Solitons and Instantons
LECTURE NOTES

Lecture notes largely based on a lectures series given by Csaba Csaki at Cornell University in 2013

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2016

Contents

Preface

If you have any corrections please let me know at ajd268@cornell.edu. Useful references for this course are *Aspects of Symmetry* by S. Coleman[1], *Solitons and Instantons* by R. Rajaraman [2], and *ABC of Instantons* by M. Shifman et al. [3]

Chapter 1

Introduction

This lecture series will be study of non-trivial field configurations in quantum field theories. Conventional quantum field theory (QFT) assumes the classical field we expand around are independent of space and time. While this is true for the bulk of effects in QFT, there are some states and effects which are due to non-trivial *topologies* for the unperturbed state. These effects are not small perturbations and thus never be extracted using a perturbative expansion. We begin by studying these effects in Minkowski space, where we show that additional particles exist in QFT which are not excited states of a trivial classical configuration. Such particles are collectively known as *solitons*. We then move on to studying such solutions in Euclidean space, which provide a description of tunneling phenomena in QFT through what are known as *instantons*. Tunneling can have profound effects producing new effective operators into the Lagrangian.

Solitons and instantons are nonperturbative solutions of the classical nonlinear equations of motion. We differentiate the two as follows:

Solitons	Instantons
Minkowski	Euclidean
Finite energy, $E < \infty$	Finite action, $S < \infty$
Non-dissapative, remain localized	Not a particle, describes tunneling effect

The similarities of the two are:

- Neither of these two involve an expansion in the coupling constant.
- Topological conservation - there will be a different kind of conserved charge but this current doesn't follow from Noether's theorem.

Chapter 2

Scalar solitons

2.1 $1+1$ dimensional solitons

2.1.1 Soliton basics

We begin our discussion with classical scalar field theory in Minkowski space. As in ordinary QFT, the classical solutions will be the vacuum for which we quantize our theory. Thus understanding the classical solutions is instrumental in understanding the full quantum theory. To this end we study the Euler-Lagrange equations of motion,

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial^\mu \phi)} = \frac{\partial \mathcal{L}}{\partial \phi} \tag{2.1}$$

As we will see solutions to this equation can vary in space. In order for this solution to have physical importance, it needs to have a finite energy,

$$E \equiv \int d^d x \mathcal{E}(\mathbf{x}, t) \tag{2.2}$$

where, $\mathcal{E} \equiv T^0_0$. Furthermore, we want vacuum states which are stable over time, instead of converting to trivial solutions at late times. This condition can be written as,

$$\lim_{t\to\infty} \max_{\text{all } x} \mathcal{E}(\mathbf{x}, t) \neq 0 \tag{2.3}$$

This is a physicists definition of the soliton. This slightly differs from a mathematicians definition which involves a requirement that the superposition of two solitons remain a soliton. They would call the less restrictive solutions that satisfy the conditions we mentioned above, solitary waves.

The crucial aspect of all these solutions is going to be, what is the manifold of vacua for the theory. If there is no nonlinear term there is nothing nontrivial. We really want to take the interaction term seriously and add it to the action.

We start with ϕ^4 theory[1],

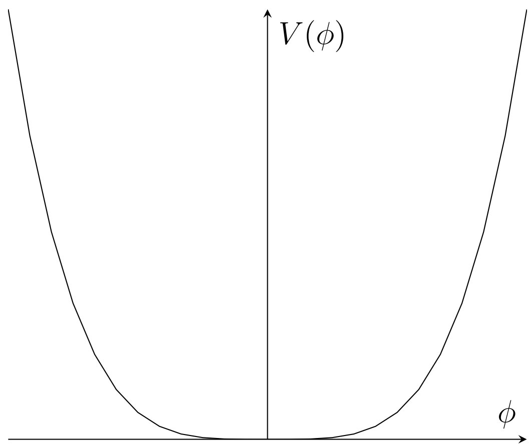
$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{1}{4}\lambda \phi^4 \tag{2.4}$$

[1] We use the negative spatial metric throughout and $g_{\mu\nu} = (1, -1)$ for two dimensions

The equation of motion for this theory is ($\Box \equiv \partial_\mu \partial^\mu$),

$$\Box\phi + \lambda\phi^3 = 0 \tag{2.5}$$

Our goal is to find solutions this partial differential equation with finite energy. Instead of attacking this problem head-on, lets first compute the energy. The potential is just a quartic,



The energy momentum tensor is found by considering the variation of the Lagrangian under space-time translations which gives,

$$T^\mu{}_\nu = \partial^\mu\phi\partial_\nu\phi - \frac{1}{2}(\partial_\mu\phi)^2\delta^\mu{}_\nu + \frac{\lambda}{4}\phi^4\delta^\mu{}_\nu \tag{2.6}$$

allowing us to compute the energy density (we use a prime to indicate a spatial derivative)

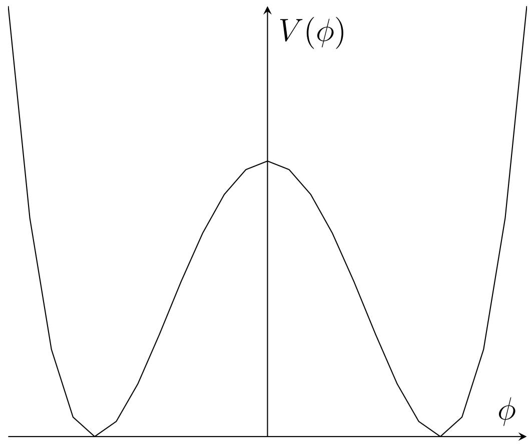
$$\mathcal{E} = \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}\phi'^2 + \frac{\lambda}{4}\phi^4 \tag{2.7}$$

The energy of the given solution is,

$$E[\phi] = \int_{-\infty}^{\infty} dx \left(\frac{1}{2}\dot{\phi}^2 + \frac{1}{2}\phi'^2 + \frac{\lambda}{4}\phi^4\right) \tag{2.8}$$

Here we are working in 1 + 1 dimensions. The energy is clearly positive definite. The energy is zero only if $\phi = 0$. But we are not requiring zero energy solutions, we are just requiring finite energy solutions. Can we find other finite energy solutions? In this case, no. Here we have only 1 classical vacuum. To get a finite energy, we must have $\phi \to 0$ for $x \to \pm\infty$. While field configurations that start and end with the same value of ϕ are solutions, it turns out there are no non-trivial solutions with these boundary conditions (we show this rigorously later on).

Now instead lets consider a theory with two vacua:



A sample potential is given by,

$$V(\phi) = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{2}\phi^2 - \frac{\lambda}{4}\phi^4 \tag{2.9}$$

Then the energy density expression then is,

$$E[\phi] = \int_{-\infty}^{\infty} dx \left(\frac{1}{2}\phi^2 + \frac{1}{2}\phi'^2 - \frac{1}{2}\phi^2 + \frac{\lambda}{4}\phi^4\right) \tag{2.10}$$

The minima occur at $\phi_\pm = \pm 1/\sqrt{\lambda}$. Then the condition to have finite energy solutions is,

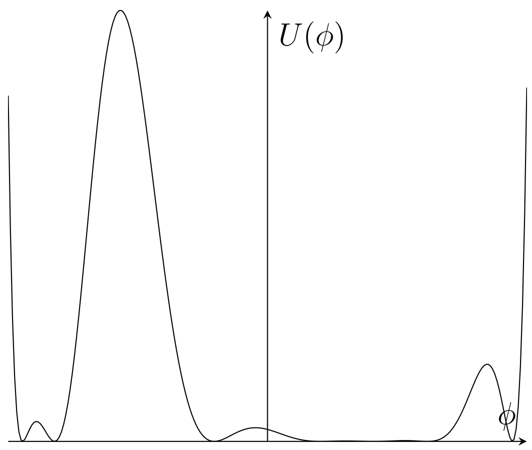
$$\lim_{x\to\pm\infty} \phi \to \phi_\pm \tag{2.11}$$

This problem has less trivial boundary conditions and can yield non-trivial solutions. Instead of study the two previous cases in detail, lets consider an arbitrary potential.

A Lagrangian with a potential, $U(\phi)$ is given by,

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 - U(\phi) \tag{2.12}$$

with $U(\phi)$ being some complicated function,



There are many possible positions where the potential is vanishing. If we denote the minima with ϕ_i then we have $U(\phi_i) = 0$. The equation of motion is given by,

$$\ddot{\phi} - \phi'' = -\frac{\partial U}{\partial \phi} \tag{2.13}$$

The energy density is given by,

$$E[\phi] = \int_{-\infty}^{\infty} dx \left[\frac{1}{2}\dot{\phi}^2 + \frac{1}{2}\phi'^2 + U(\phi)\right] \tag{2.14}$$

$E[\phi] = 0$ implies that the field is constant, $\phi = \phi_i$. Alternatively finite solutions can be acheived if scalar field asymptotically approaces on of the vacua. Therefore, $E[\phi] < \infty$ implies that,

$$\lim_{x\to+\infty} \phi = \phi_i, \lim_{x\to-\infty} \phi = \phi_j \tag{2.15}$$

The finite energy solutions **interpolate between two of the zeroes in the potential**.

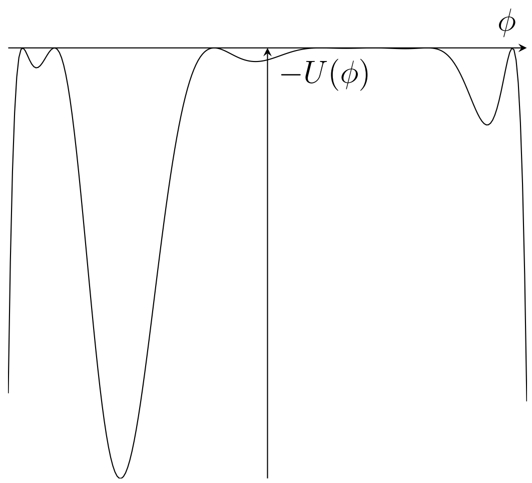
We now want to find explicit form for these solutions. Solving the partial differential equation is difficult, but we can simplify our analysis using Lorentz invariance. Instead of solving the full equations we can first look for static solutions and then add the time dependence by boosting them [2].

2.1.2 Static solutions

The equation of motion in the static case is,

$$\phi'' = \frac{\partial U}{\partial \phi} \tag{2.16}$$

This is an ordinary second order differential equation. This is reminiscent of Newton's second law for a 1D particle in a potential with the sign of the potential is reversed, $\ddot{x} = -\frac{\partial U}{\partial x}$.



[2]Its not apriori clear whether there are some additional time dependent solutions that we can't get from boosting them. While we acknowledge this as a possibility we ignore it for our purposes.

As in Newton's equation we can introduce a conserved quantity, the "total mechanical energy",

$$W = \frac{1}{2}\phi'^2 - U \tag{2.17}$$

You know that when $x \to \pm\infty$ the field has to approach a constant value (its at minimum of the potential). This implies that $\phi' \to 0$. Futhermore, at a zero of the potential we know that $U = 0$. Hence to get the finite field theory energy solutions, the mechanical energy has to be zero,

$$\lim_{x\to\pm\infty} W = 0 \tag{2.18}$$

Furthermore, since this quantity is conserved we have,

$$\frac{1}{2}\phi'^2 = U \;\Rightarrow\; \phi' = \pm\sqrt{2U(\phi)} \tag{2.19}$$

What sort of solutions could this equation yield? You are interested in the mechanical solutions in this potential that go from one maximum (in this inverted potential) to another maximum. This could be something that goes from $\phi_1^{max} \to \phi_2^{max}$, $\phi_2^{max} \to \phi_3^{max}$, etc, where we denote the different maxima by ϕ_i^{max}.

Claim 1. *In 1D, you can only have solutions that go between neighbooring vacua.*

Proof. If you try to go from $\phi_1^{max} \to \phi_3^{max}$ then you can only do it by going through ϕ_2^{max}. When you go through ϕ_2^{max} you will have $U(\phi_2^{max}) = 0$. Because this is a maximum the derivative is also vanishing, $\partial U(\phi_2^{max})/\partial\phi$. This is telling us that,

$$\frac{1}{2}\phi'^2 - U(\phi) = 0 \Rightarrow \phi' = \sqrt{2U(\phi)} \tag{2.20}$$

which implies that at the minima, $\phi' = 0$. Now

$$\phi'' = \left.\frac{\partial U}{\partial \phi}\right|_{\phi=\phi_2^{max}} = 0 \tag{2.21}$$

Furthermore,

$$\phi''' = \frac{d}{dx}\frac{\partial U}{\partial \phi} = \frac{\partial^2 U}{\partial \phi^2}\phi' \xrightarrow{\phi_2^{max}} 0 \tag{2.22}$$

and similary for higher derivatives. Thus the particle really gets stuck there. You cannot roll down on the other side. In higher dimensions you can avoid a vacua by going around it making this feature only present in the one dimensional case. □

Lets now address a claim we made earlier:

Claim 2. *Any solution with the same initial and final vacuum is a trivial solution.*

Proof. Suppose we had a solution that started and ended at the same vacuum, ϕ_0, but moved away from that vacuum away from infinity. The field starts at ϕ_0 and at some point, ϕ_1, it needs to turn around to return to ϕ_0. At the turnaround point we must have $\phi_1' = 0$, but this implies that $U(\phi_1) = 0$ by equation 2.19 and hence ϕ_1 is a vacuum and hence the field will be stuck there. □

For n minima you can have the possible transitions,

$$\overbrace{1 \to 2\,,\, 2 \to 3\,,\, \dots\,,\, n-1 \to n}^{n-1} \qquad \overbrace{n \to n-1\,,\, n-1 \to n-2\,,\, \dots\,,\, 2 \to 1}^{n-1}$$

Thus in total we have

$$2(n-1) \tag{2.23}$$

possible solutions. The ones that go one direction you call solitons and the ones going in the other direction are called antisolitons.

