzero $\langle F_g \rangle$.
- The F term is proportional to,

$$F \propto DW = \frac{\partial W}{\partial \phi} + \frac{1}{M_{Pl}^2} \frac{\partial K}{\partial \phi} W \quad (8.67)$$

[Q 15: skipped 16 min - 21 min about supergravity.]

Chapter 9

The MSSM

We now have all the ingredients to generalize the Standard Model into what is known as the Minimal Supersymmetry Standard Model (MSSM).

9.1 Particles

We need Vector superfields that provide the SM $SU(3)_C \times SU(2)_L \times U(1)_Y$ symmetry. In other words we need a set of vector superfields for $SU(3)$, another set for $SU(2)$ and a last vector superfield for $U(1)$. We also need chiral superfields that represent the matter particles. We need to have,

- Quarks:

$$\underbrace{Q_i = (3, 2, -1/6)}_{\text{left-handed}} \quad , \quad \underbrace{\bar{u}_i^c = (\bar{3}, 1, 2/3) \quad , \quad \bar{d}_i^c = (\bar{3}, 1, -1/3)}_{\text{right-handed}}$$

The notation is as follows. we want to present each field as a left chiral superfield. To do this instead of writing u_R and d_R as is usually done in the SM we write the right handed particles as left fields with both a conjugate and a bar. Furthermore we label our particles with a generation label, $i = 1, 2, 3$.

- Leptons:

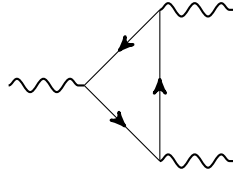
$$L_i = (1, 2, 1/2) \quad , \quad \bar{e}_i^c = (1, 1, -1) \quad , \quad \bar{\nu}_i^c = (1, 1, 0)$$

We have a newcomer here which is a right handed neutrino. We add this in because we know that neutrinos have a mass.

- Higgses:

$$H_1 = (1, 2, 1/2) \quad , \quad H_2 = (1, 2, -1/2)$$

H_2 is a completely new field. It is required in order to give mass to the Higgs. Recall that H_1 is a chiral superfield. It has the Higgs particle which is a scalar but it also has a new fermionic partner called the Higgsino. This Higgsino will contribute to anomalous diagrams (triangle diagrams) such as,



The way to cancel this anomaly is to have an extra fermion to with opposite charge.

9.2 Interactions

- We want renormalizable interactions so we have,

$$K = \Phi^\dagger e^{2qV} \Phi \tag{9.1}$$

Recall that this the the term that contains the gauge invariant vector potential functions, $\propto G_{\mu\nu}^a G_a^{\mu\nu}$. In general the Kähler potential could include a holomorphic gauge invariant function of fields in front, but due to the renormalizability requirement the function is just a constant,

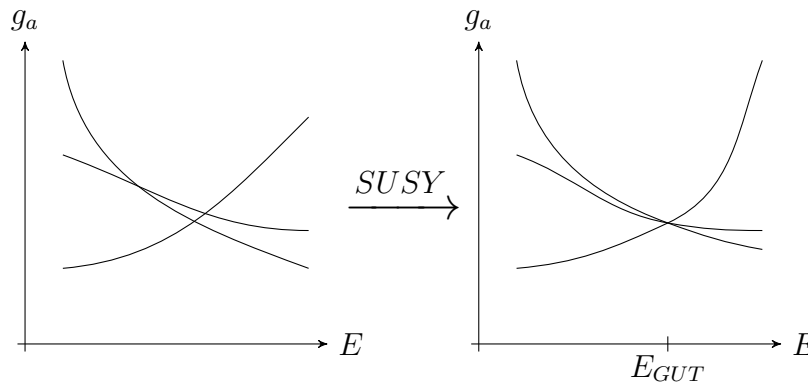
$$f_a(\Phi) = \tau_a \tag{9.2}$$

where

$$\text{Re}\tau_a = \frac{4\pi}{g_a^2} \tag{9.3}$$

where we have one coupling for each gauge group. $a = 1, \dots, 8$ for the $SU(3)$ charge, $a = 1, \dots, 3$ for the $SU(2)$ charge, and $a = 1$ for the $U(1)$ charge.

