
QUANTUM FIELD THEORY II (PHYS7652) LECTURE NOTES

LECTURE NOTES BASED ON A COURSE GIVEN BY MAXIM PERELSTEIN.
WE BEGIN WITH DISCUSSING THE PATH INTEGRAL FORMALISM IN
QUANTUM MECHANICS AND MOVE ON TO IT'S USE IN QUANTUM FIELD
THEORY. WE THEN STUDY RENORMALIZATION AND RUNNING COUPLINGS IN
ABELIAN AND NON-ABELIAN GAUGE THEORIES IN DETAIL.

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Chapter 1

Preface

This set of notes are based on lectures given by Maxim Perelstein in the *Quantum Field Theory II* course at Cornell University during Spring 2013. The course uses both Srednicki's *Quantum Field Theory* and Peskin and Schroeder's *An Introduction to Quantum Field Theory* as reference texts. I wrote these notes during lectures and as such may contain some small typographical errors and sloppy diagrams. I also make notes for myself throughout the text of the form, [Q#:...]. They are just to remind to me to go back and take a look at something and should be ignored. I've attempted to proofread and fix as many of these problems as I can. I hope to continue to revise and add to these notes until I feel they are complete. If you have any questions or would like a copy of the L^AT_EX file feel free to let me know at ajd268@cornell.edu.

1.1 Conventions

In these notes we use the conventions used by Peskin and Schroeder. We use a “West coast metric”,

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad (1.1)$$

the Weyl basis,

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \quad (1.2)$$

where $\sigma^\mu \equiv (1, \sigma^i)$ and $\bar{\sigma} \equiv (1, -\sigma^i)$.

When applying Fourier transform (FT) we put the 2π on the momenta integral such that:

$$f(x) = \int \frac{dp}{2\pi} \tilde{f}(p) e^{-ipx} \quad (1.3)$$

Chapter 2

Path Integrals in Quantum Mechanics

The primary tool of this course will be what's known as the path integral. In non-relativistic quantum mechanics (NRQM) it is an alternative formulation to using the Schrodinger equation. Consider a particle at x_a that evolves with a Hamiltonian

$$H = H(x, p; t) \quad (2.1)$$

What we want to compute is the transition amplitude for this particle to go from its initial position some other position x_b in a time $2T$. In QM we denote this amplitude with a bracket:

$$U = \langle x_b, T | x_a, -T \rangle \quad (2.2)$$

To simplify this expression consider inserting in a completeness relation $N - 1$ completeness relations:

$$= \int dx_1 dx_2 \dots dx_{N-1} \langle x_b, T | x_{N-1}, T - \Delta t \rangle \langle x_{N-1}, T - \Delta t | x_{N-2}, T - 2 \Delta t \rangle \dots \quad (2.3)$$
$$\langle x_1, -T + \Delta t | x_a, -T \rangle$$

Now if we have a bracket at two nearby times then you can evolve one side to have an equal time bracket¹:

$$\langle x, t_0 | x', t_0 + \Delta t \rangle = \langle x, t_0 | e^{-iH(t_1) \cdot (t_1 - t_0)} | x', t_0 \rangle = \langle x | e^{-iH(t_1) \Delta t} | x' \rangle \quad (2.4)$$

where in the last step we rewrote the bra and ket without time dependence as the absolute value of time has no significance.

$$= \int dx_1 dx_2 \dots dx_{N-1} \langle x_b | e^{-iH(T - \Delta t) \cdot \Delta t} | x_{N-1} \rangle \langle x_{N-1} | e^{-iH(T - 2\Delta t) \cdot \Delta t} | x_{N-2} \rangle \dots \quad (2.5)$$
$$\langle x_1 | e^{-iH(-T) \cdot \Delta t} | x_a \rangle$$

¹To do this properly with finite Δt requires Dyson's formula

Now let $t_1 = -T, t_2 = -T + \Delta t, \dots, t_N = T$. Thus we now have

$$= \int dx_1 dx_2 \dots dx_{N-1} \langle x_b | e^{-iH(t_{N-1}) \cdot \Delta t} | x_{N-1} \rangle \langle x_{N-1} | e^{-iH(t_{N-2}) \cdot \Delta t} | x_{N-2} \rangle \dots \langle x_1 | e^{-iH(t_1) \cdot \Delta t} | x_a \rangle \quad (2.6)$$

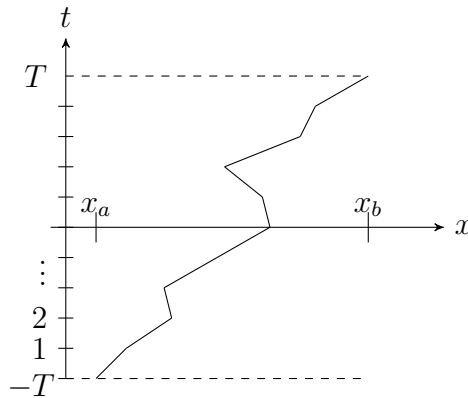
$$U = \left(\prod_{i=1}^{N-1} \int dx_i \right) \left(\prod_{i=0}^{N-1} \langle x_{i+1} | e^{-iH(t_i) \Delta t} | x_i \rangle \right) \quad (2.7)$$

$$= \left(\prod_{i=1}^{N-1} \int dx_i \right) \left(\prod_{i=0}^{N-1} U(x_i \rightarrow x_{i+1}; \Delta t) \right) \quad (2.8)$$

Note that technically these equations need two less integrals than the number of x_i values. However often people don't bother to write it like this (since it somewhat ruins the ascetic beauty in the formula). The point is that you write down each of the brackets with every position included, but you only integrate over x_1, \dots, x_{N-1} , not over $x_0 = x_a$ and $x_N = x_b$. Either way they don't matter much at the end, as long as you know not to integrate over your end points... Instead this expression is more commonly written as

$$U = \left(\prod_{i=0}^N \int dx_i U(x_i \rightarrow x_{i+1}; \Delta t) \right) \quad (2.9)$$

Pictorially this integral is shown below



This is called a path integral. For simplicity we focus on Hamiltonians are of the form

$$H = \frac{p^2}{2m} + V(x, t) \quad (2.10)$$

Consider the bracket above (if we know the value of this term we almost know everything). We take Δt to be small.