2.1.3 Explicit solution

To get an explicit solution we need to integrate over the equation of motion,

$$\frac{1}{2}\phi'^2 = U(\phi) \tag{2.24}$$

Separating variables gives,

$$x - x_0 = \pm \int_{\phi_0}^{\phi(x)} d\tilde{\phi} \frac{1}{\sqrt{2U(\tilde{\phi})}} \tag{2.25}$$

We now consider the *kink* solution. This is a 1 + 1 dimensional soliton with the potential,

$$U(\phi) = \frac{\lambda}{4}\left(\phi^2 - \frac{m^2}{\lambda}\right)^2 \tag{2.26}$$

The higher dimensional version of these are domain walls The way the constant is picked λ is still the quartic and m is the "mass" of the quantized field. The vacua are at,

$$\phi_0 = \pm\sqrt{\frac{m^2}{\lambda}} \tag{2.27}$$

We plug this into our general expression,

$$x - x_0 = \pm\sqrt{\frac{2}{\lambda}} \int_{\phi_0}^{\phi(x)} d\tilde{\phi} \left(\tilde{\phi}^2 - \frac{m^2}{\lambda}\right)^{-1} \tag{2.28}$$

Lets pick x_0 to be the center of the soliton. We can pick where exactly the center of the soliton is. We pick $\phi_0 = \phi(0) = 0$.

The integral is easy to perform using *Mathematica*,

$$\int_0^x dx \frac{1}{x^2 - a^2} = -\frac{1}{a}\tanh^{-1}\frac{x}{a} \tag{2.29}$$

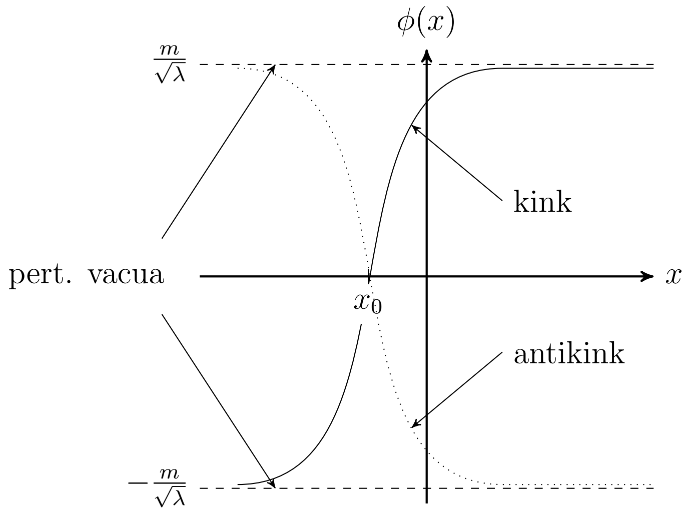
Using this result we get (with $a = m/\sqrt{\lambda}$),

$$x - x_0 = \pm\frac{\sqrt{2}}{m}\tanh^{-1}\left(\phi(x)\frac{\sqrt{\lambda}}{m}\right) \tag{2.30}$$

which gives,

$$\boxed{\phi(x) = \pm\frac{m}{\sqrt{\lambda}}\tanh\left[\frac{m}{\sqrt{2}}(x - x_0)\right]} \tag{2.31}$$

This is our first explicit soliton solution. It takes the form,



Its obvious that we can't call this field configuration a particle as its not localized in space. However, nobody really cares about the fields (perturbative vacua aren't localized either!). Its only the energy density that needs to be localized. This determines if you can have a particle-like behavior or not.

The energy density is,

$$\mathcal{E}(x) = \frac{1}{2}\phi'^2 + U(\phi) \tag{2.32}$$

We have,

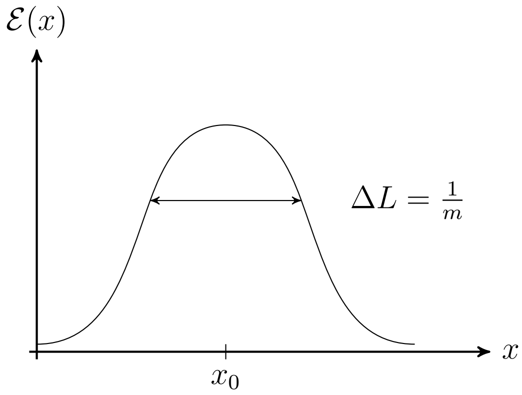
$$\phi' = \frac{m}{\sqrt{\lambda}}\frac{1}{\cosh^2\left[\frac{m}{\sqrt{2}}(x - x_0)\right]} \tag{2.33}$$

$$\frac{1}{2}\phi'^2 = \frac{m^4}{4\lambda}\frac{1}{\cosh^4\left[\frac{m}{\sqrt{2}}(x - x_0)\right]} \tag{2.34}$$

The total energy density is just,

$$\mathcal{E}(x) = \frac{m^4}{2\lambda}\frac{1}{\cosh^4\left[\frac{m}{\sqrt{2}}(x - x_0)\right]} \tag{2.35}$$

This is indeed localized as desired,



To assign some classical mass we would find the total energy,

$$M \equiv E = \int_{-\infty}^{\infty} dx \mathcal{E}(x) = \frac{2\sqrt{2}}{3} \frac{m^3}{\lambda} \tag{2.36}$$

Note that λ here is not dimensionless since we are not working in $3+1$ dimensions, so this does indeed have dimensions of mass. We see that the mass, $M \propto \frac{1}{\lambda}$. This is something you will never get in perturbation theory.

2.1.4 Particle-like properties

To get the solution for all times can boost this solution,

$$x - x_0 \to \gamma(x - x_0 - vt)\,, \tag{2.37}$$

$\gamma \equiv (1 - v^2)^{-1/2}$. Boosting the field,

$$\phi \to \pm \frac{m}{\sqrt{\lambda}} \tanh\left[\frac{\gamma m}{\sqrt{2}}(x - x_0 - vt)\right] \tag{2.38}$$

with an energy density,

$$\mathcal{E} = \frac{\gamma m^4}{2\lambda} \operatorname{sech}^4\left[\frac{\gamma m}{\sqrt{2}}(x - x_0 - vt)\right] \tag{2.39}$$

Performing the integral over space gives as expected,

$$E = \gamma M \tag{2.40}$$

2.2 Complications and topology

We now consider more complicated soliton solutions.

2.2.1 Multiple scalar fields

First suppose we have multiple scalar fields,

$$\phi^i(x,t) \quad i = 1, ..., N \tag{2.41}$$

with the Lagrangian and equations of motion:

$$\mathcal{L} = \frac{1}{2}\dot{\phi}^{i\,2} - \frac{1}{2}\phi^{i\prime 2} - U(\phi^i) \quad \Rightarrow \quad \Box\phi^i = -\frac{\partial U}{\partial \phi^i} \tag{2.42}$$

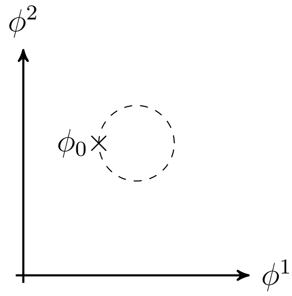
Static solutions can be found from,

$$\phi^{i\prime\prime} = \frac{\partial U}{\partial \phi^i} \tag{2.43}$$

The classical mechanical analogue is a N-dimensional motion of particle in $-U(\phi^i)$ potential. The total energy is,

$$E\left[\phi^i\right] = \int dx \sum_i \phi^{i\prime 2} + U(\phi^i) \tag{2.44}$$

The main difference here is that we can have a non-trivial solutions iwth one minimum for the potential. For example:



2.2.2 Topological charge

In both cases we've seen so far there is an important quantity which characterizes the field configuration, the difference in the vacuum at spatial infinity. We claim this is a conserved quantity suggesting the following definition:

Def. 1. Topologically Equivalent*: Two solutions are said to be topologically equivalent if they cannot be continuously* deformed *into one another without passing through a barrier of infinite action.*

Def. 2. Continuous Deformation*: For two solutions, $f_1(x)$ and $f_2(x)$, a continuous deformation, parameterized by $t \in [0,1]$, between the two solutions is a continuous function, $F(t,x)$, such that $F(0,x) = f_1(x)$ and $F(1,x) = f_2(x)$.*

Are the two kink solutions topologically equivalent? To check this we introduce a generic deformation,

$$\Phi(t,x) = \frac{m}{\sqrt{\lambda}}\left[g(t)\tanh y + h(t)\tanh(-y)\right] \quad y \equiv \frac{m}{\sqrt{2}}(x-x_0)\,, \tag{2.45}$$

$$= (g-h)\frac{m}{\sqrt{\lambda}}\tanh y\,, \tag{2.46}$$

where g,h are arbitrary continuous functions of t with $g(0)=h(1)=1$ and $g(1)=h(0)=0$. The static action is given by,

$$S(t) = \int dx \frac{1}{2}\Phi'^2 + \frac{\lambda}{4}\left(\Phi^2 - \frac{m^2}{\lambda}\right)^2 \tag{2.47}$$

$$= \frac{m^4}{4\lambda}\int dx (g-h)^2 \frac{1}{\cosh^4 y} + \left((g-h)^2\tanh^2 y - 1\right)^2 \tag{2.48}$$

The first integral is convergent, however the second diverges for $|g(t)-h(t)| \neq 1$. We have $g(0)-h(0)=1$ and $g(1)-h(0)=-1$. and since g and h are continuous, there must be a point in $t \in (0,1)$ for which the integral diverges. Thus we conclude that these two solutions are topologically inequivalent.

Thus at least for the kink the quantity,

$$\phi(\infty,t) - \phi(-\infty,t) \tag{2.49}$$

cannot be changed with a finite amount of energy and must be conserved. We will always assume that the non-trivial topological solutions we find are topologically inequivalent. Thus we can define a quantity known as the "topological charge" that quantifies the difference between vacua:

$$Q \propto \phi(\infty,t) - \phi(-\infty,t)\,, \tag{2.50}$$

where the constant of proporitionality is conventional. For the kink one can introduce,

$$Q \equiv \frac{\sqrt{\lambda}}{m}\left(\phi(\infty,t) - \phi(-\infty,t)\right) \tag{2.51}$$

Thus,

$$Q = \begin{cases} 2 & \text{(kink)} \\ 0 & \text{(trivial)} \\ -2 & \text{(anti-kink)} \end{cases} \tag{2.52}$$

Note that this charge does not follow from Noether's theorem, but instead is conserved independently from the equations of motion. Since we have a conserved charge, we expect some associated current. We can find the current by rewriting the charge,

$$Q = \frac{\sqrt{\lambda}}{m}\int dx \partial_x \phi(x,t) \tag{2.53}$$

Thus we know that $j^0 = \frac{\sqrt{\lambda}}{m}\partial_x\phi$. We want to make this into a four-vector. The only tensors we have at our disposal are $g^{\mu\nu}$, $\epsilon^{\mu\nu}$, and ∂^μ. Thus we must have,

$$j^\mu = \frac{\sqrt{\lambda}}{m}\epsilon^{\mu\nu}\partial_\nu\phi \tag{2.54}$$

This is divergenceless as required, $\partial_\mu j^\mu \propto \epsilon^{\mu\nu}\partial_\mu\partial_\nu\phi = 0$.

2.2.3 Derrick's theorem and $D > 1$

We now consider solitons in more than one spatial dimension, D. In such systems there is a theorem, known as **Derrick's theorem**, which restricts the type of systems that can have topological solutions. Consider a N-component scalar field, $\phi^i(\mathbf{x}, t)$ in D spatial dimensionn. The Lagrangian is given by,

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi^i\partial^\mu\phi^i - U(\phi^i) \tag{2.55}$$

The static equations of motion are the solutions of the equation,

$$\Box\phi^i = \frac{\partial U}{\partial\phi^i} \tag{2.56}$$

with static energy,

$$E = \underbrace{\int d^Dx\frac{1}{2}\partial_a\phi^i\partial_a\phi^i}_{\equiv E_1[\phi^i]} + \underbrace{\int d^DxU(\phi^i)}_{\equiv E_2[\phi^i]} \tag{2.57}$$

Now lets assume we find ϕ_1^i which is a solution of the equations of motion. Now consider the class of field configurations (which are generically not solutions),

$$\phi_\lambda^i(\mathbf{x}) \equiv \phi_1^i(\lambda\mathbf{x}) \tag{2.58}$$

The action for a given solution $\phi_\lambda(\mathbf{x})$ is given by,

$$\begin{aligned} S_{(\lambda)} &= -\int d^Dx\left(\frac{1}{2}\partial_a\phi_1^i(\lambda\mathbf{x})\partial_a\phi_1^i(\lambda\mathbf{x}) + U\left[\phi_1^i(\lambda\mathbf{x})\right]\right) && (2.59)\\ &= -\int d^Dy\lambda^{-D}\left(\frac{1}{2}\lambda^2\frac{\partial}{\partial y^a}\phi^i(\mathbf{y})\frac{\partial}{\partial y^a}\phi^i(\mathbf{y}) + U\left[\phi_1(\mathbf{y})\right]\right) && (2.60)\\ &= -(\lambda^{2-D}E_1 + \lambda^{-D}E_2) && (2.61) \end{aligned}$$

To have an extremum of the action,

$$\frac{dS_{(\lambda)}}{d\lambda} = (2-D)\lambda^{1-D}E_1 - D\lambda^{-D-1}E_2 = 0 \quad \Rightarrow \quad E_2 = \frac{2-D}{D}E_1 \tag{2.62}$$

Now the key observation, is that E_1 and E_2 are both positive definite quantities. This puts a restriction on the possible values of D. To this let us consider the different cases:

1. $D = 1$: In this case we find, $E_1 = E_2$. This says that in one spatial dimension the tension in the field is equal to its potential energy. This is an analogue to the viral theorem.