One can show that the running of the couplings with SUSY provides a unification scale called the GUT scale,



This is another motivation for SUSY.

- Lastly for the MSSM we take $\xi = 0$ since otherwise you break charge and colour symmetries.

- The last ingredient is the superpotential. It needs couplings among the fields we have written. The most general renormalizable superpotential is given by,

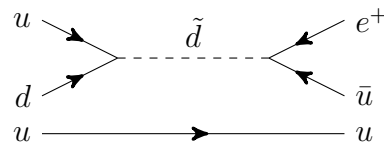
$$W = y_1 Q H_2 \bar{u}^c + y_2 Q H_1 \bar{d}^c + y_3 L H_1 \bar{e}^c + \mu H_1 H_2 + W_{\mathcal{BL}} \quad (9.4)$$

where,

$$W_{\mathcal{BL}} = \lambda_1 L L \bar{e}^c + \lambda_2 L Q \bar{d}^c + \lambda_3 \bar{u}^c \bar{d}^c \bar{d}^c + \mu' L H_2. \quad (9.5)$$

W has the typical terms you would expect. For example $Q H_2 \bar{u}^c$ in components gives a term Higgs-quark-quark as in the SM. Further, note that each term is singlet under each transformation as required for gauge invariance. For example under $SU(3)$ \bar{u}^c is a $\bar{\mathbf{3}}$ and Q is a $\mathbf{3}$ so you have a singlet, Q and H_2 are both doublets so we again have a singlet, and lastly the hypercharges add up to zero.

The extra terms in $W_{\mathcal{BL}}$ break baryon or lepton number. For example we have amplitudes of the form,



where we use the same letter for the superfield as for the field and for the corresponding superfield we denote them by a tilde. Such diagrams give rise to proton decay, $p \rightarrow e^+ + \pi^0$, within seconds assuming couplings of $\mathcal{O}(1)$. These terms are a source of embarrassment for SUSY.

We need some mechanism to forbid such contributions. The easiest thing to do is to impose a global symmetry known as R parity,

$$R_p \equiv (-)^{3(B-L)+2S} \quad (9.6)$$

Notice that $R_p = 1$ for all observed particles and $R_p = -1$ for all superpartners. This forbids all terms in $W_{\mathcal{BL}}$. Having this symmetry has some important implications.

- The lightest superpartner (LSP) is stable since it cannot decay to any particle in the SM.
- Usually, the LSP is neutral (e.g. Higgsino, photino, ...) and for this reason it is called a “neutrino”. This is the best candidate for dark matter (WIMP).
- In colliders the supersymmetric particles can only be produced in pairs which decay to LSP which gives a signal in the detectors of “missing energy”. This is one of the best ways to test supersymmetry.

9.3 Electroweak Symmetry Breaking

The scalar potential in the supersymmetric MSSM is given by,

$$V_{SUSY} = V_F + V_D \quad (9.7)$$

where

$$V_F = \sum_i |F_i|^2 = \left| \frac{\partial W}{\partial \phi_i} \right|^2 \quad (9.8)$$

$$V_D = \sum_i \frac{1}{2} D_i^a D_i^a \quad (9.9)$$

Lets focus on the vanishing baryon and lepton number sector, i.e., the Higgs sector. The relevant superpotential term is,

$$W_H = \mu H_u^T \epsilon H_d \quad (9.10)$$

from which we have,

$$\left. \begin{aligned} \frac{\partial W_H}{\partial h_u^a} &= \mu (\epsilon h_d)^a \\ \frac{\partial W_H}{\partial h_d^a} &= \mu (h_u^T \epsilon)^a \end{aligned} \right\} \Rightarrow V_F = |\mu|^2 (h_d^\dagger h_d + h_u^\dagger h_u) \quad (9.11)$$

There are a two gauge symmetries and hence two types of D_i contributions. The $SU(2)_L$ and $U(1)_Y$ D terms are,

$$D_{SU(2)_L}^a = -\frac{g}{2} (h_u^\dagger \tau^a h_u + h_d^\dagger \tau^a h_d) \quad (9.12)$$

$$D_Y = -\frac{g'}{2} (h_u^\dagger h_u - h_d^\dagger h_d) \quad (9.13)$$

where τ^a are the Pauli matrices.