$$\langle x_{i+1}|e^{-iH\Delta t}|x_i\rangle = \langle x_{i+1}|x_i\rangle - i\Delta t \langle x_{i+1}|\frac{p^2}{2m} + V(x, t_i)|x_i\rangle \quad (2.11)$$

$$= \delta(x_{i+1} - x_i) - i\Delta t \langle x_{i+1}|\frac{p^2}{2m}|x_i\rangle - i\Delta t \delta(x_{i+1} - x_i)V(x_i, t_i) \quad (2.12)$$

$$= \int \frac{dp_i}{2\pi} e^{-ip_i(x_{i+1}-x_i)} - i\Delta t \langle x_{i+1}|\frac{p^2}{2m}|x_i\rangle - i\Delta t V(x_i, t_i) \quad (2.13)$$

$$\times \int \frac{dp_i}{2\pi} e^{-ip_i(x_{i+1}-x_i)} \quad (\text{no sum})$$

By insertion of complete set of states (note also that $\langle p_i|x_i\rangle = e^{-x_i p_i}$) we have,

$$\langle x_{i+1}|p^2|x_i\rangle = \int dp' dp'' e^{ix_{i+1}p'} e^{-ix_i p''} p''^2 \langle p'|p''\rangle \quad (2.14)$$

$$= \int \frac{dp'}{2\pi} e^{ix_{i+1}p'} e^{-ix_i p'} p'^2 \quad (2.15)$$

$$\langle x_{i+1}|p^2|x_i\rangle = \int \frac{dp_i}{2\pi} e^{i(x_{i+1}-x_i)p_i} p_i^2 \quad (\text{no sum}) \quad (2.16)$$

Therefore,

$$\langle x_{i+1}|e^{-iH(t_i)\Delta t}|x_i\rangle = \int \frac{dp_i}{2\pi} e^{-ip_i(x_{i+1}-x_i)} \left(1 - i\Delta t \left(\frac{p_i^2}{2m} + V(x_i, t_i) \right) \right) \quad (2.17)$$

$$= \int \frac{dp_i}{2\pi} e^{-ip_i(x_{i+1}-x_i)} e^{-i\Delta t H(x_i, p_i, t_i)} \quad (2.18)$$

$$= \int \frac{dp_i}{2\pi} e^{-ip_i \Delta t \frac{(x_{i+1}-x_i)}{\Delta t}} e^{-i\Delta t H(x_i, p_i, t_i)} \quad (2.19)$$

Note that in the last step we have combined our infinitesimal result into our exponential (in the end we take $\Delta t \rightarrow 0$). Notice that we no longer have operators; p_i and H_i are numbers! The expression has only numbers, we have moved away from regular quantum mechanics. now taking the $\Delta t \rightarrow 0$ limit we have

$$\langle x_{i+1}|e^{-iH_i \Delta t}|x_i\rangle = \int \frac{dp_i}{2\pi} e^{-i\Delta t(p_i x_i - H_i)} \quad (2.20)$$

Thus we have

$$U = \left(\prod_{i=1}^{N-1} \int dx_i \right) \left(\prod_{i=0}^{N-1} \int \frac{dp_i}{2\pi} e^{-i\Delta t(\hat{x}_i p_i - H_i)} \right) \quad (2.21)$$

$$= \int \frac{dp_0}{2\pi} \left(\prod_{i=1}^{N-1} \int dx_i \int \frac{dp_i}{2\pi} \right) e^{-i \sum_{i=0}^{N-1} \Delta t(\hat{x}_i p_i - H_i)} \quad (2.22)$$

Taking the limit we finally have

$$U = \left(\prod_{i=1}^{N-1} \int dx_i \int \frac{dp_i}{2\pi} \right) e^{-i \int_{-T}^T dt (\dot{x}_i p_i - H_i)} \quad (2.23)$$

where now that we have taken the limit we have just dropped the $\int dp_0$ integral and the value of p_0 in the exponential. Since we have an infinite number of integrals, adding one more shouldn't effect the final result. Only when we discretize our integrals should such a distinction be of any importance.

For convenience we define

$$\int \mathcal{D}x = \lim_{N \rightarrow \infty} \prod_{i=1}^{N-1} \int_{-\infty}^{\infty} dx_i \quad ; \quad \int \mathcal{D}p = \lim_{N \rightarrow \infty} \prod_{i=1}^{N-1} \int_{-\infty}^{\infty} dp_i \quad (2.24)$$

We can now write

$$U = \int \mathcal{D}x \mathcal{D}p e^{-i \int_{-T}^T dt (\dot{x}_i p_i - H_i)} \quad (2.25)$$

Note that the momenta integral has nothing to do with the particular dynamics of the problems and can be done in general since they are just Gaussian integrals. The integral we need is (we now go back to the discrete limit for a moment, see equation 2.22)

$$\int \frac{dp_i}{2\pi} e^{i\Delta t (p_i \dot{x}_i - \frac{p_i^2}{2m} - V_i)} = \int \frac{dp_i}{2\pi} e^{-i \frac{\Delta t}{2m} (p_i - m\dot{x}_i)^2} e^{i \frac{\Delta t}{2m} (m\dot{x}_i)^2 - i\Delta t V_i} \quad (2.26)$$

$$= \int \frac{dp_i}{2\pi} e^{-i \frac{\Delta t}{2m} (p_i)^2} e^{i \frac{\Delta t}{2m} (m\dot{x}_i)^2 - i\Delta t V_i} \quad (2.27)$$

$$= C(\Delta t, m) e^{i \frac{\Delta t}{2m} (m\dot{x}_i)^2 - i\Delta t V_i} \quad (2.28)$$

$$= C(\Delta t, m) e^{i\Delta t (\frac{m}{2} \dot{x}_i^2 - V_i)} \quad (2.29)$$

$$= \mathcal{C}(\Delta t, m) e^{i\Delta t L[x_i, t_i]} \quad (2.30)$$

where we have denoted the value of the momenta integral by $\mathcal{C}(\Delta t, m)$. Though not obvious, it's value will turn out to have no impact on physical quantities. Nevertheless we include for completeness,

$$\mathcal{C}(\Delta t, m) = \sqrt{\frac{2\pi m}{i\Delta t}} \quad (2.31)$$

Thus we have

$$\begin{aligned} U &= \mathcal{C} \prod_{i=1}^{N-1} \int dx_i \exp \left(i\Delta t \sum_{j=1}^N L(x_j, t) \right) = \mathcal{C} \int \mathcal{D}x \exp \left(i \int_{-T}^T L[x(t), t] dt \right) \\ &= \mathcal{C} \int \mathcal{D}x e^{iS} \Big|_{x(-T)=x_a, x(T)=x_b} \end{aligned} \quad (2.32)$$

where \mathcal{S} is the action. This can be thought of as

$$U = \sum_{paths} e^{\mathcal{S}(path)} \quad (2.33)$$

We worked in natural units. Going back to regular units we have

$$U = \sum_{paths} e^{\mathcal{S}(path)/\hbar} \quad (2.34)$$

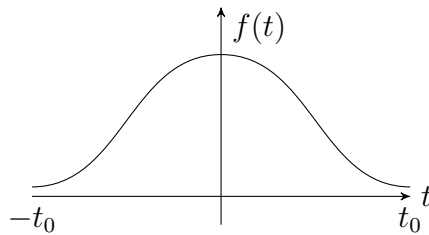
A classical system is one that has a large action (alternatively one can take the Bohr limit of $\hbar \rightarrow 0$). We are dealing with a rapidly oscillating integrand. The trajectory that ends up being the dominant contribution is the one that the smallest phase ($\mathcal{S}(path)/\hbar$). This is the one that has minimum \mathcal{S} , as in classical mechanics.