2. $D = 2$: This says that $E_2 = 0$ or equivalently, $U = 0$. The only known possibility where you have a non-trivial solution is in the non-linear σ model where $\phi^i\phi^i = a^2$.

3. $D > 2$: $E_2 < 0$! since E_2 is positive definite, there are no non-trivial solutions for $D > 2$.

Thus, the somewhat dissapointing conclusion is that in the $3+1$ dimensions scalar fields cannot have instanton solutions. Note that its still possible to lift lower dimensional solutions to $3+1$ dimensions but these won't be finite energy and hence do not have a particle interpretation (e.g., domain walls).

Chapter 3

Solitons in gauge theories

3.1 Review of gauge fields

Before plunging into solitons in gauge theories we quickly review the basics of non-abelian gauge theories. A group G is charactorized by the structure constants, f^{abc}, which are related to the generators of the group by,

$$\left[T^a, T^b\right] = -if^{abc}T^c \tag{3.1}$$

and $a = 1, 2, ..., \dim(G)$. The gauge fields are denoted by,

$$A_\mu \equiv A_\mu^a T^a \tag{3.2}$$

and the fields transform under the gauge symmetry through,

$$A_\mu \to gA_\mu g^{-1} + \frac{i}{e}(\partial_\mu g)g^{-1} \tag{3.3}$$

where e denotes the coupling constant, and g is a group element in the adjoint representation,

$$g \equiv e^{i\alpha_a T_a} \tag{3.4}$$

Explicitly we can write the gauge transformation in terms of the generators:

$$\begin{aligned} gA_\mu g^{-1} &= e^{i\alpha_a T_a}T^b e^{-\alpha_c T_c}A^b & (3.5)\\ &= A_\mu + i\alpha_a\left(T_aT_b - T_bT_a\right)A_b + ... & (3.6)\\ &= A_\mu + \alpha_a f_{abc}T_c A_\mu^b & (3.7)\end{aligned}$$

Furthermore,

$$\begin{aligned} \frac{1}{2}(\partial_\mu g)g^{-1} &= \frac{i}{e}\left(i\partial_\mu\alpha_a T_a\right)\left(1 - \alpha_a T_a\right) & (3.8)\\ &= -\frac{1}{e}\partial\alpha_a T_a & (3.9)\end{aligned}$$

Therefore,

$$A_\mu \to A_\mu - \frac{1}{e}\partial\alpha_a T_a - \epsilon_a f_{abc} T_c A_b \tag{3.10}$$

Under a gauge transformation the field strength tensor,

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - e f^{abc} A^b_\mu A^c_\nu \tag{3.11}$$

transformations in the adjoint representation,

$$F_{\mu\nu} \to g F_{\mu\nu} g^{-1} \tag{3.12}$$

3.2 Non-abelian Higgs theory

We now consider a non-abelian gauge theory with a Higgs field in a generic representation. The Lagrangian is given by,

$$\mathcal{L} = -\frac{1}{4}\mathrm{Tr} F_{\mu\nu}F^{\mu\nu} + |D_\mu\phi|^2 + U[\phi] \tag{3.13}$$

with the covariant derivative,

$$(D_\mu\phi)_m = \partial_\mu\phi_m + ieA^a_\mu t^a_{mn}\phi_n \tag{3.14}$$

Computing the equations of motion is straightforward. The final result is,

$$D_\mu F^{a\,\mu\nu} = ie\phi t^a D^\nu\phi + h.c. \tag{3.15}$$

$$D_\mu D^\nu \phi_m = \left(\frac{\partial U}{\partial \phi_m}\right)^* \tag{3.16}$$

We now look for static solutions,

$$\partial_0\phi = 0 \quad \text{(static)} \tag{3.17}$$

Claim 3. *There exists a gauge such that $A_0 = 0$. This is known as the temporal gauge.*

Proof. Consider a generic gauge configuration and apply a gauge transformation,

$$A_0 \to gA_0g^{-1} + \frac{i}{e}(\partial_0 g)g^{-1} \tag{3.18}$$

We need to show there exists a g such that,

$$gA_0g^{-1} = -\frac{i}{e}(\partial_0 g)g^{-1} \tag{3.19}$$

for any A_0. This condition can be rewritten,

$$g^{-1}\frac{\partial g}{\partial t} = ieA_0 \tag{3.20}$$

This is a well known diffential equation, familiar from constructing correlation functions in QFT, and has the solution,

$$g = \mathcal{T}\exp\left[ie\int A_0(t')dt'\right] \tag{3.21}$$

□

Thus for static solutions we can always work in a gauge such that,

$$D_0\phi = 0 \quad \text{(static)} \tag{3.22}$$

Note that this still does not completely fix the gauge as we can still make t-indepent gauge transformations.

In this gauge fixing,

$$F^a_{0i} = \partial_0 A^a_i - \partial_i A^a_0 - ef^{abc}A^b_0 A^c_i = \partial_0 A^a_i \tag{3.23}$$

which gives the action,

$$S = \int d^Dx\left[\frac{1}{2}(\partial_0 A^a_i)^2 + |\partial_0\phi|^2\right] + \int d^Dx\left[\frac{1}{4}(F^a_{ij})^2 + |D_i\phi|^2 + U\left[\phi\right]\right] \tag{3.24}$$

$$= 0 + E_1\left[F^2\right] + E_2\left[D_i\phi\right] + E_3\left[U\right] \quad \text{(static)} \tag{3.25}$$

Using this we can now rebuild the scaling arguement in the theory.

Assume we found a static solution to the equations of motion,

$$\phi_1(\mathbf{x})\,,\ A^\mu_1(\mathbf{x}) \tag{3.26}$$

Now consider the family of functions which are not solutions to the equations of motion, except at $\lambda = 1$:

$$\phi_\lambda(\mathbf{x}) = \phi_1(\lambda\mathbf{x})\,,\ A^\mu_\lambda(\mathbf{x}) = \lambda A^\mu_1(\lambda\mathbf{x}) \tag{3.27}$$

which implies,

$$F_{ij}(\mathbf{x}) \to \lambda^2 F_{ij}(\lambda\mathbf{x})\,,\ D_i\phi \to \lambda D_i\phi(\lambda\mathbf{x}) \tag{3.28}$$

This gives a rescaled action,

$$S\left[\phi_L, A^\mu_\lambda\right] = -\lambda^{-D}\int d^D(\lambda x)\frac{1}{4}\lambda^4 F_{ij}(\lambda\mathbf{x})F_{ij}(\lambda\mathbf{x}) + \lambda^2\left|D_i\phi\right|^2 + U(\phi) \tag{3.29}$$

$$= -\lambda^{-D}\left(\lambda^4 E_1 + \lambda^2 E_2 + E_3\right) \tag{3.30}$$

As before we should have an extremum in the action when $\lambda = 1$:

$$-D\lambda^{-D-1}\left(\lambda^4 E_1 + \lambda^2 E_2 + E_3\right) + \lambda^{-D}\left(4\lambda^3 E_1 + 2\lambda E_2\right) = 0 \tag{3.31}$$

or

$$(4-D)E_1 + (2-D)E_2 - DE_3 = 0 \tag{3.32}$$

As before, lets consider the different cases

1. $D = 1$: Gauge fields don't propagate with one spatial dimension, so we don't expect any interesting instanton effects here

2. $D = 2$: Our master equation tells us that $E_1 = E_3$. For this peculiar possibility, the scalar potential energy is exactly equal to the gauge field energy. This is known as the "vortex" solution.

3. $D = 3$: The master equations says that $E_1 - E_2 - 3E_3 = 0$. This corresponds to a monopole solution.

4. $D = 4$: Here the master equations gives, $-2E_2 - 4E_3 = 0$. This can only be a solution if $E_2 = E_3 = 0$. In other words in four spatial dimensions soliton solutions can exist only if the scalar field has vanishing energy $\Rightarrow$ solution involves only the gauge field. These will be of fundamental importance when we discuss instantons but we postpone this discussion until then.

3.3 Vacua of spontaneously broken gauge theories

We now study the vacuum of a gauge theory with a manifold of vacua, instead of finite set of discrete vacua. The concept of a manifold of vacua is familiar from spontaneously broken symmetries. Since we are interested in $D = 3$, we study theories with scalar and gauge fields. Both of these will have a non-trivial vacuum configurations. The perturbative vacuum is simmply,

$$\phi_0 = \text{const}\,,\; A^a_\mu = 0 \tag{3.33}$$

However, our interest will be in more intricate vacua.

Consider the gauge group, G (with elements, g), broken to a subgroup, H (with elements h). We denote a vacuum state by ϕ_0. Since the vacuum is invariant under the subgroup we have,

$$h\phi_0 = \phi_0 \tag{3.34}$$

Gauge fields in H will remain massless and the other gauge bosons (those in G/H) pick up a mass. $U(\phi)$ must be invariant under, $\phi \to g\phi$, which in turn also implies,

$$U(g\phi_0) = 0 \tag{3.35}$$

Thus the set of vacua related to ϕ_0 by a full gauge transformation correspond to minima. While $g\phi_0$ always corresponds to a minima its non-trivial to know whether this will correspond to **all** minima.

Claim 4. *If $g\phi_0$ can generate all minima than the minimal set of generators that can generate all minima, is the "coset", G/H, defined by $g_1 = g_2 h$.*

Proof. Let us parameterize all minima by a vector, θ: $\phi_0(\theta)$. By assumption it is given by a group element acting on a single vacuum,

$$\phi_0(\theta) = g_1\phi_0(0) \tag{3.36}$$

Generically this choice of g will not be unique and so we'll also have

$$\phi_0(\theta) = g_2\phi_0(0) \tag{3.37}$$

This implies that,

$$g_2^{-1}g_1\phi_0(0) = \phi_0(0) \tag{3.38}$$

which is the defining characteristic of the unbroken subgroup. Thus,

$$g_2^{-1}g_1 = h \quad \Rightarrow \quad g_1 = g_2 h \tag{3.39}$$

Thus the elements g which are identified with $g_1 = g_2 h$ can generate all the minima. □

One subtlety to the above is sometimes the invariance of the potential ($\bar{G}$) is a bit bigger than that of the full gauge group, G. Then subgroup may also be a bit smaller, $\bar{H}$. What you are really after is $\bar{G}/\bar{H}$. We will see an example of this shortly when consider SU(5).

3.3.1 Georgi-Glashow model

The Georgi-Glashow model was a candidate for the weak interactions before the confirmation of the Z boson. The model comprises of $SU(2)$ gauge symmetry with a Higgs triplet, ϕ_a and a potential,

$$U = \frac{\lambda}{4}\left(\phi^i\phi^i - a^2\right)^2 \tag{3.40}$$

The symmetry of the potential is, $SU(2)$.

Claim 5. *SU(2) is isomorphic to S^3 (3D sphere embedded in 4D)*

Proof. An arbitrary element of SU(2) can be written as,

$$\begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix} \tag{3.41}$$

with the condition $\left|a^2\right|^2 + \left|b\right|^2 = 1$. Writing $a = y_1 + iy_2$ and $b = y_3 + iy_4$ gives the condition,

$$y_1^2 + y_2^2 + y_3^2 + y_4^2 = 1 \tag{3.42}$$

which is the defining equation for 3-sphere. □

The Higgs gains a vacuum expectation value (VEV) which by gauge symmetry can be taken to be in the first component,

$$\phi = \begin{pmatrix} h+a \\ \phi_2 \\ \phi_3 \end{pmatrix} \tag{3.43}$$

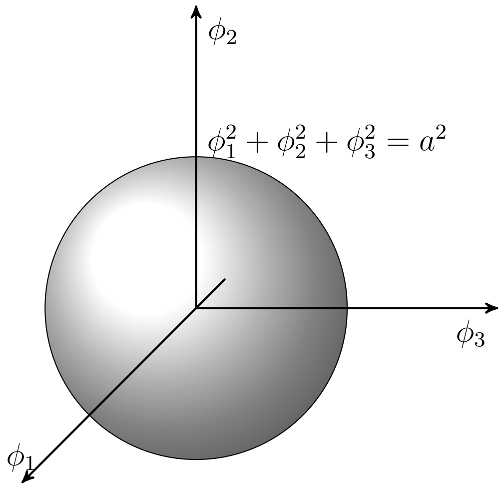
Then,

$$V = \frac{\lambda}{4}\left(\phi_2^2 + \phi_3^2 + 2ha\right)^2 \tag{3.44}$$

The h field is massive and ϕ_2 and ϕ_3 are massless giving 2 goldstone bosons (GB). The invariant subgroup is $H = SO(2)$ and corresponds to rotations between ϕ_2 and ϕ_3. The vacuum manifold is given by $SU(2)/U(1)$ which is a 2-sphere, S^2 (a sphere embedded in a 3 dimensional space). This is because the vacua are characterized with,

$$\phi_1^2 + \phi_2^2 + \phi_3^2 = a^2 \tag{3.45}$$

which is the defining equation of a 2-sphere.