To find the $SU(2)_L$ contribution to the potential we use the identity,

$$\frac{\tau_{ij}^a \tau_{kl}^a}{2} = \frac{1}{2} \left(\delta_{il} \delta_{jk} - \frac{1}{2} \delta_{ij} \delta_{kl} \right) \quad (9.14)$$

This gives,

$$D_{SU(2)_L}^a D_{SU(2)_L}^a = \frac{g^2}{2} (h_{u,i}^* h_{u,j} + h_{d,i}^* h_{d,j}) \left(\delta_{il} \delta_{jk} - \frac{1}{2} \delta_{ij} \delta_{kl} \right) (h_{u,k}^* h_{d,\ell}^* + h_{d,k}^* h_{d,\ell}) \quad (9.15)$$

$$\begin{aligned} &= \frac{g^2}{2} \left[(h_u^\dagger h_u)^2 - \frac{1}{2} (h_u^\dagger h_u)^2 + |h_u^\dagger h_d|^2 - \frac{1}{2} (h_u^\dagger h_u)(h_d^\dagger h_d) \right. \\ &\quad \left. + |h_d^\dagger h_u|^2 - \frac{1}{2} (h_d^\dagger h_d) h_u^\dagger h_u + (h_d^\dagger h_d)^2 - \frac{1}{2} (h_d^\dagger h_d)^2 \right] \quad (9.16) \end{aligned}$$

$$= \frac{g^2}{2} \left[\frac{1}{2} (|h_u|^2 - |h_d|^2)^2 + 2 (h_d^\dagger h_u)^2 \right] \quad (9.17)$$

Furthermore,

$$D_Y D_Y = \frac{g'^2}{4} (|h_u|^2 - |h_d|^2)^2 \quad (9.18)$$

which gives,

$$V_D = \frac{g^2}{2} |h_d^\dagger h_u|^2 + \frac{g^2 + g'^2}{8} (|h_u|^2 - |h_d|^2)^2 \quad (9.19)$$

and

$$V_{SUSY} = |\mu|^2 (h_d^\dagger h_d + h_u^\dagger h_u) + \frac{g^2}{2} |h_d^\dagger h_u|^2 + \frac{g^2 + g'^2}{8} (|h_u|^2 - |h_d|^2)^2 \quad (9.20)$$

In order to have a real Lagrangian we must have, $g^2, g'^2 > 0$. However, then we necessarily has a single minima at the origin. This shows that in the MSSM electroweak symmetry can't be broken without SUSY breaking. This is a general trend which is a consequence of SUSY invariant theories requiring the potential to be positive definite.

But we have thus far omitted an important contribution, soft SUSY breaking terms. The most general superrenormalizable terms are,

$$V_{SUSY} = m_u^2 h_u^\dagger h_u + m_d^2 h_d^\dagger h_d + \text{Re} \{ b_\mu h_u^T \epsilon h_d \} \quad (9.21)$$

In general b_μ, h_u , and h_d each contain a phase. We can write,

$$b_\mu h_u^T \epsilon h_d = |b_\mu| h_u^T \epsilon h_d e^{i\phi_{b_\mu}} \quad (9.22)$$

$$= |b_\mu| h_u'^T \epsilon h_d' \quad (9.23)$$

where

$$h_u' \equiv e^{i\frac{1}{2}\phi_{b_\mu}} h_u \quad (9.24)$$

$$h_d' \equiv e^{i\frac{1}{2}\phi_{b_\mu}} h_d \quad (9.25)$$

is a simple field redefinitions that absorbs the phase of b_μ . This gives,

$$V = (|\mu|^2 + m_u^2) |h_u|^2 + (|\mu|^2 + m_d^2) |h_d|^2 + \frac{g^2}{2} |h_d^\dagger h_u|^2 + \frac{g^2 + g'^2}{8} (|h_u|^2 - |h_d|^2)^2 + b_\mu \text{Re} \{ h_u'^T \epsilon h_d' \} \quad (9.26)$$