2.1 Harmonic Oscillator

We will now do an example. Consider a simple harmonic oscillator with an external force, $f(t)$. The Hamiltonian is given by

$$H = \underbrace{\frac{p^2}{2} + \frac{\omega^2 x^2}{2}}_{H_0} - \frac{\omega}{2} \overbrace{f(t)x}^{H'} \quad (2.35)$$

(set $m = 1$). We consider a force that is turned on and then turned off at some time later:



Initially the system is in the ground state ($|0\rangle$), what's the probability to remain in $|0\rangle$ after t_0 ? In standard quantum mechanics we compute probabilities as

$$P = |\langle 0, t_0 | 0, -t_0 \rangle|^2 \quad (2.36)$$

It's unclear how this relates to the path integral formulation since so far we've only discussed probabilities of a particle moving from some x eigenstate to another x eigenstate. **[Q 1: At the end repeat this problem using standard PT.]** Consider (we use Dyson's formula discussed above)

$$U(x_a \rightarrow x_a, 2T) \equiv \langle x_a | e^{-i \int_{-T}^T H dt} | x_a \rangle \quad (2.37)$$

with $T > t_0$. We split the bracket into 3 parts:

$$U(x_a \rightarrow x_a, 2T) = \langle x_a | e^{-2iH_0T} e^{-i \int_{-t_0}^{t_0} H' dt} | x_a \rangle \quad (2.38)$$

$$= \langle x_a | e^{-iH_0(T-t_0)} e^{-i \int_{-t_0}^{t_0} H' dt} e^{-iH_0(t_0-(-T))} | x_a \rangle \quad (2.39)$$

$$= \langle x_a | e^{-iH_0T} e^{-i \int_{-t_0}^{t_0} H dt} e^{-iH_0T} | x_a \rangle \quad (2.40)$$

where we have used the fact that H' is only applied between $-t_0$ and t_0 and in the last step we were careful to write H' in Dysons formula not H_0 . We now insert two complete set of states:

$$U(x_a \rightarrow x_a, 2T) = \int dx_+ dx_- \langle x_a | e^{-iH_0T} | x_+ \rangle \langle x_+ | e^{-i \int_{t_0}^{t_0} H dt} | x_- \rangle \langle x_- | e^{iH_0T} | x_a \rangle \quad (2.41)$$

We now try to simplify the $\langle x_- | e^{-iH_0T} | x_a \rangle$ term. Consider,

$$e^{-iH_0T} | x_a \rangle = \sum_{n=0}^{\infty} e^{-iH_0T} | n \rangle \langle n | x_a \rangle \quad (2.42)$$

$$= \sum_{n=0}^{\infty} \psi_n(x_a) e^{-iE_nT} | n \rangle \quad (2.43)$$

where we have inserted the set of eigenstates of H_0 . Now comes a “dirty trick”. Take $H_0 \rightarrow (1 - i\epsilon)H$ (at the end we will take $\epsilon \rightarrow 0$).

$$\lim_{\epsilon \rightarrow 0} \lim_{T \rightarrow \infty} e^{-iH_0(1-i\epsilon)T} | x_a \rangle = \lim_{\epsilon \rightarrow 0} \lim_{T \rightarrow \infty} \sum_{n=0}^{\infty} \psi_n(x_a) e^{-iE_nT} e^{-\epsilon T E_n} | n \rangle \quad (2.44)$$

Choose spectrum as $E_n > 0, n > 0$ and $E_0 = 0$. If we take $T \rightarrow \infty$ then the only term that contributes from the sum is the one that isn't effected by the damping term, $e^{-\epsilon T E_n}$ (the $|0\rangle$ term).

$$\lim_{\epsilon \rightarrow 0} \lim_{T \rightarrow \infty} e^{-iH_0(1-i\epsilon)T} | x_a \rangle = \lim_{\epsilon \rightarrow 0} \psi_0(x_a) | 0 \rangle \quad (2.45)$$

$$= \psi_0(x_a) | 0 \rangle \quad (2.46)$$

This key step helps us take position eigenstates and relate them to what we really want which is energy eigenstates. Define \bar{U} as U with $H_0 \rightarrow H_0(1 - i\epsilon)$

$$\lim_{\epsilon \rightarrow 0} \lim_{T \rightarrow \infty} \bar{U}(x_a \rightarrow x_a, 2T) = |\psi_0(x_a)|^2 \int dx_+ dx_- \langle 0 | x_+ \rangle \langle x_+ | e^{-i \int_{-t_0}^{t_0} H dt} | x_- \rangle \langle x_- | 0 \rangle \quad (2.47)$$

$$= |\psi_0(x_a)|^2 \langle 0 | e^{-\int_{-t_0}^{t_0} H dt} | 0 \rangle \quad (2.48)$$

$$= |\psi_0(x_a)|^2 \langle 0, t_0 | 0, -t_0 \rangle \quad (2.49)$$

$$\int_{-\infty}^{\infty} dx_a \lim_{\epsilon \rightarrow 0} \lim_{T \rightarrow \infty} \bar{U}(x_a \rightarrow x_a, 2T) = \langle 0, t_0 | 0, -t_0 \rangle \quad (2.50)$$

due to the normalization in the wavefunction. We now go back to our formula for the path integral. Note that don't go to our "final" form of our path integral equation since to simplify the momentum integral we assumed H of the form $\frac{p^2}{2m} + V$ (which is not the case now that we threw in the $i\epsilon$). Instead we go to our earlier form (see equation 2.25).