3.3.2 SU(5)

Next consider G =SU(5), a candidate for grand unification of the SM, broken by a scalar adjoint Higgs, ϕ, to $SU(3) \times SU(2) \times U(1)$. A scalar adjoint can be formed from a fundamental and anti-fundamental, forming a hermitian tracless 2×2 matrix transforming as $\phi \to g\phi g^{-1}$. Usually the potential which gives rise to this breaking has an accidental Z_2 symmetry under the transformation,

$$\phi \to \begin{pmatrix} -1 & & & & \\ & -1 & & & \\ & & -1 & & \\ & & & -1 & \\ & & & & -1 \end{pmatrix} \phi \tag{3.46}$$

This is not an element of $SU(5)$ (it doesn't have determinant equal to 1). Thus the potential is invariant under

$$\bar{G} = SU(5) \times Z_2 \tag{3.47}$$

An adjoint of the VEV,

$$\phi_0 = v \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & -3/2 & \\ & & & & -3/2 \end{pmatrix} \tag{3.48}$$

can produce the desire symmetry pattern. H is made of,

$$\left\{ M_1 \equiv \left(\begin{array}{c|c} g_1 & \\ \hline & \mathbb{1}_{2\times 2} \end{array} \right) , \; M_2 \equiv \left(\begin{array}{c|c} \mathbb{1}_{3\times 3} & \\ \hline & g_2 \end{array} \right) , \; M_3 \equiv \left(\begin{array}{c|c} e^{2i\theta} \mathbb{1}_{3\times 3} & \\ \hline & e^{-3\theta/2} \mathbb{1}_{2\times 2} \end{array} \right) \right\} \tag{3.49}$$

Clearly,

$$H\phi_0 = h\phi h^{-1} = \phi_0 \tag{3.50}$$

as desired. However, not all of the vacua at distinct, e.g., $g_1 = e^{-2\pi/3}\mathbb{1}_{3\times 3}$, $g_2 = -\mathbb{1}_{2\times 2}$, $\theta = \pi/3$. Then

$$M_1 M_2 M_3 = \mathbb{1}_{5\times 5} \tag{3.51}$$

Therefore, the invariance group is not $SU(3) \times SU(2) \times U(1)$ but smaller. One can show that the actual subgroup is $SU(3) \times SU(2) \times U(1)/Z_6$. This will effect properties of the monopole!

3.4 Topological solutions with SSB

We now repeat the procedure to find topological solutions. As before we work in the temporal gauge. Furthermore, it will be convenient to work in polar coordinates. We claim without proof that one can use the remaining gauge freedom to fix the gauge such that,

$$A_r^a = 0 \tag{3.52}$$

In order to have a finite energy density we must have,

$$\int d^D x \, |D_i \phi|^2 \xrightarrow{r\to\infty} 0 \tag{3.53}$$

Due to our gauge fixing condition this implies,

$$\partial_r \phi \xrightarrow{r\to\infty} 0 \tag{3.54}$$

and so ϕ will asymptotically be r independent,

$$\phi_\infty(r, \theta_i, t) = \phi_\infty(\theta_i, t) \tag{3.55}$$

The field values at ∞ are called the "sphere at infinity". We learned from our experience with solitons that different vacuum configurations of the field at the boundaries, yields different topological solutions. Thus this sphere at (spatial!) infinity, should somehow be mapped to the different vacua. In D spatial dimensions this implies a mapping,

$$S^{D-1} \rightarrow G/H \tag{3.56}$$

The mathematical tool to compute these mappings is known as "homotopy theory", which we'll study in more detail soon. However, in many cases you can use physical arguments to compute most of what you need to understand the topology, as we did for $SU(2) \rightarrow U(1)$ where we found we needed a mapping from $S^2 \rightarrow S^2$.

Our goal is to find **non-trivial** classical solutions to the equations of motion. Thus we want the derivatives of the fields to be non-vanishing. However, equation 3.53 combined with equation 3.54, greatly restrict the possible solutions! We must have,

$$\lim_{r\to\infty} D_{\theta_i}\phi = \lim_{r\to\infty} \frac{1}{r}\frac{1}{\sin\theta ...}\frac{\partial\phi_\infty}{\partial\theta_i} + iA_{\theta_i}\phi_\infty = 0 \tag{3.57}$$

Since ϕ is asymptotically r-independent, we know how A_{θ_i} behaves at large r:

$$\lim_{r\to\infty} A_{\theta_i} \propto \frac{1}{r} \tag{3.58}$$

Recall that we are interested in finite action solutions. For a potential falling off as $1/r$ is the energy in the E and B fields finite? Equation 3.24 shows that the energy in our gauge is,

$$\frac{1}{4}\int d^D x (F^a_{ij})^2 \sim \int d^{D-1}\theta \int dr r^{D-1}\frac{1}{r^4} \tag{3.59}$$

which is convergent if $D - 5 \leq -2$ or $D \leq 3$. Thus we conclude:

Finite action, **non-trivial**, solutions to the classical equations of motion can only exist in spontaneously broken gauge theories if $D \leq 3$.

3.5 Vortex solution

The *vortex* is a topological solution of a spontaneously broken abelian gauge symmetry, also known as "Landau-Ginsburg" theory. We will focus on the case of 2+1 dimensions, known as the Neilson-Oloson vertex. It has important physical consequences as it forms a theory of superconductivity through a dynamical breaking of $\mathrm{U}(1)_{\mathrm{EM}}$ which occurs throgh condensation of electron pairs giving a non-zero vacuum expectation value,

$$\langle\Omega|e^- e^-|\Omega\rangle \neq 0 \tag{3.60}$$

The unbroken group is $G = \mathrm{U}(1)$ with a complex scalar, ϕ, which in superconductivity, forms the electron pair: $\phi \equiv \frac{1}{f^2} e_L^- e_R^-$, where f is some dimensionful scale. We will take the scalar potential to be,

$$V(\phi) = \frac{\lambda}{4} \left(|\phi|^2 - a^2 \right)^2 \quad \lambda > 0 \tag{3.61}$$

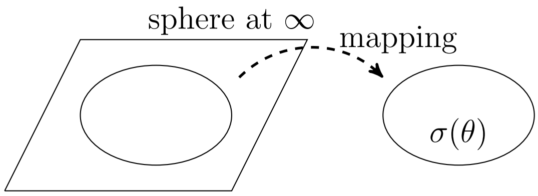
The vacuum manifold (parametrized by $G/H = \mathrm{U}\ (1) \simeq S^1$) is given by,

$$\left\{ \phi_0(\sigma) = a e^{i\sigma} \right\} \tag{3.62}$$

Since we are working in 2 spatial dimensions, the sphere at infinity is **also** a circle,

$$\phi(\infty, \theta) \, : \, S^1 \to S^1 \tag{3.63}$$

To describe the boundary conditions we need a mapping between $\theta \to \sigma$, i.e., we need $\sigma(\theta)$. Pictorally,



We are interested in homotopically inequivalent mappings.

Def. 3. Homotopically Equivalent: *Two mappings, $\sigma_1(\theta_i)$ and $\sigma_2(\theta_i)$ are homotopically equivalent if there exists a linear combination of the mappings with a parameter t such that at $t = 0$ its equal to σ_1 and at $t = 1$ its equal to σ_2 while keeping the base-point fixed.*

Def. 4. Based point: *A base point of a mapping, $\sigma(\theta_i)$ is reference point of the mapping where $\sigma(\theta_{i,0})$ is always equal to some constant, σ_0*

Before studying this model in detail lets give a few examples. First consider the mappings,

$$\sigma_1(\theta) = \theta \quad \sigma_2(\theta) = k\theta, k \in \mathbb{Z} \tag{3.64}$$

The basepoint we have chosen is $\theta_0 = 2\pi p$, $\sigma_0 = 2\pi p, p \in \mathbb{Z}$. Clearly both these solutions obey this condition. Are these homotopically equivalent? Consider,

$$t\sigma_1(\theta) + (1 - t)\sigma_2(\theta) = [k + t(1 - k)]\, \theta \tag{3.65}$$

This homotopy does not keep the basepoint fixed unless $k = 1$ for all t and hence these solutions are not homotopically equivalent. We see that solutions which go around the sphere at infinity a different number of times are homotopically equivalent.

Alternatively, consider two mappings which both go around the sphere at infinity with different "speeds":

$$\sigma_1(\theta) = \theta \quad \sigma_2(\theta) = 2\pi \sin\frac{\theta}{4} \tag{3.66}$$

These are homotopically eqiuvalent since,

$$t\sigma_1(2\pi) + (1-t)\sigma_2(2\pi) = 2\pi \tag{3.67}$$

and hence keeps the basepoint fixed.

Lastly consider the trivial mapping and a mapping which doesn't go once around the vacuum:

$$\sigma_1(\theta) = 0 \quad \sigma_2(\theta) = \begin{cases} \theta & 0 \geq \theta \geq \pi \\ (2\pi - \theta) & \pi \geq \theta \geq 2\pi \end{cases} \tag{3.68}$$

The linear combination:

$$t\sigma_1(2\pi) + (1-t)\sigma_2(2\pi) = 0 \tag{3.69}$$

and so these are homotopically equivalent.

We see that the key concept in finding charatorizing mappings is the number of times we go around the circle in coset space as we go once around the circle at infintiy. This is known as the *winding number.* The above consider show[1] that mappings are homotopically inequivalent **if and only if** they have the different winding numbers. As we'll see there is an infinite number of such solutions, each having a distinct energy.

Explicitly, the winding number can be written,

$$n = \frac{\sigma(2\pi) - \sigma(0)}{2\pi} \tag{3.70}$$

$$= \frac{1}{2\pi}\int_0^{2\pi} d\theta \frac{d\sigma(\theta)}{d\theta} \tag{3.71}$$

We can compute the action explicitly for the vacuum manifold, $\phi = ae^{i\theta}$. The action is given by,

$$-\int d^2x \left[\frac{1}{4}F_{ij}F^{ij} + |D_i\phi|^2 + U(\phi)\right] \tag{3.72}$$

For static solutions the energy is always equal to the negative of the action. Thus to avoid keeping track of a pesky negative sign, we'll always work with the energy and look for extrema in the energy instead. Since we are working with an abelian 2D gauge theory we can simplify the gauge field:

$$F_{12} = \partial_x A_y - \partial_y A_x = \epsilon_{ij}\partial_i A_j = B \tag{3.73}$$

At $r \to \infty$,

$$\phi(\infty, \theta) = ae^{i\sigma(\theta)} \tag{3.74}$$

[1] Technically we didn't actually show that solutions with different winding numbers are homotopically inequivalent since we only tried one candidate mapping.

Inserting this into equation 3.57 we get,

$$\lim_{r\to\infty} D_\theta \phi = \lim_{r\to\infty} \left(\frac{a}{r} i\partial_\theta \sigma - ieA_\theta a \right) e^{i\sigma} = 0 \tag{3.75}$$

or,

$$\frac{d\sigma}{d\theta} = e \lim_{r\to\infty} rA_\theta \tag{3.76}$$

Note that earlier we concluded that $A_\theta \propto \frac{1}{r}$ so this is a finite quantity. This can be used to compute the winding number:

$$n = \frac{e}{2\pi} \int_0^{2\pi} d\theta \lim_{r\to\infty} rA_\theta \tag{3.77}$$

$$= \frac{e}{2\pi} \int \mathbf{A} \cdot d\mathbf{s} \tag{3.78}$$

$$= \frac{e}{2\pi} \int \mathbf{B} \cdot d\mathbf{a} \tag{3.79}$$

This is just the magnetic flux passing through the circle at infinity giving the quantization condition,

$$\Phi = \frac{2\pi}{e} n \tag{3.80}$$

Thus we conclude that for $n \neq 0$ this theory has quantized magnetic charge. This is indeed the case in a superconductor.

Lets pause and summarize what happened here. We studied the classical solutions to a 2+1 dimensional spontaneously broken U(1) gauge theory and looked for solutions with non-trivial vacua and finite energy. We found that there exists an infinite set of such solutions, described by a discrete quantum number: the winding number. There will also be solutions with trivial vacua, such that the quantum theory contains excitations of an ordinary scalar field, an ordinary massive gauge field, and of a novel ground state, whose gauge field configuration produces a non-zero magnetic flux. In a quantum theory there will be small fluctuations about this vacuum. However, these small fluctuations (by definition of being small) won't be able to take your out of this vacuum. Thus this quantity is a conserved quantity often referring to as a *topological charge*.

Since we have a charge there should be a corresponding current. We want something which obeys, $\partial_\mu j^\mu = 0$ and which yields,

$$\int d^2x j^0 = -\frac{e}{2\pi} \int d^2x \epsilon_{ij} \partial_i A_j \tag{3.81}$$

$$= -\frac{e}{2\pi} \int d^2x \epsilon^{0\mu\nu} \partial_\mu A_\nu \tag{3.82}$$

which suggests,

$$j^\mu \equiv \frac{e}{2\pi} \epsilon^{\mu\nu\rho} \partial_\nu A_\rho \tag{3.83}$$

$\partial_\mu j^\mu = 0$ as a consequence of the antisymmetry of the levi-cevita desired.

3.5.1 Critical Vortex

Obtaining solutions to the coupled, second order, partial differential equations for ϕ and A_μ is a difficult problem. It would be much nicer if we could reduce these equations to first order as we did for the kick solution. Unfortunately, this cannot be done in general but we can recast the solutions into a more convenient form in a certain limit. The idea is to use that solutions will minimize the action. If we write the action in terms of positive definite quantities we can have a better idea of when the action will be minimized.