For this system to be physical, the potential must be bounded in all directions. This is not generically the case for the direction that,

$$h_u = \begin{pmatrix} \phi \\ 0 \end{pmatrix}, h_d = \begin{pmatrix} 0 \\ \phi \end{pmatrix} \quad (9.27)$$

for $\phi \rightarrow \infty$. In this case

$$V \rightarrow (2|\mu|^2 + m_u^2 + |\mu|^2 + m_d^2 - b_\mu) \phi^2 \quad (9.28)$$

This is positive definite if

$$b_\mu \leq 2|\mu|^2 + m_u^2 + m_d^2 \quad (9.29)$$

as we will see this is the first of two conditions on b_μ .

We now study the VEV's in greater detail. First we note that the potential has an extrema at the origin, $h_u^+ = h_u^0 = h_d^- = h_d^0 = 0$. This is a consequence of having no linear terms in the potential. This implies that the charged fields, h_u^+ and h_d^- , don't need to gain a VEV. We assume $\langle h_u^+ \rangle = \langle h_d^- \rangle = 0$ and allow the neutral scalars to gain a VEV,

$$h_u^0 = v_u + \phi_u \quad (9.30)$$

$$h_d^0 = v_d + \phi_d \quad (9.31)$$

We want to study the VEV's of the theory. For this purpose we can focus on the neutral parts,

$$V_{neut} = \frac{g^2 + g'^2}{8} (|\phi_u + v_u|^2 + |\phi_d + v_d|^2)^2 + (|\mu|^2 + m_u^2) |m_u + v_u|^2 + (|\mu|^2 + m_d^2) |\phi_d + v_d|^2 - b_\mu \text{Re} \{ (\phi_u + v_u)(\phi_d + v_d) \} \quad (9.32)$$

In order to have a minima the terms linear in $\text{Re}\phi_u, \text{Re}\phi_d, \text{Im}\phi_u, \text{Im}\phi_d$ must vanish. This yields two equations (with real and imaginary parts),

$$\left(\frac{g^2 + g'^2}{4} (|v_u|^2 - |v_d|^2) + |\mu|^2 + m_u^2 \right) v_u^* - \frac{1}{2} b_\mu v_d = 0 \quad (9.33)$$

$$\left(-\frac{g^2 + g'^2}{4} (|v_u|^2 - |v_d|^2) + |\mu|^2 + m_d^2 \right) v_d^* - \frac{1}{2} b_\mu v_u = 0 \quad (9.34)$$

So far we have been working in a generic gauge. To simplify these equations we perform a $U(1)_Y$ gauge transformation to make v_u real. This implies that v_d is also real.

We now rewrite these equations using two useful definitions,

$$\tan \beta \equiv \frac{v_u}{v_d}, \quad v^2 \equiv v_u^2 + v_d^2 \quad (9.35)$$

which combined imply,

$$v_u = v \sin \beta, \quad v_d \equiv v \cos \beta \quad (9.36)$$

This gives,

$$\left(-\frac{g^2 + g'^2}{4} v^2 c_{2\beta} + |\mu|^2 + m_u^2 \right) s_{2\beta} - b_\mu c_\beta^2 = 0 \quad (9.37)$$

$$\left(\frac{g^2 + g'^2}{4} v^2 c_{2\beta} + |\mu|^2 + m_d^2 \right) s_{2\beta} - b_\mu s_\beta^2 = 0 \quad (9.38)$$

Adding the two equations gives,

$$b_\mu = (2|\mu|^2 + m_u^2 + m_d^2) s_{2\beta} \quad (9.39)$$

$$b_\mu = m_A^2 s_{2\beta} \quad (9.40)$$

Subtracting the two gives,

$$\left(\frac{g^2 + g'^2}{2}v^2c_{2\beta} + m_d^2 - m_u^2\right) s_{2\beta} + b_\mu c_{2\beta} = 0 \tag{9.41}$$