$$\lim_{T \rightarrow \infty} \bar{U} = \int \mathcal{D}x \mathcal{D}p \exp \left(i \int_{-\infty}^{\infty} dt (p\dot{x} - (1 - i\epsilon)H_0 + f(t)x) \right) \quad (2.51)$$

$$= \int \mathcal{D}x \int \prod \frac{dp}{2\pi} \exp \left(i\Delta t \frac{(1 - i\epsilon)}{2} \left\{ -p^2 + 2p\dot{x}(1 + i\epsilon) - \dot{x}^2(1 + i\epsilon)^2 + \dot{x}^2(1 + i\epsilon)^2 + 2(1 + i\epsilon)f(t)x - \omega^2 x^2 \right\} \right) \quad (2.52)$$

where we have used $1 + i\epsilon \approx (1 - i\epsilon)^{-1}$, and we ignore the constant term in H_0 since it has no impact on the final result. The term in the exponential can be factored into a constant term (with respect to dp) and a $-(p - \dot{x}(1 + i\epsilon))^2$ term. We define $p' \equiv p - \dot{x}(1 + i\epsilon)$. We have the integral

$$\int \frac{dp}{2\pi} \exp \left(-i\Delta t \frac{1 - i\epsilon}{2} p^2 \right) = \frac{1}{2\pi} \sqrt{\frac{2\pi}{i\Delta t}} (1 + i\epsilon)^{1/2} \quad (2.53)$$

$$= \sqrt{\frac{-i}{2\pi\Delta t}} (1 + i\epsilon) \quad (2.54)$$

Finally we have

$$\lim_{T \rightarrow \infty} \bar{U} = \mathcal{C} \int \mathcal{D}x \exp \left(i \int_{-\infty}^{\infty} dt \frac{\dot{x}^2}{2} (1 + i\epsilon) - \frac{\omega^2 x^2}{2} (1 - i\epsilon) + f(t)x \right) \quad (2.55)$$

To do this integral we need a way to put \dot{x} and x in a similar form. We do this using a Fourier transform:

$$x(t) = \int_{-\infty}^{\infty} \frac{dE}{2\pi} e^{-iEt} \tilde{x}(E) \quad (2.56)$$

$$\dot{x}(t) = \int \frac{dE}{2\pi} (-iE) e^{-iEt} \tilde{x}(E) \quad (2.57)$$

$$f(t) = \int \frac{dE}{2\pi} e^{-iEt} \tilde{f}(E) \quad (2.58)$$

The integration measure can be changed to $\mathcal{D}\tilde{x}$. This is because every variable in x space has a one-to-one correspondence with the variables in k space and the Jacobian to switch spaces is one (this is shown in appendix 2.A). We have

$$\int dt f(t)x = \int \frac{dE}{2\pi} \tilde{f}(E) \int \frac{dE'}{2\pi} \tilde{x}(E') 2\pi \delta(E + E') \quad (2.59)$$

$$= \int \frac{dE}{2\pi} \tilde{f}(E) \tilde{x}(-E) \quad (2.60)$$

$$= \frac{1}{2} \left(\int \frac{dE}{2\pi} \tilde{f}(E) \tilde{x}(-E) + \tilde{f}(-E) \tilde{x}(E) \right) \quad (2.61)$$

Similarly,

$$\int dt \dot{x}^2 = \int \frac{dE}{2\pi} \tilde{x}(E) \int \frac{dE'}{2\pi} \tilde{x}(E') 2\pi \delta(E + E') (-1) E E' \quad (2.62)$$

$$= \int \frac{dE}{2\pi} \tilde{x}(E) \tilde{x}(-E) E^2 \quad (2.63)$$

$$= \frac{1}{2} \int \frac{dE}{2\pi} E^2 \left(\tilde{x}(E) \tilde{x}(-E) + \tilde{x}(-E) \tilde{x}(E) \right) \quad (2.64)$$

$$\lim_{T \rightarrow \infty} \bar{U} = \mathcal{C} \int \mathcal{D}\tilde{x} \exp \left(i \frac{1}{2} \int \frac{dE}{2\pi} \overbrace{\left(E^2 (1 + i\epsilon) - \omega^2 (1 - i\epsilon) \right)}^{E^2 - \omega^2 + i\epsilon} \tilde{x}(E) \tilde{x}(-E) + \tilde{f}(E) \tilde{x}(-E) + \tilde{f}(-E) \tilde{x}(E) \right) \quad (2.65)$$

The idea is to complete the square for each $d\tilde{x}$. The variable that does that is $\tilde{y}(E) = \tilde{x}(E) + \frac{\tilde{f}(E)}{E^2 - \omega^2 + i\epsilon}$. The argument in the exponential turns into,

$$(\dots) = \frac{i}{2} \int \frac{dE}{2\pi} (E^2 - \omega^2 + i\epsilon) \left(\tilde{y}(E) - \frac{\tilde{f}(E)}{E^2 - \omega^2 + i\epsilon} \right) \left(\tilde{y}(-E) - \frac{\tilde{f}(-E)}{E^2 - \omega^2 + i\epsilon} \right) \quad (2.66)$$

$$+ \tilde{f}(E) \left(\tilde{y}(-E) - \frac{\tilde{f}(-E)}{E^2 - \omega^2 + i\epsilon} \right) + \tilde{f}(-E) \left(\tilde{y}(E) - \frac{\tilde{f}(E)}{E^2 - \omega^2 + i\epsilon} \right) \quad (2.67)$$

$$= \frac{i}{2} \int \frac{dE}{2\pi} (E^2 - \omega^2 + i\epsilon) \tilde{y}(-E) \tilde{y}(E) - \frac{\tilde{f}(E) \tilde{f}(-E)}{E^2 - \omega^2 + i\epsilon} \quad (2.68)$$

Thus we can write,

$$\lim_{T \rightarrow \infty} \bar{U} = \mathcal{C} \int \mathcal{D}\tilde{y} \exp \left(i \frac{1}{2} \int \frac{dE}{2\pi} \tilde{y}(E) (E^2 - \omega^2 + i\epsilon) \tilde{y}(-E) \right) \exp \left(\frac{i}{2} \int \frac{dE}{2\pi} \frac{\tilde{f}(E) \tilde{f}(-E)}{-E^2 + \omega^2 - i\epsilon} \right) \quad (2.69)$$

Now consider the special case of $f \rightarrow 0$. Then we have no perturbation and the amplitude to stay in the ground is just 1. So in this case we have (the second exponential is trivial)

$$1 = \mathcal{C} \int \mathcal{D}\tilde{y} \exp \left(i \frac{1}{2} \int \frac{dE}{2\pi} \tilde{y}(E) (E^2 - \omega^2 + i\epsilon) \tilde{y}(-E) \right) \quad (2.70)$$