Consider the action of the static solution,

$$E = \int d^2x \left[\frac{1}{2} F_{12}^2 + |D_1\phi|^2 + |iD_2\phi|^2 + \frac{\lambda}{4}\left(|\phi|^2 - a^2\right)^2 \right] \tag{3.84}$$

where we write the kinetic term in a form convenient for the discussion below. The masses of the radial mode and the gauge boson are, $m_A^2 = 2e^2a^2$ and $m_\phi^2 = \frac{1}{2}\lambda a^2$, respectively. Shifting terms around in the action gives,

$$\begin{aligned} E = \int d^2x \Bigg[&\frac{1}{2}\left(F_{12} + \frac{\sqrt{\lambda}}{2}(|\phi|^2 - a^2)\right)^2 + |(D_1 + iD_2)\phi|^2 - i\left((D_1\phi)^\dagger D_2\phi - (D_2\phi)^\dagger D_1\phi\right) \\ &- \frac{\sqrt{\lambda}}{2} F_{12}(|\phi|^2 - a^2) \Bigg] \end{aligned} \tag{3.85}$$

Using,[2]

$$\int d^2x (D_1\phi)^\dagger D_2\phi - (D_2\phi)^\dagger D_1\phi = ie \int d^2x \phi^\dagger F_{12}\phi \tag{3.86}$$

we have,

$$E = \int d^2x \left[\frac{1}{2}\left(F_{12} + \frac{\sqrt{\lambda}}{2}(|\phi|^2 - a^2)\right)^2 + |(D_1 + iD_2)\phi|^2 + \left(e - \frac{\sqrt{\lambda}}{2}\right)|\phi|^2 F_{12} \right] + \frac{\sqrt{\lambda}}{2} a^2 \Phi \tag{3.87}$$

The first two terms are positive definite. The last term is a (quantized) topological term. The third term gives the coupling between the scalar and the gauge boson and its sign, which can be positive or negative depending on the relative value of λ and e, has a crucial

[2] Note that,

$$\int d^2x (D_i\phi)^\dagger D_j\phi = \int d^2x (-\phi^\dagger \partial_i + ie\phi^\dagger A_i) D_j\phi = -\int d^2x \phi^\dagger D_i D_j \phi$$

and

$$F_{12} = \frac{i}{e}\left[D_1, D_2\right]$$

effect on the qualitative features of the model. The relevant ratio is an order parameter and can be written in terms of the masses,

$$\frac{m_\phi}{m_A} = \frac{\sqrt{\lambda}/2}{e} \tag{3.88}$$

If $m_\phi < m_A$ then we have a type I supeconductor. Here the vortices attact each other destropying superconductivity relatively quickly. If $m_\phi > m_A$ then we have a type II superconductor. Here the vortices repel and don't combine together. This leads to a much larger critical magnetic field.

To get the rough behavior of the solutions its interesting to consider the regime that $m_\phi = m_A$, known as the *critical vortex*. In this case the total energy is just,

$$E = \int d^2x \left[\frac{1}{2} \left(F_{12} + \frac{\sqrt{\lambda}}{2}(|\phi|^2 - a^2) \right)^2 + |(D_1 + iD_2)\phi|^2 \right] + 2\pi n a^2 \tag{3.89}$$

We can now reduce the corresponding second order equations by making a bold assumption: the solutions to the equations of motion will minimize these positive definite quantities, lets assume that the terms are identically zero. In this case we get the corresponding equations known as the *BPS* equations:

$$F_{12} + e\left(|\phi|^2 - a^2\right) = 0 \tag{3.90}$$

$$(D_1 + iD_2)\phi = 0 \tag{3.91}$$

For simplicity lets focus on the case where $n = 1$. We want to use an satz consistent with our boundary conditions,

$$\phi(\infty, \theta) = ae^{i\theta} \tag{3.92}$$

$$\lim_{r\to\infty} rA_\theta = \frac{1}{e} \tag{3.93}$$

and ϕ, A_θ finite as $r \to 0$. This gives the generic form,[3]

$$\phi(r, \theta) = ae^{i\theta}\varphi(r) \tag{3.94}$$

$$A_\theta(r, \theta) = \frac{1}{e}\frac{1 - f(r)}{r} \quad \text{or} \quad A_i = \frac{1}{e}\frac{\epsilon_{ij}x_j}{2r^2}(1 - f(r)) \tag{3.95}$$

[3]Recall that to convert from cartesian to polar coordinates:

$$\begin{pmatrix} A_x \\ A_y \end{pmatrix} = \begin{pmatrix} c_\theta & -s_\theta \\ s_\theta & c_\theta \end{pmatrix} \begin{pmatrix} A_r \\ A_\theta \end{pmatrix}$$

$$\begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix} = \begin{pmatrix} c_\theta & -s_\theta \\ s_\theta & c_\theta \end{pmatrix} \begin{pmatrix} \partial_r \\ \frac{1}{r}\partial_\theta \end{pmatrix}$$

The field strength is given by,

$$eF_{12} = e\epsilon_{ij}\partial_i A_j \tag{3.96}$$
$$= \frac{1}{2}\partial_k \left[x_k \frac{1-f(r)}{r^2} \right] \tag{3.97}$$
$$= -\frac{1}{2}\frac{1}{r}\frac{\partial f}{\partial r} \tag{3.98}$$

and

$$(D_1 + iD_2)\phi = e^{i\theta}\left(D_r + iD_\theta\right) ae^{i\theta}\varphi(r) \tag{3.99}$$
$$= ae^{2i\theta}\left[\partial_r\varphi(r) - \left(\frac{1}{r} - eA_\theta\right)\varphi\right] \tag{3.100}$$
$$= ae^{2i\theta}\left[\partial_r\varphi(r) - \frac{f(r)}{r}\varphi\right] \tag{3.101}$$

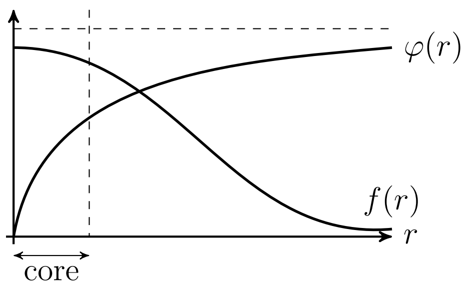
which leads to the BPS equations:

$$\frac{1}{r}f' - 2e^2a^2\left(\varphi^2 - 1\right) = 0 \tag{3.102}$$
$$\varphi' - \frac{f}{r}\varphi = 0 \tag{3.103}$$

The boundary conditions are,

$$\left\{\begin{array}{l} \lim_{r\to\infty}\varphi(r) = 1 \\ \lim_{r\to\infty} f(r) = 0 \\ \lim_{r\to 0}\varphi(r) = 0 \\ \lim_{r\to 0} f(r) = 1 \end{array}\right\} \tag{3.104}$$

Pictorially it looks like,



At the core of the vortex, the Higgs field vanishes and the gauge symmetry is unbroken. However, at large distances it will go its usual value, $|\phi| = a$. Roughly we have,

$$f(r) \sim e^{-m_\phi r}, 1 - \varphi(r) \sim e^{-m_\phi r} \tag{3.105}$$

Chapter 4

Homotopy Theory

4.1 Basics

In our discussions of topological solutions in theories with broken continuous symmetries, we need to know how to map the sphere at infinity to the vacuum manifold, i.e.,

$$S^n \to \mathcal{M} \tag{4.1}$$

As our base point in the n-dimensional sphere(p_0) we take the north pole, and the corresponding base point in the vacuum manifold as, m_0. In other words we need to find a function,

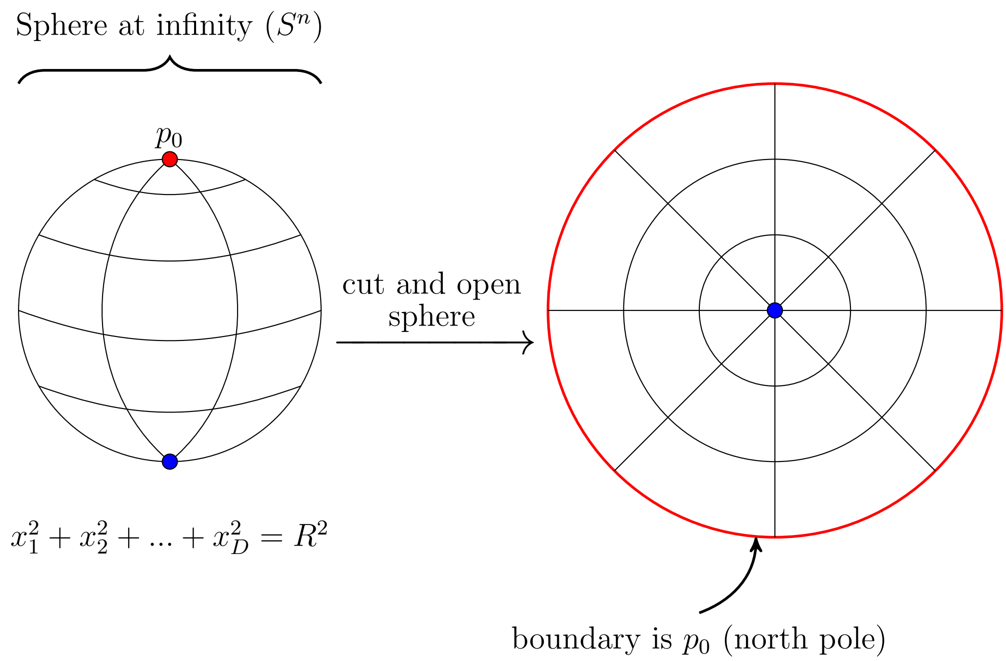
$$\{f: \quad S^n \to \mathcal{M} | f(p_0) = m_0\} \tag{4.2}$$

There is a set of such functions which are homotopically equivalent. We denote this set by $[f]$.

Def. 5. Homotopy class: *The set of all mappings between $S^n \to \mathcal{M}$ which are homotopically equivalent.*

Def. 6. Homotopy group: *The set of homotopy classes. We denote this group by, $\pi^n(\mathcal{M}, m_0)$.*

The fact that the homotopy group forms a group is not obvious. We will show this soon. In order to prove such statements in homotopy theory pictorally it is convenient to first define some useful concepts.



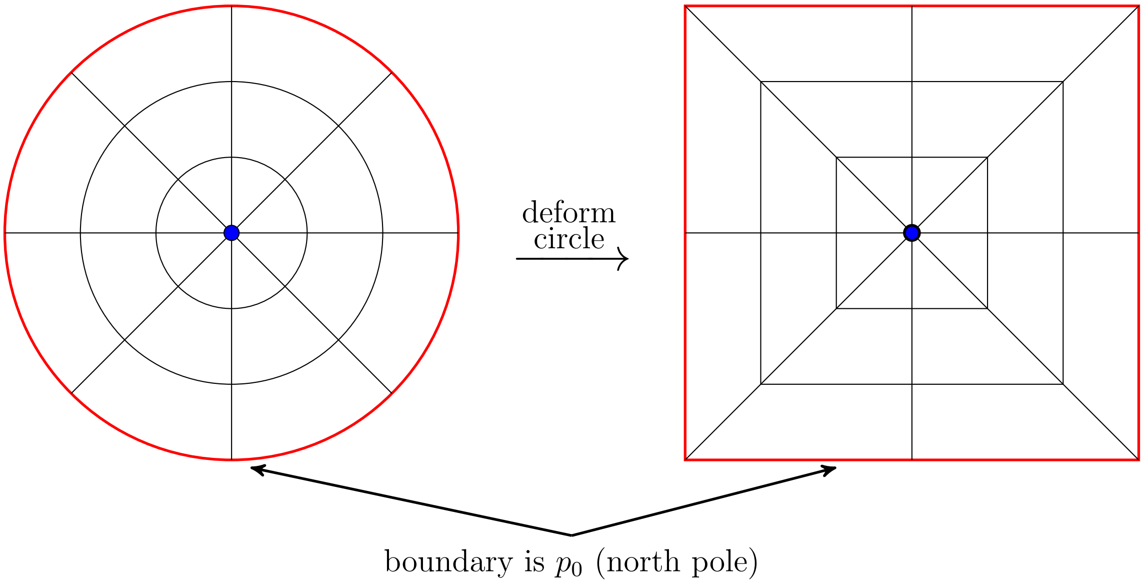
The flattened sphere has 1 less coordinate but an inequality instead of an equality as its defining equation. In particular we can choose x_D to be the distance from the south pole. Then $\{x_1, x_2, ..., x_{D-1}\}$ define the direction in flattened sphere. The D-dimensional coordinates are given by,

$$y_i = \frac{1}{2}(1 + \frac{x_D}{R})\frac{x_i}{x} \tag{4.3}$$

where,

$$\bar{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_{D-1} \end{pmatrix} \quad x \equiv \sqrt{x_1^2 + x_2^2 + ... + x_{D-1}^2} \tag{4.4}$$

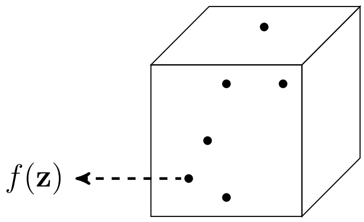
To simplify the topology of the mapping, we deform this map to be the interior of a square in 3D, or more generally, a hypercube:



In particular we use the coordinates,

$$z_i = \frac{1}{2}\left(1 + \frac{r}{M} y_i\right) \quad r \equiv \sqrt{\sum_i y_i^2}\,, i = 1, 2, ..., D-1 \tag{4.5}$$

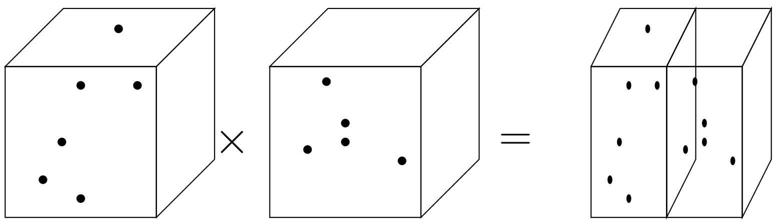
where the z_i now denote the distances along each direction in this hypercube. In $D = 3$ (3 spatial dimensions), its the two directions in a square. These are what we will call the "standard coordinates". A given map can be represented pictorally as,



With this machinery we can define group product $f = f_2 \cdot f_1$ as

$$f(z_1, z_2, ..., z_{D-1}) = \begin{cases} f_1(z_1, ..., z_{D-2}, 2z_{D-1}) & z_{D-1} \leq 1/2 \\ f_2(z_1, ..., z_{D-2}, 2z_{D-1}) & z_{D-1} \geq 1/2 \end{cases} \tag{4.6}$$

In words: you take two maps and squash them together such that the product map is equal to the first map for $z_{D-1} \leq 1/2$ and the second map for $z_{D-1} \geq 1/2$. The rescaling by a constant of z_{D-1} shows that the maps are "squashed". Further, we emphasize that for each map p_0 is always on the boundary and so the product map still has p_0 on the boundary. Pictorally we have,



One can show that we can apply this product to equivalence classes:

$$[f_2] \cdot [f_1] = [f_2 \cdot f_1] \tag{4.7}$$

i.e., $[f_2] \cdot [f_1]$ is independent of choice on f_2 and f_1. Furthermore, one can show this product defines a group, i.e., that it contains an identity element, is associative, and has an inverse. Furthermore, one can show that $n \geq 2$, the group is commutative (abelian). The group is denoted,

$$\pi^n(\mathcal{M}) \tag{4.8}$$

where for $n = D - 1$ for the sphere at infinity.