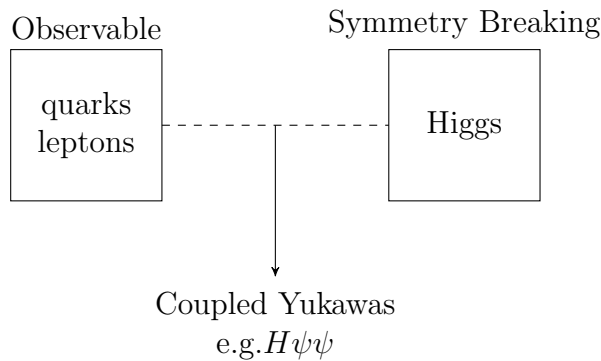
$$(m_Z^2 c_{2\beta} + m_d^2 - m_u^2) + m_A^2 c_{2\beta} = 0 \tag{9.42}$$

$$-(m_A^2 + m_Z^2) c_{2\beta} = m_d^2 - m_u^2 \tag{9.43}$$

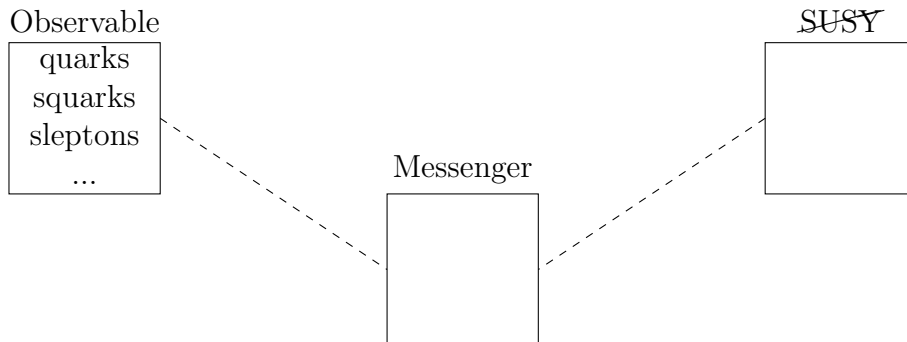
[Q 16: This is off by a minus sign from Weinberg]

9.4 Supersymmetry Breaking

Recall that in the SM we have two sectors,



In SUSY the situation is a bit different as in general we need 3 sectors,

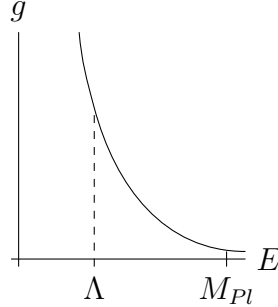


Its possible to break supersymmetry directly instead as with supersymmetry breaking however, this turns out not to work very well. Most of the uncertainty of supersymmetric model is not as much from the supersymmetry breaking sector but from the messenger sector.

9.4.1 SUSY Sector

We discussed previously that you can break supersymmetry at tree level (and hence at any order in perturbation theory), however that can be very bad because you get light particles that we should have seen by now. Thus we must break SUSY in a non-perturbative way.

To understand what this means consider the following example. Suppose you have a SUSY breaking sector with the group $G = SU(N)$ with matter such that it is asymptotically free,



In asymptotically free theories it is well known that fermions can condensate non-perturbatively at low energies (this phenomena is often discussed in the context of chiral symmetry breaking). The condensation is such that the vacuum expectation value of two fermions paired together is non-zero. In particular the gauginos will pair and develop the VEV,

$$\langle \lambda\lambda \rangle \neq 0 \quad (9.44)$$

(analogous to Cooper pairs in superconductivity). The fermions can only get close to each other at large enough couplings. By dimensional analysis we have,

$$\langle \lambda\lambda \rangle \sim \Lambda^3 \quad (9.45)$$

Depending on the model one can show that this can break supersymmetry,

$$M_{SUSY} \sim \Lambda = M_{Pl} e^{-a/g^2} \quad (9.46)$$

due to the renormalization group running (a is some constant). So Λ , the natural scale of the theory, can be hierarchically much smaller than the Planck energy.