This is a mathematical relation completely independent of the particular system that we consider. Thus we can insert it into equation 2.69 to get a final succinct expression for the amplitude,

$$\langle 0, t_0 | 0, -t_0 \rangle = \lim_{\epsilon \rightarrow 0} \exp \left(\frac{i}{2} \int \frac{dE}{2\pi} \frac{\tilde{f}(E) \tilde{f}(-E)}{-E^2 + \omega^2 + i\epsilon} \right) \quad (2.71)$$

The $(E^2 - \omega^2)^{-1}$ factor is reminiscent of the inverse of the simple harmonic oscillator equation. This hints that this can be reexpressed in terms of the Green's function for simple harmonic motion defined by:

$$(\partial_t^2 + \omega^2) G(t - t') = \delta(t - t') \quad (2.72)$$

Writing both sides out using a Fourier decomposition,

$$\int \frac{dE}{2\pi} (-E^2 + \omega^2) \tilde{G}(E) e^{-iE(t-t')} = \int \frac{dE}{2\pi} e^{-iE(t-t')} \quad (2.73)$$

and so,

$$\tilde{G} = \frac{1}{E^2 - \omega^2} \quad (2.74)$$

Now taking the inverse Fourier Transform,

$$G(t' - t) = \int \frac{dE}{2\pi} \frac{e^{-iE(t-t')}}{-E^2 + \omega^2} \quad (2.75)$$

As always there is an ambiguity on the path of integration. We choose the Feynman decomposition:

$$G = \lim_{\epsilon \rightarrow 0} \int \frac{dE}{2\pi} \frac{e^{-iE(t-t')}}{-E^2 + \omega^2 - i\epsilon} \quad (2.76)$$

Using this expression we can rewrite the amplitude in equation 2.71. The argument in the exponential is

$$(\dots) = \frac{i}{2} \int \frac{dE}{2\pi} \int dt dt' f(t) f(t') \frac{e^{iEt} e^{-iEt'}}{E^2 - \omega^2 + i\epsilon} \quad (2.77)$$

$$= \frac{i}{2} \frac{dE}{2\pi} \int dt dt' f(t) G(t - t') f(t') \quad (2.78)$$

where we have used the symmetric property of the Green's functions: $G(t - t') = G(t' - t)$. Applying this into our equation above we have

$$\langle 0, t_0 | 0, -t_0 \rangle = \exp \left(\frac{i}{2} \int_{-t_0}^{t_0} dt dt' f(t) G(t - t') f(t') \right) \quad (2.79)$$

Alternatively we can express the amplitude using the definition,

$$W[f(t)] \equiv \lim_{T \rightarrow \infty} \int_{x(T)=x(-T)} \mathcal{D}x \exp \left[i \int_{-T}^T dt (\bar{\mathcal{L}}_0 + fx) \right] = \int \mathcal{D}x \exp \left[i \int_{-\infty}^{\infty} dt (\bar{\mathcal{L}}_0 + fx) \right] \quad (2.80)$$

where $\bar{\mathcal{L}}_0$ is the free Lagrangian with all the ϵ inserted. Using Eq. 2.55 we see that we can write,

$$\langle 0, t_0 | 0, -t_0 \rangle = \lim_{\epsilon \rightarrow 0} CW[f] \quad (2.81)$$

if $f = 0$ then

$$\langle 0, t_0 | 0, -t_0 \rangle = 1 \quad \Rightarrow \mathcal{C} = \frac{1}{W[0]} \quad (2.82)$$

So in our new notation the amplitude is given by,

$$\langle 0, t_0 | 0, -t_0 \rangle = \frac{W[f]}{W[0]} \quad (2.83)$$

where,

$$W[f] = W[0] \exp \left[\frac{i}{2} \int \frac{dE}{2\pi} \frac{\tilde{f}(E)\tilde{f}(-E)}{-E^2 + \omega^2 - i\epsilon} \right] \quad (2.84)$$

$$= W[0] \exp \left[\frac{i}{2} \int dt dt' f(t)G(t-t')f(t') \right] \quad (2.85)$$

since $W[0]$ just cancels away in the calculation of a transition amplitude we don't need to calculate it explicitly.

2.2 Anharmonic Oscillator

Up until now the Hamiltonian of our system was the simply quadratic free Hamiltonian with a time dependent perturbation. The eigenstates were left unchanged but the $f(t)x$ term allowed the possibility of a state transitioning into another state. The fact that we were able to complete the square and exactly integrate the exponential was instrumental in finding a simple form for the amplitude. This was a consequence of not having any terms in the exponential that are more than quadratic in x . We now consider modifying the Hamiltonian to include such interactions.

Consider the anharmonic oscillator with a driving force:

$$H_0 = \frac{p^2}{2} + \frac{\omega^2 x^2}{2} \quad (2.86)$$

$$H_a = \frac{\lambda x^4}{4!} \quad (2.87)$$

and

$$H = H_0 + H_a - f(t)x \quad (2.88)$$

We have some ground state, $|\Omega\rangle$. We add a constant to the Hamiltonian such that $E_\omega = 0$. Our discussion earlier was rather general and we can reuse some of our earlier results

$$\langle \Omega, t_0 | \Omega, -t_0 \rangle = \frac{W_\lambda[f]}{W_\lambda[0]}, \quad (2.89)$$

where

$$W_\lambda[f] \equiv \int \mathcal{D}x \exp \left[i \int_{-\infty}^{\infty} dt (\tilde{\mathcal{L}}_0 - (\frac{\lambda x^4}{4!})(1 - i\epsilon) + f(t)x) \right] \quad (2.90)$$

Due to the quartic term we cannot simply do this integral as before. There is not general way to integrate this Gaussian. What we can do, is do a small λ expansion (perturbation theory).