4.2 Important examples

We now present some relevant examples which we will refer back to throughout these notes. Consider the mappings from a circle to a 2-sphere order higher dimensional space:

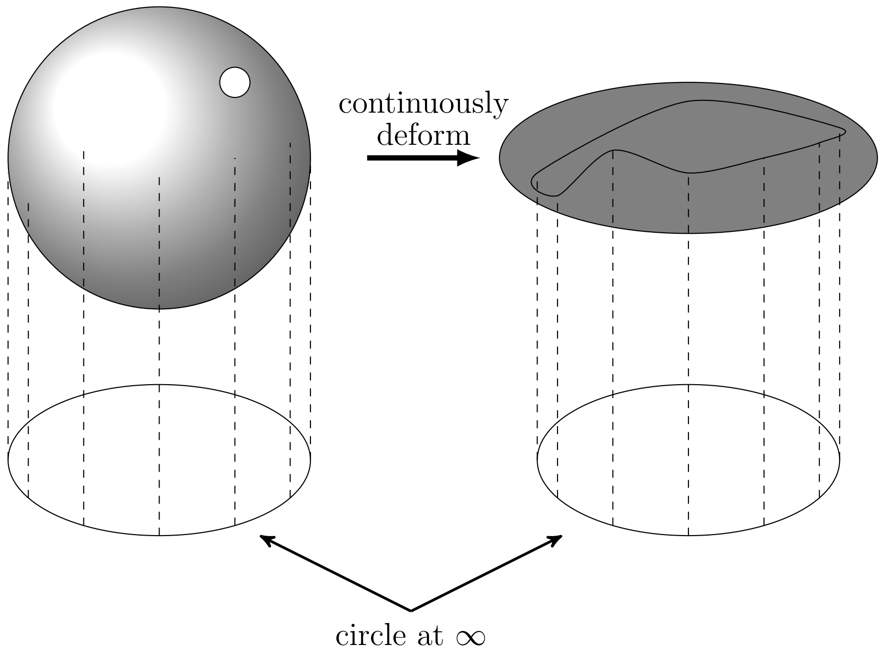
$$\pi^1(S^m) \quad m \geq 2 \tag{4.9}$$

This can be written as a mapping:

$$S^1 \to S^m \tag{4.10}$$

Claim 6. *The only element in $\pi^1(S^m)$ is the unit element.*

Proof. Since S^m is a larger space then S^1 there must at least one point in S^m which is not mapped to in S^1. Thus we are free to consider the mapping removing this point in S^m. This mapping is homotopically equivalent to a disk (i.e., it can always be continuously deformed to a disk) so pictorally:



Any circle will always map to some closed path on the disk. This path can always be continuously deformed to a point. This is easy to show explicitly for a circular image path, charactorized by an angle θ. We denote the point in the image by parameters, x and y. In this case we can consider the homotopy,

$$\mathbf{F}(t,\theta) = r_0 \begin{pmatrix} t\cos\theta + (1-t) \\ t\sin\theta \end{pmatrix} \tag{4.11}$$

where we have chosen as or base point $\mathbf{F}(t,2\pi) = (r_0, 0)$. For $t = 1$ this is just a circular path and for $t = 1$ this is just a point in the image and this always has the basepoint, $x(t,2\pi) = r_0, y(t,2\pi) = 0$.

While we only show this explicitly for a circular path its easy to convince yourself this will hold for a generic path as long x and y are valid coordinates (i.e., there are no topological defects, e.g., a donut shape where clearly x, y coordinates would not make sense. Thus this mapping is homotopically equivalent to the mapping to a constant and hence $\pi^1(S^m)$ is the unit element. □

More generally we have,

Claim 7. *For any $m > n$: $\pi^n(S^m) = \emptyset$*

We defer proof of this claim to a later lecture once we've developed a bit more machinery.

Claim 8. $\pi^1(S^1) = \mathbb{Z}$ *(the homotopy group of the circle at infinity to a circle is the group integers under addition).*

Proof. Recall that in equation 3.65 we considered this exact mapping:

$$\begin{array}{cccc} f: & S^1 & \rightarrow & S^1 \\ & 0 \le \theta \le 2\pi & & 0 \le \sigma \le 2\pi \end{array} \tag{4.12}$$

We found the mappings, $k_1\theta$ and $k_2\theta$, were homotopically equivalent only if $k_1 = k_2$. Therefore, we conclude that the homotopy group is composed of an infinite set of homotopies charactorized by an integer. □

Claim 9. *We can generalize the previous relation: $\pi^n(S^n) = \mathbb{Z}$*

Proof. We can charactorize the sphere at infinity and the coset sphere by two sets of angles:

$$\boldsymbol{\theta} = (\theta_1, \theta_2, ...) \tag{4.13}$$

$$\boldsymbol{\sigma} = (\sigma_1(\boldsymbol{\theta}), \sigma_2(\boldsymbol{\theta}), ...) \tag{4.14}$$

The volume of the domain is given by,

$$V_{\rm dom} = \int_{S^n} \sqrt{g(\boldsymbol{\theta})} d\theta_1 d\theta_2 ... d\theta_n \tag{4.15}$$

The volume of the range is,

$$V_{\text{ran}} = \int_{S^n} \sqrt{g(\boldsymbol{\sigma})} d\sigma_1 d\sigma_2 ... d\sigma_n \tag{4.16}$$

$$= \int_{S^n} \sqrt{g(\boldsymbol{\sigma})} \det \left(\frac{d\sigma_i}{d\theta_j} \right) d\theta_1 d\theta_2 ... d\theta_n \tag{4.17}$$

where the integral runs over the full range of the sphere, i.e., its an integral including the potential multiple coverings of the group. The ratio of these volumes will be given by the "covering number" or the number of times the range covers the domain:

$$k \equiv \frac{V_{\text{ran}}}{V_{\text{dom}}} \tag{4.18}$$

As we've seen the σs form an inequivalent homotopy if they cover the θs a different number of times. Thus, k is an integer. □

As an example of the above consider $\pi^1(S^1)$ where $\sigma = k\theta$:

$$V_{\text{dom}} = \int_0^{2\pi} d\theta = 2\pi \tag{4.19}$$

$$V_{\text{ran}} = \int_0^{2\pi k} d\sigma = k \int_0^{2\pi} d\theta = 2\pi k \tag{4.20}$$

Lets now consider the cases where the image is "smaller" than the sphere at infinity. One can show:

Claim 10. $\pi^2(S^1) = \emptyset$

Proof. The generic mapping from the 2-sphere to the circle with two sections, $z > 0$ and $z < 0$:

$$\sigma(x, y) \tag{4.21}$$

where z defined as $\pm\sqrt{R^2 - x^2 - y^2}$. We show that for each half we can shrink the half-sphere into a point and so we can shrink the whole sphere. Consider the homotopy,

$$\sigma(tx + 1 - t, yt) \tag{4.22}$$

which has the base point, $x = 1, y = 0$ with the image base point $\sigma(1, 0)$. Clearly this is a continuous deformation keeping the basepoint fixed. We can trivially do the same for the $z < 0$ half and so this mapping is equivalent to the constant mapping:

$$\sigma\Big|_{t=0} = \sigma(1, 0) \tag{4.23}$$

□

More generally we have,

Claim 11. $\pi^n(S^1) = \emptyset$, $n > 1$

Unlike the lasso'ing a basketball these results are not intuitive. One might expect the logic we used for $\pi^2(S^1)$ to transfer over to $\pi^3(S^2)$ however it does not.

4.3 Exact homotopic sequence

4.3.1 Basics

Computing homotopy groups is a difficult task. A very useful tool is called "exact homotopy sequence. As usual our goal to find a mapping,

$$M: \quad G_1 \to G_2 \tag{4.24}$$

Before we introduce the sequences, we need some definitions:

Def. 7. $\ker M$*: the set of all elements in* G_1 *that are mapped to the identity element of* G_2*:*

$$\ker(M) = \{g \in G_1, M(g_1) = \mathbb{1}\} \tag{4.25}$$

Def. 8. $\operatorname{Im} M$*: the set of all elements in* G_2 *which are mapped to by something in* G_1*:*

$$\operatorname{Im} M = \{g_2 \in G_2 \mid \exists g_1 \in G_1, M(g_1) = g_2\} \tag{4.26}$$

Def. 9. Exact*: A given sequence,*

$$G_1 \xrightarrow{M_1} G_2 \xrightarrow{M_2} G_2 \xrightarrow{M_3} \cdots \tag{4.27}$$

is exact if

$$\operatorname{Im} M_i = \ker(M_{i+1}) \quad \forall i \tag{4.28}$$

Claim 12. *There exists a sequence,*

$$\pi^{n+1}(G/H) \to \pi^n(H) \xrightarrow{M_1} \pi^n(G) \xrightarrow{M_2} \pi^n(G/H) \xrightarrow{M_3} \pi^{n-1}(H) \xrightarrow{M_1'} \cdots \tag{4.29}$$

Proving this claim is feasible but somewhat technical. Instead we take this for granted and proceed to applications of this theorem.

4.3.2 Examples

Perhaps the simplest application of this is for the sequence:

$$\emptyset \xrightarrow{M_1} G \xrightarrow{M_2} \emptyset \tag{4.30}$$

Claim 13. *If this is an exact sequence (one where* $\operatorname{Im} M_1 = \ker M_2$*) then* $\Rightarrow G = \emptyset$*.*

Proof. An $\emptyset$ can only map into $\emptyset$ and so $\operatorname{Im} M_1 = \emptyset$. Furthermore, by the definition of the kernal of M_2 (all elements in G which are mapped to the identity, $\ker M_2 = G$). Since we have an exact sequence we also have, $\ker M_2 = \emptyset$. Therefore, $\ker M_2 = G = \emptyset$ □

Claim 14. *If the sequence,*

$$\emptyset \xrightarrow{M_1} A \xrightarrow{M_2} B \xrightarrow{M_3} \emptyset \tag{4.31}$$

is exact then,

$$A = B \tag{4.32}$$

Proof. Again we have,

$$\operatorname{Im} M_1 = \emptyset = \ker M_2 \tag{4.33}$$
$$\ker M_3 = B = \operatorname{Im} M_2 \tag{4.34}$$

M_2 is a map from $A \to B$ whose kernal is trivial (only identity of A maps to the identity of B) and whose image is B (the set of elements of A which map to B are equal to B). Thus there are no "extra elements" in A and the map is one-to-one and so,

$$A = B \tag{4.35}$$

□

Now lets consider some applications of these ideas into actual groups we will be interested in.

Claim 15. $\pi^3(S^2) = \mathbb{Z}$

Proof. Recall that for the breaking of SU(2) $\to$ U(1), the vacuum manifold is equal to a 2-sphere: SU(2)/U (1) $= S^2$. Thus we can instead study $\pi^3(\mathrm{SU}(2)/\mathrm{U}(1))$. This is much simpler since now we can use our exact homotopic sequence. We have,

$$\begin{array}{lcccccccr} & \pi^3(H) & \to & \pi^3(G) & \to & \pi^3(G/H) & \to & \pi^2(H) & (4.36) \\ \text{or} & \pi^3(\mathrm{U}(1)) & \to & \pi^3(\mathrm{SU}(2)) & \to & \pi^3(\mathrm{SU}(2)/\mathrm{U}(1)) & \to & \pi^2(\mathrm{U}(1)) & (4.37) \end{array}$$

We know that $\pi^3(\mathrm{U}(1)) = \pi^3(S^1) = \emptyset$ and similarly, $\pi^2(\mathrm{U}(1)) = \emptyset$. Thus we can use the trick we just learned to say,

$$\pi^3(\mathrm{SU}(2)/\mathrm{U}(1)) = \pi^3(\mathrm{SU}(2)) \tag{4.38}$$

since $\mathrm{SU}(2) = S^3$, $\mathrm{SU}(2)/\mathrm{U}(1) = S^2$, and earlier we showed that $\pi^3(S^3) = \mathbb{Z}$ so,

$$\pi^3(S^2) = \mathbb{Z} \tag{4.39}$$

□

Claim 16. $\pi^1(\mathrm{SO}(3)) = \mathbb{Z}_2$

Proof. Consider $G =$ SU(2) and $H = \mathbb{Z}_2$ with the sequence,

$$\underbrace{\pi^1(\mathrm{SU}(2))}_{\emptyset} \to \pi^1(\mathrm{SU}(2)/\mathbb{Z}_2) \to \pi^0(\mathbb{Z}_2) \to \underbrace{\pi^0(\mathrm{SU}(2))}_{\emptyset} \tag{4.40}$$

$\pi^0(G)$ measures how many connected components G has. Thus $\pi^0(\mathbb{Z}_2) = \mathbb{Z}_2$. Furthermore, $\pi^1(\mathrm{SU}(2)/\mathbb{Z}_2) = \pi^1(\mathrm{SO}(3))$. Therefore,

$$\pi^1(\mathrm{SO}(3)) = \mathbb{Z}_2 \tag{4.41}$$

□

Claim 17. $\pi^1(\mathrm{SO(N)}) = \mathbb{Z}_2$ *for* $N \geq 3$.