9.4.2 Messenger Sector

This is the most model dependent part of supersymmetry. There are several ways of transmitting the information that SUSY was broken:

1. Gravity mediation: You break SUSY in one sector which doesn't couple to the observable sector. However, gravity couples to everybody which in turn transmits the breaking to the observable sector. This is perhaps the most popular approach. It has several interesting properties. Since it is gravity mediated we know that the mass splitting in the observable sector must be very small if M_{Pl} is very large (in the limit that $M_{Pl} \rightarrow \infty$ there is no gravity).

$$\Delta m = \frac{\text{something}}{M_{Pl}} \quad (9.47)$$

Δm is due to SUSY breaking and so this is the only other scale that can be involved. So by dimensional analysis we must have,

$$\Delta m = \frac{M_{SUSY}^2}{M_{Pl}} \quad (9.48)$$

We want $\Delta m \sim 1\text{TeV}$ to be able to fix the hierarchy problem. We know that $M_{Planck} \sim 10^{18}\text{GeV}$. Thus we know that,

$$M_{SUSY} = \sqrt{M_{EW}M_{Pl}} \sim 10^{11}\text{GeV} \quad (9.49)$$

So there is an intermediate scale between the electroweak scale and the Planck scale that gives SUSY breaking. Furthermore, the gravitino gets a mass given by,

$$m_{3/2} = \frac{M_{SUSY}^2}{M_{Pl}} \sim \text{TeV} \quad (9.50)$$

(Aside: the gravitino gets a mass from what's known as the Super-Higgs effect; the gravitino "eats" the goldstino to get a mass. This is different from the supersymmetric version of the Higg's mechanism where a whole vector superfield eats a whole chiral superfield.)

2. Gauge Mediation: With the total gauge group equal to the gauge group of the SM multiplied by sector that breaks supersymmetry,

$$G = \underbrace{(SU(3) \times SU(2) \times U(1))}_{G_0} \times G_{SUSY} \quad (9.51)$$

The matter fields are charged under both G_0 and G_{SUSY} . In the gravity mediation case the particles introduced in the SUSY breaking sector were all singlets under the gauge group of the SM. However, this is not the case here. By dimensional analysis we have,

$$\Delta m \sim M_{SUSY} \quad (9.52)$$

which we demand to be of order 1TeV . On the other hand the gravitino mass, $m_{3/2}$ will still be equal to

$$\frac{M_{SUSY}^2}{M_{Pl}} \sim 10^{-3}\text{eV} \quad (9.53)$$

This would imply that the gravitino is the LSP.

3. Anomaly Mediation: The auxillary fields of the supergravity models get a VEV. This is always present but suppressed by loop effects.

In each of these mechanisms we have an effective Lagrangian,

$$\mathcal{L} = \mathcal{L}_{SUSY} + \mathcal{L}_{SUSY} \quad (9.54)$$

for the observable sector. The \mathcal{L}_{SUSY} terms are called soft breaking terms. They take the form of,

$$\mathcal{L}_{SUSY} = \underbrace{(M_\lambda \lambda^a \lambda^a + h.c.)}_{\text{gaugino masses}} + \overbrace{m_0^2 \phi^{a*} \phi^a}^{\text{scalar masses}} + \underbrace{(A \phi \phi \phi + h.c.)}_{\text{trilinear}} \quad (9.55)$$

Given a model for the messenger sector you need to determine what each of these terms are. When people talk about the MSSM they talk about the supersymmetric part plus these soft-breaking terms. They are called soft-breaking since they don't effect the good ultraviolet properties of supersymmetry.

9.5 Hierarchy Problem

In the Standard Model we know that

- The gauge particles are massless due to gauge invariance.
- Fermions are also massless since gauge invariance forbids $m\psi\psi$.
- There is no symmetry that protects bosons from getting a mass.

In SUSY

- Bosons have the same mass as fermions which implies that their mass is also protected by symmetry
- Gives miraculous calculations

The biggest problem that SUSY faces is it doesn't solve the cosmological constant problem. You may wonder if there is a way to address both problems at the same time. One possible solution to solve this problem is known as the "Landscape". The idea is to consider a potential that has many minima $\sim 10^{100}$ with each minima corresponding to some possible universe. Then since very few universes could support life as we know it, our universe is just one of the very few possible choices making the fine-tuning in some sense natural.