The $1 - i\epsilon$ is important in the initial Lagrangian ($\bar{\mathcal{L}}_0$) to pick out the ground state and relate W to the amplitude of transition from the ground state. However it turns out that the $1 - i\epsilon$ has no effect on the anharmonic term. [Q 2: Check Peskin.]. Thus we can take the ϵ that's there to zero. We do a small λ expansion (i.e. Taylor expansion):

$$W_\lambda [f] = \int \mathcal{D}x \exp \left[i \int_{-\infty}^{\infty} dt (\bar{\mathcal{L}}_0 - fx) \right] \exp \left[\frac{-i\lambda}{4!} \int_{-\infty}^{\infty} dt x^4 \right] \quad (2.91)$$

$$\approx \int \mathcal{D}x \exp \left[i \int_{-\infty}^{\infty} dt (\bar{\mathcal{L}}_0 - fx) \right] \left\{ 1 - \frac{i\lambda}{4!} \int dt x^4(t) + \frac{\lambda^2}{(4!)^2} \int dt_1 dt_2 x^4(t_1) x^4(t_2) + \dots \right\} \quad (2.92)$$

We can write this discretely as

$$W_\lambda [f] \approx \prod_{i=1}^{N-1} \int dx_i \exp \left[i \sum_n \Delta t (\bar{\mathcal{L}}_{0,n} - f_n x_n) \right] \left\{ 1 - \frac{i\lambda}{4!} \Delta t \sum_j x_j^4 + \frac{\lambda^2}{(4!)^2} (\Delta t)^2 \sum_{j_1, j_2} x_{j_1}^4 x_{j_2}^4 + \dots \right\} \quad (2.93)$$

or compactly as

$$W_\lambda [f] = W_0 [f] - \frac{i\lambda}{4!} \int_{-\infty}^{\infty} dt \int \mathcal{D}x x^4(t) \exp \left[i \int_{-\infty}^{\infty} dt (\bar{\mathcal{L}}_0 + fx) \right] + \frac{(-i)^2}{2!} \left(\frac{\lambda}{4!} \right)^2 \int dt_1 dt_2 \int \mathcal{D}x x^4(t_1) x^4(t_2) \exp \left[i \int dt (\bar{\mathcal{L}}_0 + fx) \right] + \dots \quad (2.94)$$

First consider $W_0 [f]$. We already know this function since we calculated it last class (we called it $W [f]$ before we had a perturbation):

$$W_0 [f] = \int \mathcal{D}x \exp \left[i \int_{-\infty}^{\infty} dt (\bar{\mathcal{L}}_0 + fx) \right] \quad (2.95)$$

$$= \prod_{i=1}^N \int dx_i \exp \left[i \sum_{j=1}^N \Delta t (\bar{\mathcal{L}}_{0,j} + f_j x_j) \right] \quad (2.96)$$

where $f_j = f(-T + j\Delta t)$ and $x_j = x(-T + j\Delta t)$. We know how to do this integral explicitly (see Eq. 2.85). We hope to rewrite the other terms in terms of $W_0 [f]$. To do this we define something called a ‘‘functional derivative’’:

$$\frac{\delta W_0}{\delta f(t_{j'})} \equiv \frac{1}{\Delta t} \frac{\partial W_0}{\partial f_{j'}} \quad (2.97)$$

In our case this is given by

$$\frac{\delta W_0}{\delta f(t_{j'})} = \prod_{i=1}^N \int dx_i i x_{j'} \exp \left[i \sum_{j=1}^N \Delta t (\bar{\mathcal{L}}_{0,j} + f_j x_j) \right] \quad (2.98)$$

$$= i \int \mathcal{D}x x(t_{j'}) \exp \left[i \int dt (\bar{\mathcal{L}}_0 + f x) \right] \quad (2.99)$$

So at least in this case taking the functional derivative of W_0 with respect to f at a time $t_{j'}$ does the thing you would expect; it pulls out a factor of $i x(j')$.

With this in mind we can write

$$\begin{aligned} W_\lambda[f] &= W_0[f] - \frac{i\lambda}{4!} \int_{-\infty}^{\infty} dt \left(\frac{1}{i} \frac{\delta}{\delta f(t)} \right)^4 W_0[f] \\ &+ \frac{(-i)^2}{2!} \left(\frac{\lambda}{4!} \right)^2 \int dt_1 dt_2 \left(\frac{1}{i} \frac{\delta}{\delta f_1} \right)^4 \left(\frac{1}{i} \frac{\delta}{\delta f_2} \right)^4 W_0[f] + \dots \end{aligned} \quad (2.100)$$

We begin by considering $\mathcal{O}(\lambda)$:

$$- \frac{i\lambda}{4!} W_0[0] \int dt \left(\frac{1}{i} \frac{\delta}{\delta f(t)} \right)^4 \exp \left[\frac{i}{2} \int d\tau d\tau' f(\tau) G(\tau - \tau') f(\tau') \right] \quad (2.101)$$

For brevity we write,

$$\exp[\dots] \equiv \exp \left[\frac{i}{2} \int d\tau d\tau' G(\tau - \tau') f(\tau) f(\tau') \right] \quad (2.102)$$

We now take the functional derivatives:

$$\frac{1}{i} \frac{\delta}{\delta f(t)} \exp[\dots] = \frac{1}{i \Delta t} \frac{\partial}{\partial f_{j'}} \exp \left[\frac{i}{2} \sum_{i,j} \Delta t \Delta t G(\tau_i - \tau_j) f_i f_j \right] \quad (2.103)$$

$$= \frac{1}{i} \frac{1}{\Delta t} \frac{i}{2} \Delta t^2 \left(\sum_i G(\tau_i - t_{j'}) f_i + \sum_j G(t_{j'} - \tau_j) f_j \right) \exp[\dots] \quad (2.104)$$

$$= \frac{1}{2} \Delta t \left(\sum_i G(\tau_i - t_{j'}) f_i + \sum_j G(t_{j'} - \tau_j) f_j \right) \exp[\dots] \quad (2.105)$$

$$= \sum_j \Delta t G(t_{j'} - \tau_j) f_j \exp[\dots] \quad (2.106)$$

$$= \int d\tau' G(t - \tau') f(\tau') \exp[\dots] \quad (2.107)$$

So again the functional derivative does what one would expect. It takes the derivative of the arguement of the exponential and uses the product rule on it. Continuing on gives,

$$\left(\frac{1}{i} \frac{\delta}{\delta f(t)} \right)^2 \exp[\dots] = \frac{1}{i} G(t - t) \exp[\dots] + \int d\tau' d\tau'' G(t - \tau') G(t - \tau'') f(\tau') f(\tau'') \exp[\dots] \quad (2.108)$$

$$\left(\frac{1}{i} \frac{\delta}{\delta f(t)}\right)^3 \exp[\dots] = \left\{ \frac{1}{i} G(t-t) \int d\tau' G(t-\tau') f(\tau') + \frac{2}{i} G(t-t) \int d\tau' G(t-\tau') f(\tau') \right. \\ \left. + \int d\tau' d\tau'' d\tau''' G(t-\tau') G(t-\tau'') G(t-\tau''') f(\tau') f(\tau'') f(\tau''') \right\} \exp[\dots] \quad (2.109)$$