Proof. We can again use our Higgs intuition to simplify the problem. For a general SO(N) symmetry broken with a scalar ϕ, a fundamental of SO(N),

$$\langle \phi \rangle = \begin{pmatrix} v \\ 0 \\ \vdots \\ 0 \end{pmatrix} \tag{4.42}$$

After ϕ gains a VEV, the unbroken subgroup is SO($N-1$) given a coset,

$$G/H = SO(N)/SO(N-1) = S^{N-1} \tag{4.43}$$

Now consider the exact sequence,

$$\underbrace{\pi^2(\mathrm{SO}(N)/\mathrm{SO}(N-1))}_{\emptyset} \to \pi^1(\mathrm{SO}(N-1)) \to \pi^1(\mathrm{SO}(N)) \to \underbrace{\pi^1(\mathrm{SO}(N)/\mathrm{SO}(N-1))}_{\emptyset} \tag{4.44}$$

where the endpoints are both trivial as long as $N \geq 4$. This implies that,

$$\pi^1(\mathrm{SO}(N)) = \pi^1(\mathrm{SO}(N-1)) \tag{4.45}$$

For $N = 4$ we have,

$$\pi^1(\mathrm{SO}(4)) = \pi^1(\mathrm{SO}(3)) = \mathbb{Z}_2 \tag{4.46}$$

where we have used our previous result. Applying this relation consecutively we get,

$$\pi^1(\mathrm{SO}(N)) = \mathbb{Z}_2 \quad N \geq 3 \tag{4.47}$$

□

Claim 18. $\pi^1(SU(N)) = \emptyset$ *for* $N \geq 2$

Proof. Again we use our knowledge of spontaneous symmetry breaking to help us extract the homotopy group. For a fundamental scalar charged under at SU(N) we have,

$$|\phi|^2 = \overbrace{(\phi_1^R)^2 + (\phi_1^I)^2 + (\phi_2^R)^2 + ...}^{2N} \tag{4.48}$$

Thus the vacuum manifold is a $2N-1$ sphere:

$$\mathrm{SU}(N)/\mathrm{SU}(N-1) = S^{2N-1} \tag{4.49}$$

Now consider the exact sequence,

$$\underbrace{\pi^2(\mathrm{SU}(N)/\mathrm{SU}(N-1))}_{\emptyset} \to \pi^1(\mathrm{SU}(N-1)) \to \pi^1(\mathrm{SU}(N)) \to \underbrace{\pi^1(\mathrm{SU}(N)/\mathrm{SU}(N-1))}_{\emptyset} \tag{4.50}$$

where the end points are the trivial group if $N \geq 2$. Thus we have,

$$\pi^1(\mathrm{SU}(N)) = \pi^1(\mathrm{SU}(N-1)) \tag{4.51}$$

since $\pi^1(\mathrm{SU}(2)) = \emptyset$ we can iteratively show that $\pi^1(\mathrm{SU}(N)) = 0$ for all $N \geq 2$. □

One can similarly show two other important homotopy groups:

Claim 19. $\pi^2(SU(N)) = \emptyset$

This will be used for studying soliton solutions in $3 + 1$ dimensions.

Claim 20. $\pi^3(SU(N)) = \mathbb{Z}$

This will be important when we discuss instantons since instantons in D dimensions are equivalent to solitons in $D - 1$ dimensions.

Chapter 5

Magnetic Monopoles

5.1 Monopoles in general

As we'll see in $3+1$ dimensions soliton solutions of spontaneously broken gauge theories consist of a tower of monopoles, each with different topological charge. However, before applying our knowledge of topological field theory, let us first discuss the properties of monopoles in general.

An electric monopole in electromagnetism is simply a charged particle with charge q whose E field takes the form,[1]

$$\mathbf{E} = e\frac{\mathbf{r}}{r^3} \tag{5.1}$$

One can form an analogue of this for a magnetic monopole with magnetic charge, g,

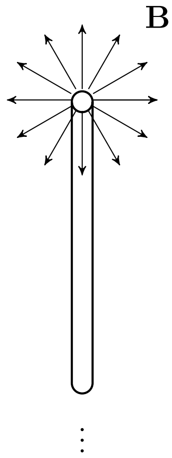
$$\mathbf{B} = g\frac{\mathbf{r}}{r^3} \tag{5.2}$$

5.1.1 Dirac's derivation

How will such a monopole behave? Dirac's insight: If we can come up with a find a magnetic field configuration in EM for which there does not exist a test that would differentiate it from a monopole than that system should share properties with a monopole.

Consider an half-infinite solenoid:

[1]The prefactor here is of course just conventional, but we attempt to keep gaussian units throughtout this discussion.



This is called a Dirac string and has B field lines that look just like a monopole. The only difference between the Dirac string and the monopole is the Dirac string has a flux along the tube,

$$\Phi = \int \mathbf{B} \cdot d\mathbf{a} = 4\pi g \frac{r^3}{r^3} = 4\pi g \tag{5.3}$$

where we used that the flux in the tube should also be equal to the flux emitted by the end. Is this flux an observable quantity? It can be observed through the Abrahav-Bohm effect.

Recall that the Schrodinger equation with a magnetic field is given by,

$$\frac{-\hbar^2}{2m}\left(\nabla + \frac{ie}{\hbar c}\mathbf{A}(\mathbf{r})\right)^2 \psi(\mathbf{r}) + V(\mathbf{r})\psi(\mathbf{r}) = E\psi(\mathbf{r}) \tag{5.4}$$

If we consider a change of variables,

$$\psi(\mathbf{r}) = \exp\left(-\frac{ie}{\hbar c} f(\mathbf{r})\right)\psi_0(\mathbf{r}) \tag{5.5}$$

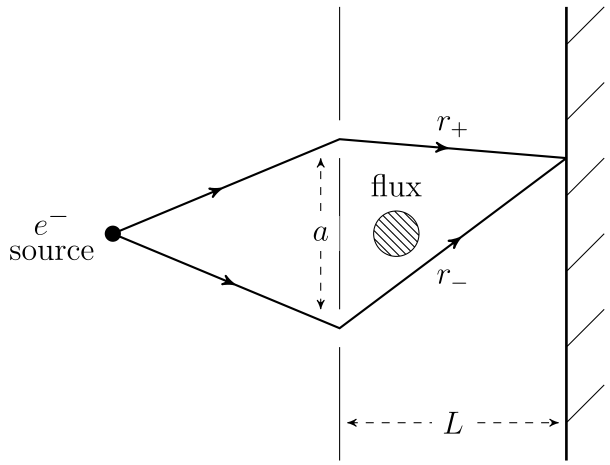
with

$$f(\mathbf{r}) = \int_{\mathbf{r}_0}^{\mathbf{r}} \mathbf{A}(\mathbf{r}') \cdot d\mathbf{r}' \quad \Rightarrow \nabla f(\mathbf{r}) = \mathbf{A}(\mathbf{r}) \tag{5.6}$$

Then $\psi_0(\mathbf{r})$ satisfies,

$$\left(-\frac{\hbar^2}{2m}\nabla^2 + V\right)\psi_0(\mathbf{r}) = E\psi_0(\mathbf{r}) \tag{5.7}$$

Therefore, the presence of $\mathbf{A}$ gives an additional position dependent phase to ψ. This can be detected through interference. Consider the following setup:



where the flux tube represents the center of the Dirac string, far away from the monopole-like end. The magnetic flux modifies the interference pattern due to the additional phase. This phase is independent of path (since $\mathbf{A}$ is a conservative vector field), and so the phase difference of the bottom and top wavefunctions (assuming they start in phase) is,

$$\exp\left(\frac{1}{\lambda}(r_+ - r_-) + \frac{e}{\hbar c}\oint \mathbf{A}\cdot d\mathbf{r}\right) \tag{5.8}$$

where L is the distance from the slits to the screen, λ is the wavelength, and

$$r_\pm = \sqrt{L^2 + (y \pm a/2)^2} \tag{5.9}$$

The first term gives the standard formula for the double slit interference and the second is the new addition due to the fluxtube. The new term can be written,

$$\frac{e}{\hbar c}\oint \mathbf{A}\cdot d\mathbf{r} = \frac{e}{\hbar c}\Phi \tag{5.10}$$

For this term to be unphysical we must have,

$$\frac{e}{\hbar c}\Phi = 2\pi n \quad n \in \mathbb{Z} \tag{5.11}$$

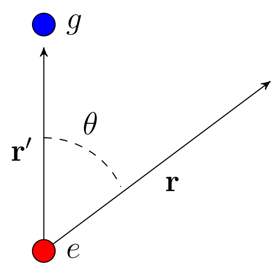
Using our expression for the flux computed earlier we have,

$$g = \frac{\hbar c}{2e} n \tag{5.12}$$

Thus a Dirac string with quantized charge with quantized charge, shares properties with a monopole. Thus we conclude that a monopole can't have arbitrarily charged monopoles but only quanta of $\hbar c/2e$.

5.1.2 Derivation using quantization of angular momentum

Now consider an alternative derivation of the same result using angular momentum. Consider an electric monopole at the origin with a magnetic monopole a distance r' away. For simplicity we place the magnetic monopole along the $\hat{z}$ direction:



The electromagnetic and magnetic fields of the system are,

$$\mathbf{E} = e\frac{\mathbf{r}}{r^3} \quad \mathbf{B} = g\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3} \tag{5.13}$$

giving an angular momentum,

$$\int d\mathbf{J} = \frac{1}{4\pi c}\int \mathbf{r}\times\overbrace{(\epsilon_0\mathbf{E}\times\mathbf{B}\,dV)}^{\text{mom density}} \tag{5.14}$$

$$= -\frac{ge}{4\pi c}\int drr^2\int d\cos d\phi\frac{\mathbf{r}\times(\mathbf{r}\times\mathbf{r}')}{r^3\,|\mathbf{r}-\mathbf{r}'|^3} \tag{5.15}$$

The numerator can be simplified using the triple product identity,

$$\textbf{numerator} = (\mathbf{r}\cdot\mathbf{r}')\mathbf{r} - r^2\mathbf{r}' \tag{5.16}$$

$$= r^2 r'(\cos\theta\hat{r} - \hat{z}) \tag{5.17}$$

which gives,

$$\mathbf{J} = -\frac{ge}{4\pi c}\int dr\,rr'\int d\cos\int d\phi\frac{(\cos\theta\hat{r}-\hat{z})}{(r^2+r'^2-2rr'\cos\theta)^{3/2}} \tag{5.18}$$

Note that since $\hat{x}\propto\sin\phi, \hat{y}\propto\cos\phi$:

$$\int d\phi f(r,\theta)\hat{r} = 2\pi f(r,\theta)\cos\theta\hat{z} \tag{5.19}$$

and so we can write the above as,

$$\mathbf{J} = \frac{ge}{2c}\hat{z}\int dr\,rr'\int_{-1}^{1}dc_\theta\frac{1-c_\theta^2}{(r^2+r'^2-2rr'c_\theta)^{3/2}} \tag{5.20}$$

The integral is easy done using *Mathematica*:

$$\int_{-1}^{1} dx \frac{1-x^2}{(r^2+r'^2-2rr'x)^{3/2}} = \frac{4}{3}\begin{cases} \frac{1}{r'^3} & r<r' \\ \frac{1}{r^3} & r>r' \end{cases} \tag{5.21}$$

and so,

$$\mathbf{J} = \frac{ge}{2c}\frac{4}{3}\hat{z}\left[\int_0^{r'} dr\,\frac{r}{r'^2} + \int_{r'}^{\infty}\frac{r'}{r^2}\right] \tag{5.22}$$

$$= \frac{ge}{c}\hat{z} \tag{5.23}$$

Since angular momentum is quantized, with a minimum value of $\hbar/2$:

$$g = \frac{\hbar c}{2e}n \tag{5.24}$$

in agreement with the result we derived completely independently above.

Both the derivations above required an electric charge to derive the quantization condition. We have taken the charge to be electron-like, but we know that we have other objects with smaller charges then electrons, quarks. If we consider the derivations above with smaller charges, then the lowest magnetic charge must be larger. However, this conclusion is a bit too quick. If a particle exists, like quarks, that has additional charges as well as electromagnetic charges the phase from the Aharonov-Bohm effect (or angular momentum in the second derivation) can be at least partially cancelled canelled by phases due to additional interactions. For example we can find the phase in the Aharonov-Bohm effect using a “down-quark Dirac string“ is,

$$\exp\left[-i\left(\frac{e}{3\hbar c} - \frac{g_s}{\hbar c}\right)\right] \tag{5.25}$$

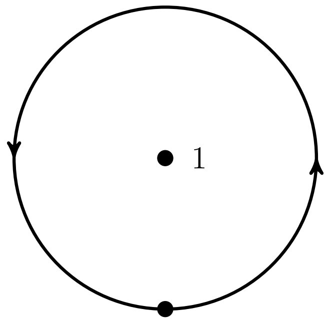
where the sign is due to color being attractive. This type of cancellation occurs for GUT monopoles.