One may wonder if you could use this same idea to solve the hierarchy problem. This has been applied for example in a model called Split-SUSY.

Appendix A

Two Component Spinor Techniques

[Q 17: This section is under construction]

In this set of notes we primarily use two component spinors. However, quantum field theory is conventionally taught in four component Dirac spinors. In practice one can always move from one convention to another to compute a calculation. However, this is an unnecessary hurdle. In this chapter we summarize how to perform calculations directly through two component fermion techniques. This discussion primarily follows Ref. [1].

A.1 Transformations

A two component spinor transforms as,

$$\psi^\alpha \rightarrow \left(e^{-\frac{i}{2} \boldsymbol{\alpha} \cdot \boldsymbol{\sigma}} \right)_\beta^\alpha \psi^\beta = \left(\frac{\sqrt{p \cdot \boldsymbol{\sigma}}}{\sqrt{m}} \right)_\beta^\alpha \psi^\beta \quad (\text{A.1})$$

where,

$$\sqrt{p \cdot \boldsymbol{\sigma}} = \frac{E_p + m - \boldsymbol{\sigma} \cdot \mathbf{p}}{\sqrt{2(E_p + m)}} = \frac{p_\mu \sigma^\mu + m}{\sqrt{2(E_p + m)}} \quad (\text{A.2})$$

Similarly,

$$\sqrt{p \cdot \bar{\boldsymbol{\sigma}}} = \frac{p_\mu \bar{\sigma}^\mu + m}{\sqrt{2(E_p + m)}} \quad (\text{A.3})$$

In these formula it is understood that $(\sigma^0)^{\alpha\dot{\alpha}}$ or $(\bar{\sigma}^0)^{\dot{\alpha}\alpha}$ must be added when indices are added.

As an consistency check we consider the square of the square root,

$$\sqrt{p \cdot \bar{\boldsymbol{\sigma}}}_\alpha^\gamma \sqrt{p \cdot \boldsymbol{\sigma}}_\gamma^\beta = \frac{1}{2(E_p + m)} (p \cdot \sigma \bar{\sigma}^0 p \cdot \sigma \bar{\sigma}^0)_\alpha^\beta + m^2 + 2m [p \cdot \sigma \bar{\sigma}^0]_\alpha^\beta \quad (\text{A.4})$$

To simplify this we use the identity,

$$\sigma^\mu \bar{\sigma}^0 \sigma^\nu = g^{\mu 0} \sigma^\nu - g^{\mu\nu} \sigma^0 + g^{0\nu} \sigma^\mu = i \epsilon^{\mu 0 \nu \rho} \sigma^\rho \quad (\text{A.5})$$

$$\Rightarrow p \cdot \sigma \bar{\sigma}^0 p \cdot \sigma \bar{\sigma}^0 = p^0 (p \cdot \sigma) \bar{\sigma}^0 - p^2 + p^0 (p \cdot \sigma) \bar{\sigma}^0 \quad (\text{A.6})$$

This gives,

$$(\sqrt{p \cdot \sigma} \sqrt{p \cdot \bar{\sigma}})_\alpha^\beta = \frac{1}{2(E_p + m)} (2p \cdot \sigma \bar{\sigma}^0 (E_p + m))_\alpha^\beta \quad (\text{A.7})$$

$$= (p \cdot \sigma) \bar{\sigma}^0 \quad (\text{A.8})$$

as desired.

A.2 Identities

A.3 Quantization

The fermion Lagrangian is given by,

$$\mathcal{L} = i\xi^\dagger \bar{\sigma}_\mu \partial^\mu \xi - \frac{m}{2} (\xi \xi + h.c.) \quad (\text{A.9})$$

We now find the equations of motion,

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \xi)} = i\xi^\dagger \bar{\sigma}^\mu, \quad \frac{\partial \mathcal{L}}{\partial \xi} = -m\xi \quad (\text{A.10})$$

$$\Rightarrow \not{p} \xi^\dagger = -m\xi \quad (\text{A.11})$$

where \not{p} is understood to be either $p \cdot \sigma$ or $p \cdot \bar{\sigma}$ depending on the necessary index structure.