$$= \left\{ \frac{3}{i} G(t-t) \int d\tau' G(t-\tau') f(\tau') \right. \\ \left. + \int d\tau' d\tau'' d\tau''' G(t-\tau') G(t-\tau'') G(t-\tau''') f(\tau') f(\tau'') f(\tau''') \right\} \exp[\dots] \quad (2.110)$$

$$\left(\frac{1}{i} \frac{\delta}{\delta f(t)}\right)^4 \exp[\dots] = \left\{ -3G^2(t-t) + \frac{3}{i} G(t-t) \int d\tau' d\tau'' G(t-\tau') G(t-\tau'') f(\tau') f(\tau'') \right. \\ \left. + \frac{3}{i} G(t-t) \int d\tau' d\tau'' G(t-\tau') G(t-\tau'') f(\tau') f(\tau'') \right. \quad (2.111)$$

$$+ \int d\tau' d\tau'' d\tau''' d\tau'''' G(t-\tau') G(t-\tau'') G(t-\tau''') G(t-\tau'''')) \\ \left. \times f(\tau') f(\tau'') f(\tau''') f(\tau''') \right\} \exp[\dots] \quad (2.112)$$

$$= \left\{ -3G^2(t-t) + \frac{6}{i} G(t-t) \int d\tau' d\tau'' G(t-\tau') G(t-\tau'') f(\tau') f(\tau'') \right. \\ \left. + \int d\tau' d\tau'' d\tau''' d\tau'''' G(t-\tau') G(t-\tau'') G(t-\tau''') G(t-\tau'''')) \right. \\ \left. \times f(\tau') f(\tau'') f(\tau''') f(\tau''') \right\} \exp[\dots] \quad (2.113)$$

We now consider $W_\lambda[0]$ (i.e. the above expression with $f = 0$). This is the normalization term for every amplitude.

$$W_\lambda[0] = W_0[0] + i \frac{3\lambda}{4!} W_0[0] \int_{-\infty}^{\infty} G^2(t-t) dt \quad (2.114)$$

and in summary we have,

$$W_\lambda[f] = W_0[f] - W_0[f] \frac{i\lambda}{4!} \int_{-\infty}^{\infty} dt \left\{ -3G^2(t-t) \right. \\ \left. + \frac{6}{i} G(t-t) \int d\tau' d\tau'' G(t-\tau') G(t-\tau'') f(\tau') f(\tau'') \right. \\ \left. + \int d\tau' d\tau'' d\tau''' d\tau'''' G(t-\tau') G(t-\tau'') G(t-\tau''') G(t-\tau'''')) \right. \\ \left. \times f(\tau') f(\tau'') f(\tau''') f(\tau''') \right\} \quad (2.115)$$

Recall that the transitions amplitudes are,

$$\langle 0, -T|0, T\rangle = \frac{W_\lambda[f]}{W_\lambda[0]} \quad (2.116)$$

So to first order the transition amplitude is,

$$\begin{aligned} \langle 0, -T|0, T\rangle = 1 - \frac{i\lambda}{4!} \int_{-\infty}^{\infty} dt \left\{ -3G^2(t-t) + \frac{6}{i}G(t-t) \int d\tau' d\tau'' G(t-\tau')G(t-\tau'') \right. \\ \times f(\tau')f(\tau'') + \int d\tau' d\tau'' d\tau''' d\tau'''' G(t-\tau')G(t-\tau'')G(t-\tau''')G(t-\tau'''') \\ \left. \times f(\tau')f(\tau'')f(\tau''')f(\tau'''') \right\} \quad (2.117) \end{aligned}$$

This calculation was quite tedious and was only to first order. Nevertheless there are two main independent ingredients that we can identify. The order of the interaction (which is always accompanied by an integration of t) and the Green's function. You may think that the factors $f(\tau)d\tau$ are also independent but that's not true. The number of these factors and their arguments are completely determined by the form of the Green's functions. Furthermore, as we will see for our purposes we will take these to zero at the end of the calculations. So we can now identify the pieces that will always be part of the transition amplitude calculations. These will be our "Feynman Rules":

$$\begin{aligned} \times & \rightarrow -i\lambda \int_{-\infty}^{\infty} dt \\ \text{---} & \rightarrow G(t_1 - t_2) \end{aligned}$$

Furthermore, a time will be given to each intersection point in a diagram and a symmetry factor is also required for each diagram. So for example we can write,

$$W_\lambda[0] = W_0[0] \left(1 + \text{diagram 1} + \text{diagram 2} + \text{diagram 3} + \text{diagram 4} + \dots \right)$$

2.3 Correlation Functions

Now that we have an idea of how to generalize our discussion to more complicated potentials we quickly go back again to consider the simple harmonic oscillator. Consider the following derivative with $t_1 < t_2$:

$$\frac{1}{i^2} \frac{\delta^2 W_0[f]}{\delta f(t_1) \delta f(t_2)} \Big|_{f \rightarrow 0} = W_0[0] G(t_1 - t_2) \quad (2.118)$$

Alternatively we can rewrite this expression in terms of the Lagrangian,

$$\frac{1}{i^2} \frac{\delta^2 W_0}{\delta f_1 \delta f_2} \Big|_{f \rightarrow 0} = \int \mathcal{D}x \, x(t_1)x(t_2) \exp \left(i \int dt \bar{\mathcal{L}}_0 \right) \quad (2.119)$$

$$= \lim_{T \rightarrow \infty} \int_{t_2}^T \mathcal{D}x \int dx_2 \int_{t_1}^{t_2} \mathcal{D}x \int dx_1 \int_{-T}^{t_1} \mathcal{D}x \, x_2 x_1 \times \exp \left[i \left(\int_{-T}^{t_1} + \int_{t_1}^{t_2} + \int_{t_2}^T \right) dt \bar{\mathcal{L}} \right] \quad (2.120)$$

$$= \lim_{T \rightarrow \infty} \int_{t_2}^T \mathcal{D}x \int dx_2 \, x_2 \int_{t_1}^{t_2} \mathcal{D}x \int dx_1 \, x_1 \int_{-T}^{t_1} \mathcal{D}x \times \exp \left[i \left(\int_{-T}^{t_1} + \int_{t_1}^{t_2} + \int_{t_2}^T \right) dt \bar{\mathcal{L}} \right] \quad (2.121)$$

$$= \int dx_1 \int dx_2 x_1 x_2 \langle 0, T | x_2, t_2 \rangle \langle x_2, t_2 | x_1, t_1 \rangle \langle x_1, t_1 | 0, -T \rangle W_0 [0] \quad (2.122)$$

where in the second line we have split the integration interval into the different parts. This is trivial in the exponential. However, on the outside extra spatial integrals over dx_1 and dx_2 appear due to the way the $\mathcal{D}x$ are defined; they have the end points fixed so to split them up you need to have an integral over each midpoint.