5.2 Dyons

Above we considered particles with only electric or only magnetic charge. Dyons are (hypothetical) particles that carry both charges. What is there quantization condition? Consider two such particles with charges,

$$(q_1, g_1) \quad (q_2, g^2) \tag{5.26}$$

Now consider particle one going around particle two,



The phase of $-4\pi i q_1 g_2/\hbar c$ and $4\pi i q_2 g_1 \hbar c$ respectively giving a relative phase shift of,

$$\exp\left[-\frac{i4\pi}{\hbar c}\left(q_1 g_2 - q_2 g_1\right)\right] . \tag{5.27}$$

This phase shift is unobservable if,

$$q_1 g_2 - q_2 g_1 = \frac{1}{\hbar c}\frac{n}{2} \tag{5.28}$$

This is known as the Schwinger-Dyson quantization condition. For $q_1 = e$, $q_2 = 0$, $g_1 = 0$, $g_2 = g$ we recover the earlier result. However, for dyons electric charges are not all integer multiples of a fundamental charge. In particle consider two particle with charges,

$$\frac{1}{\hbar c}\left(n_1 - \frac{\theta}{2\pi}m_1, m_1\right) , \frac{1}{\hbar c}\left(n_2 - \frac{\theta}{2\pi}m_2, m_2\right) \tag{5.29}$$

for $n_1, n_2, m_1, m_2 \in \mathbb{Z}$. The Dirac quantization condition gives,

$$q_1 g_2 - q_2 g_1 = \frac{1}{\hbar c}\left(n_1 m_2 - n_2 m_1\right) \tag{5.30}$$

(the θ term cancels) satisfying the quantization condition. Hence there is a continuos set of charges parameterized by a parameter θ that can exist. The Witten effect [4] shows that this parameter is equal to the θ angle of the gauge theory.

5.3 Duality transformation of Maxwell's equations

Maxwell's equations are given in their differential form as,

$$\nabla \cdot \mathbf{E} = 4\pi\rho_e \tag{5.31}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{5.32}$$

$$\nabla \times \mathbf{E} = -\frac{1}{c}\partial_t \mathbf{B} \tag{5.33}$$

$$\nabla \times \mathbf{B} = \frac{1}{c}\partial_t \mathbf{E} + \frac{4\pi}{c}\mathbf{j}_e \tag{5.34}$$

where ρ_e and $\mathbf{j}_e$ are the electric charge and current respectively. Generically there is no symmetry relating the $\mathbf{E}$ and $\mathbf{B}$ fields. However, when the sources vanish, $\rho_e, \mathbf{j}_e = 0$ the equations are symmetric under an SO(2) transformation:

$$\begin{pmatrix} \mathbf{E} \\ \mathbf{B} \end{pmatrix} \rightarrow \begin{pmatrix} c_\theta & s_\theta \\ -s_\theta & c_\theta \end{pmatrix} \begin{pmatrix} \mathbf{E} \\ \mathbf{B} \end{pmatrix} \tag{5.35}$$

This is known as the duality transformation of Maxwell's equations. Now suppose we had electric and/or magnetic charges. You can always redefine your fields by rotating away one of the charges using a duality transformation:

$$(q, g) \rightarrow (qc_\theta + gs_\theta, -qs_\theta + gc_\theta) \tag{5.36}$$

Thus there is nothing special about magnetism. The statement that "magnetic charges don't carry charges" is a statement that we can always define our electric and magnetic fields such that the "one that has a source" is the electric field and the other one is the magnetic field.

How would Maxwell's equation look like with a magnetic source? To see this consider the simplest electromagnetic duality, $\alpha = 90^o$. In this case,

$$\mathbf{E} \rightarrow \mathbf{B} \quad \mathbf{B} \rightarrow -\mathbf{E} \tag{5.37}$$

Recall that Maxwell's equation can be written in terms of the field strength tensor,

$$\partial^\mu F_{\mu\nu} = \frac{4\pi}{c} j_{e,\nu} \quad \partial_\nu \tilde{F}^{\mu\nu} = 0 \tag{5.38}$$

where $\tilde{F}_{\alpha\beta} \equiv \epsilon_{\alpha\beta\mu\nu} F^{\mu\nu}$. In matrix form,

$$F = \left(\begin{array}{c|ccc} 0 & & \mathbf{E}^T & \\ \hline & 0 & B_z & -B_y \\ -\mathbf{E} & -B_z & 0 & B_x \\ & B_y & -B_x & 0 \end{array} \right) \quad \tilde{F} = \left(\begin{array}{c|ccc} 0 & & -\mathbf{B}^T & \\ \hline & 0 & E_z & -E_y \\ \mathbf{B} & -E_z & 0 & E_x \\ & E_y & -E_x & 0 \end{array} \right) \tag{5.39}$$

Thus under such a rotation we have, $F \rightarrow -\tilde{F}$ and $\tilde{F} \rightarrow F$. Adding in magnetic charges we expect,

$$\partial^\mu F_{\mu\nu} = \frac{4\pi}{c} j^e_\nu \quad \partial^\mu \tilde{F}_{\mu\nu} = \frac{4\pi}{c} j^m_\nu \tag{5.40}$$

We will show that the topological current in monopole-type soolitons will endeed enter in j^m_μ.

5.4 Topological Monopoles

Thus far we have studied generic properties of monopoles. We now consider a particle example.

5.4.1 ’t Hooft-Polyakov monopole

Consider the Georgi-Glashow model of an $SO(3)$ symmetry with a triplet ϕ^a and potential,

$$U(\phi) = \lambda(\boldsymbol{\phi}^2 - v^2)^2 \tag{5.41}$$

The symmetry breaking pattern is,

$$\mathrm{SU}(2) \to \mathrm{U}(1) \tag{5.42}$$

Thus we have two broken generators (in the toy SM, these would give mass to the $W^\pm$ with mass $m_W = ev$). The relevant homotopy group is,

$$\pi^2(\mathrm{SU}(2)/\mathrm{U}(1)) = \pi^2(S^2) = \mathbb{Z} \tag{5.43}$$

Thus we expect a conserved integer charge. What is the conserved current? We won’t derive the form of the current but instead reveal the answer and then show that it is indeed conserved. The desired current is,

$$K_\mu = \frac{1}{8\pi}\epsilon_{\mu\nu\rho\sigma}\epsilon_{abc}(\partial^\nu\hat{\phi}_a)(\partial^\rho\hat{\phi}^b)(\partial^\sigma\hat{\phi}^c) \tag{5.44}$$

where $\hat{\phi}^a \equiv \phi^a/v$. This is clearly conserved since,

$$\partial^\mu K_\mu \propto \epsilon_{\mu\nu\rho\sigma}(\partial^\mu\partial^\nu\hat{\phi}^a)(...) + ... = 0 \tag{5.45}$$

Note that this current is a non-Noether current, but of topological origin. The conserved charge is,

$$\begin{aligned}
Q &= \int d^3x K_0 &(5.46)\\
&= \frac{1}{8\pi}\epsilon_{abc}\int d^3x\left(\phi^1\hat{\phi}^a\partial^2\hat{\phi}^b\partial^3\hat{\phi}^c - \partial^2\hat{\phi}^a\partial^1\hat{\phi}^b\partial^3\hat{\phi}^c + ...\right) &(5.47)\\
&= \frac{1}{8\pi}\epsilon_{abc}\epsilon_{ijk}\int d^3x\partial^i\hat{\phi}^a\partial^j\hat{\phi}^b\partial^k\hat{\phi}^c &(5.48)\\
&= \frac{1}{8\pi}\epsilon_{abc}\epsilon_{ijk}\int d^3x\partial^i\left[\hat{\phi}^a\partial^j\hat{\phi}^b\partial^k\hat{\phi}^c\right] - \cancel{\hat{\phi}^a(\partial^i\partial^j\hat{\phi}^b)\partial^k\hat{\phi}^c} + ... &(5.49)\\
&= \frac{1}{8\pi}\epsilon_{abc}\epsilon_{ijk}\int d^3x\partial^i\left[\hat{\phi}^a\partial^j\hat{\phi}^b\partial^k\hat{\phi}^c\right] &(5.50)
\end{aligned}$$

To simplify this integral note that,

$$\int d^Dx\partial_i f(x) = \int d\Omega \lim_{r\to\infty}\frac{x_i}{r}f(x) \tag{5.51}$$

which can be derived as a particular case of Gausses law by considering a spherical integration region. Furthermore,

$$\begin{aligned}
d\Omega &= \frac{1}{(D-1)!}d\theta_1 d\theta_2 ... \frac{1}{r^{D-1}}\left[\frac{r}{x_1}\det\frac{\partial x}{\partial\theta}_1 - \frac{r}{x_2}\det\frac{\partial x}{\partial\theta}_2 + ...\right] &(5.52)\\
&= \frac{1}{(D-1)!}d\theta_1 d\theta_2 ... \frac{1}{r^{D-1}}\epsilon_{im_1m_2...}\epsilon_{p_1p_2...}\frac{r}{x_i}\left(\frac{\partial x_{m_1}}{\partial\theta_{p_1}}\frac{\partial x_{m_2}}{\partial\theta_{p_2}}...\right) &(5.53)
\end{aligned}$$

or for $D = 3$:

$$d\Omega = \frac{1}{2} d\alpha_1 d\alpha_2 \epsilon_{imn} \epsilon_{pq} \frac{r}{x_i} \frac{\partial x_m}{\partial \alpha_p} \frac{\partial x_n}{\partial \alpha_q} \tag{5.54}$$

where α_1 and α_2 are the angles (usually denoted θ and ϕ). Thus we conclude,

$$\int d^3x \partial_i f(x) = \epsilon_{imn} \epsilon_{pq} \frac{1}{2} d^2\alpha \lim_{r\to\infty} \frac{\partial x_m}{\partial \alpha_p} \frac{\partial x_n}{\partial \alpha_q} f(\alpha) \tag{5.55}$$

and

$$Q = \frac{1}{16\pi} \epsilon_{abc} \epsilon_{ijk} \epsilon_{imn} \epsilon_{pq} \int d^2\alpha \lim_{r\to\infty} \hat{\phi}^a \frac{\partial \hat{\phi}^b}{\partial \alpha_w} \frac{\partial \hat{\phi}^c}{\partial \alpha_v} \frac{\partial \alpha_w}{\partial x_j} \frac{\partial \alpha_v}{\partial x_k} \frac{\partial x_m}{\partial \alpha_p} \frac{\partial x_n}{\partial \alpha_q} \tag{5.56}$$

$$= \frac{1}{16\pi} \int d^2\alpha \epsilon_{abc} \epsilon_{pq} \hat{\phi}^a \left(\frac{\partial \hat{\phi}^b}{\partial \alpha_p} \frac{\partial \hat{\phi}^c}{\partial \alpha_q} - \frac{\partial \hat{\phi}^b}{\partial \alpha_q} \frac{\partial \hat{\phi}^c}{\partial \alpha_p} \right) \tag{5.57}$$

$$= \frac{1}{8\pi} \int d^2\alpha \epsilon_{abc} \epsilon_{pq} \hat{\phi}^a \frac{\partial \hat{\phi}^b}{\partial \alpha_p} \frac{\partial \hat{\phi}^c}{\partial \alpha_q} \tag{5.58}$$

Now convert the integral to coordinates in the internal $\hat{\phi}^a$ space, (ξ_1, ξ_2) which gives,

$$Q = \frac{1}{4\pi} \int d^2\xi \frac{1}{2} \epsilon_{abc} \epsilon_{rs} \frac{\partial \hat{\phi}^b}{\partial \xi_r} \frac{\partial \hat{\phi}^c}{\partial \xi_s} \hat{\phi}^a \tag{5.59}$$

The factor in the center is just the differential of the surface in the internal space and so we can write,

$$Q = \frac{1}{4\pi} \int dV_{\text{int}} \partial_a \hat{\phi}^a \tag{5.60}$$

where the volume integral now runs over the internal (image) space.

We want to argue that this topological current will play the role of the magnetic current. For these topological solutions the VEV is not pointing in the smae direction all the time. This implies that the unbroken U(1) direction is changing. Given this how should you define $F_{\mu\nu}$?

't Hooft showed that,

$$F^{\text{EM}}_{\mu\nu} = \hat{\phi}^a F^a_{\mu\nu} - \frac{1}{e} \epsilon^{abc} \hat{\phi}^a D_\mu \hat{\phi}^b D_\nu \hat{\phi}^c \tag{5.61}$$

is gauge invariant. Note that if $\hat{\phi} = (0\,,\,0\,,\,1)^T$ then $D_\mu \hat{\phi}^a D^2_\nu \hat{\phi} = 0$ and so $F^{\text{EM}}_{\mu\nu} = F^3_{\mu\nu}$. Far out since $D_\mu \hat{\phi}$ contributes to the energy density we have,

$$D_\mu \hat{\phi} \to 0 \;\Rightarrow\; F^{\text{EM}}_{\mu\nu} \simeq \hat{\phi}^a F^a_{\mu\nu} \tag{5.62}$$

With lots of algebra one can show that,

$$\frac{1}{2} \epsilon_{\mu\nu\rho\sigma} \partial^\nu F^{\rho\sigma}_{\text{EM}} = \frac{4\pi}{e} K_\mu \tag{5.63}$$

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