We now expand the spinors in terms of creation annihilation operators,

$$\xi^\alpha e \sum_s \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} (x^\alpha(\mathbf{p}, s) a(\mathbf{p}, s) e^{-ip \cdot x} + y^\alpha(\mathbf{p}, s) a^\dagger(\mathbf{p}, s) e^{ip \cdot x}) \quad (\text{A.12})$$

$$\bar{\xi}^{\dot{\alpha}} = \sum_s \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} (\bar{x}^{\dot{\alpha}}(\mathbf{p}, s) a^\dagger(\mathbf{p}, s) e^{ip \cdot x} + \bar{y}^{\dot{\alpha}}(\mathbf{p}, s) a(\mathbf{p}, s) e^{-ip \cdot x}) \quad (\text{A.13})$$

where x^α, y^α are momentum dependent two component **commuting** spinors which are similar to u and v from four-component notation. The anticommuting nature of ξ^α is now embedded in the creation and annihilation operators,

$$\{a(\mathbf{p}, s)^\dagger, a(\mathbf{p}', s')\} = \delta^{(3)}(\mathbf{p} - \mathbf{p}') \delta_{ss'} \quad (\text{A.14})$$

$$\{a(\mathbf{p}, s), a(\mathbf{p}', s')\} = \{a^\dagger(\mathbf{p}, s), a^\dagger(\mathbf{p}', s')\} = 0 \quad (\text{A.15})$$

We use relativistic normalization for our states,

$$|\mathbf{p}, s\rangle = (2\pi)^3 \sqrt{2E_p} a_{\mathbf{p}, s}^\dagger |0\rangle \quad (\text{A.16})$$

A.4 Feynman Rules

An incoming fermion line contributes,

$$\langle 0 | \xi^\alpha | \mathbf{p}, s \rangle = \sum_s \int \frac{d^3 p'}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \langle 0 | a_{\mathbf{p}', s'} (2\pi)^3 \sqrt{2E_p} a_{\mathbf{p}, s}^\dagger | 0 \rangle e^{-ip \cdot x} x^\alpha \quad (\text{A.17})$$

$$= e^{ip \cdot x} x^\alpha \quad (\text{A.18})$$

A.5 Charged Fermions

Next we want to study a charged fermions. Recall from QFT that charged particles interact through a conserved current and these currents are associated with a symmetry. To this end we study a model with a $SO(N)$ a symmetry ¹,

$$\mathcal{L} = i \xi^\dagger \bar{\sigma}_\mu \partial^\mu \xi - \frac{m}{2} (\xi_i \xi_i + h.c.) \quad (\text{A.19})$$

below we use the conventions that spinors have a lowered flavor index and conjugate spinor have an upper flavor index. A transformation on the spinors takes the form,

$$\psi_{\alpha, i} \rightarrow U_i^j \psi_{\alpha, j} \quad (\text{A.20})$$

and conjugation gives,

$$(\psi_{\alpha, i})^\dagger = \bar{\psi}_{\dot{\alpha}}^i \quad (\text{A.21})$$

More generally, conjugation flips the height of any flavor index.

Under $SO(N)$ the fields transform as,

$$\xi_i \rightarrow \mathcal{O}_i^j \xi_j \quad (\text{A.22})$$

$$\bar{\xi}_i \rightarrow \bar{\xi}_j \mathcal{O}_i^j \quad (\text{A.23})$$

The $SO(N)$ condition imposes that

$$\mathcal{O}_i^j \mathcal{O}_j^k = \delta_i^k \quad (\text{A.24})$$

and note that since we are working with real transformation matrices we are free to raise or lower the flavor indices at will (equivalently, the metric in this space is just the identity).

¹One may wonder why we didn't suggest studying an $SU(N)$ symmetry. The reason that the Majorana mass term isn't invariant under such a transformation.

Bibliography

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