We now switch to the Heisenberg picture. We choose our reference time $-T$. By definition then we have

$$|0\rangle \equiv |0, -T\rangle \quad (2.123)$$

likewise we define

$$|x\rangle \equiv |x, -T\rangle . \quad (2.124)$$

With this reference point we can move into the Heisenberg interperotation. The Heisenberg form of the position operator is, $\hat{x}(t) = e^{-iH_0(t+T)} \hat{x}(-T) e^{iH_0(t+T)}$. We want to think of x_1 as an eigenvalue of this operator. In other words

$$x_i |x_i\rangle = \hat{x}(t_i) |x_i\rangle \quad (2.125)$$

This enables us to get rid of the x integrals:

$$\frac{1}{i^2} \frac{\delta^2 W}{\delta f_1 \delta f_2} \Big|_{f \rightarrow 0} = W_0 [0] \langle 0 | \hat{x}(t_2) \hat{x}(t_1) | 0 \rangle \quad (2.126)$$

This expression holds if $t_2 > t_1$. If $t_1 > t_2$ everything is the same by symmetry but $t_1 \leftrightarrow t_2$ and so we have,

$$\frac{1}{W_0 [0]} \frac{1}{i^2} \frac{\delta^2 W}{\delta f_1 \delta f_2} \Big|_{f \rightarrow 0} = \langle 0 | \mathcal{T} (\hat{x}(t_2) \hat{x}(t_1)) | 0 \rangle \quad (2.127)$$

where \mathcal{T} denotes the time ordering operator which puts the operator with the earliest time to the right. So we see that correlation functions (key objects in Quantum Field

Theory!) can be related to W by taking derivatives. In the case of the anharmonic oscillator we have already taken the first derivative of $W_0[f]$ (see Eq 2.107) we just need to take a second at a different time,

$$\langle 0|\mathcal{T}(\hat{x}(t_2)\hat{x}(t_1))|0\rangle = \frac{1}{W_0[0]} \frac{1}{i} \frac{\delta}{\delta f_1} \left(\int d\tau' G(t_2 - \tau') f(\tau') \exp[\dots] \right) \Big|_{f \rightarrow 0} \quad (2.128)$$

$$= \frac{1}{i} G(t_2 - t_1) \quad (2.129)$$

Our discussion did not depend much on the form of the Hamiltonian. If we had Eq 2.113 with or without $\lambda = 0$ the results would be the same as long as we label our ground state by $|\Omega\rangle$ (the ground state of the full system (in our case the anharmonic theory),

$$\langle \Omega|\mathcal{T}(\hat{x}(t_1)\hat{x}(t_2))|\Omega\rangle = \frac{1}{W_\lambda[0]} \frac{1}{i^2} \frac{\delta^2 W_\lambda[f]}{\delta f_1 \delta f_2} \Big|_{f \rightarrow 0} \quad (2.130)$$

$$= \frac{1}{W_\lambda[0]} \frac{1}{i^2} \frac{\delta^2}{\delta f_1 \delta f_2} \left(W_0 - \frac{i\lambda}{4!} \int dt \left(\frac{1}{i} \frac{\delta}{\delta f} \right)^4 W_0 + \dots \right) \Big|_{f \rightarrow 0} \quad (2.131)$$

We already sketched out $W_\lambda[0]$. Calculating

$$\frac{\delta^2 W_\lambda}{\delta f_1 \delta f_2} \Big|_{f \rightarrow 0} \quad (2.132)$$

explicitly gives any possible diagram with two external points (two unconnected times, t_1 and t_2). We show this to first order. The expression for W_λ is given in Eq 2.113. Taking its derivatives gives to first order,

$$\begin{aligned} \frac{1}{i^2} \frac{\delta^2 W_\lambda}{\delta f_1 \delta f_2} \Big|_{f \rightarrow 0} &= \left\{ \frac{1}{i} G(t_1 - t_2) - \frac{i\lambda}{4!} \int dt 3iG^2(0)G(t_1 - t_2) \right. \\ &\quad \left. + 12iG(0)G(t - t_1)G(t - t_2) \right\} W_0[0] \end{aligned} \quad (2.133)$$

The zero'th order contribution is a Green's function starting at the initial point and ending at the second (i.e. from t_1 to t_2). The first order contributions are the a double bubble diagram (due to the $G^2(0)$ factor) with the zero'th order Green's function and the second term is a bubble and Green's function going in and one going out. Now to find the amplitudes we need to divide by $W_\lambda[0]$ to get the correlation functions. We calculated this quantity earlier on and to first order is given by,

$$W_\lambda[0] = W_0[0] \left(1 - \frac{3\lambda i}{4!} \int dt G^2(0) \right) \quad (2.134)$$

So to first order we have,

$$\langle \Omega | \mathcal{T} (\hat{x}(t_1) \hat{x}(t_2)) | \Omega \rangle = \frac{\frac{1}{i} G(t_1 - t_2) + \frac{\lambda}{4!} \int dt 3G^2(0)G(t_1 - t_2) + 12G(0)G(t - t_1)G(t - t_2)}{1 - \frac{3i\lambda}{4!} \int dt G^2(0)} \quad (2.135)$$

$$= \frac{(\frac{1}{i} G(t_1 - t_2) + \frac{i\lambda}{4!} \int 4G(0)G(t - t_1)G(t - t_2)) (1 - \frac{i\lambda}{4!} \int dt 3G^2(0))}{1 - \frac{3i\lambda}{4!} \int dt G^2(0)} \quad (2.136)$$

$$= \frac{1}{i} G(t_1 - t_2) + \frac{i\lambda}{4!} \int 4G(0)G(t - t_1)G(t - t_2) \quad (2.137)$$

To all orders we have,

$$\begin{aligned} & \text{---} + \text{---} \cup \text{---} + \text{---} \cap \text{---} + \text{---} \cup \cup \text{---} + \text{---} \cup \cap \text{---} + \text{---} \cup \cap \cup \text{---} + \dots \\ & \hline & 1 + \cap + \cup \cup + \dots \\ & = \text{---} + \text{---} \cup \text{---} + \text{---} \cup \cup \text{---} + \dots \end{aligned}$$

We now briefly summarize what we have done. We have the correlation functions for anharmonic oscillator (x^4):

$$\langle \Omega | \mathcal{T} (\hat{x}(t_1) \hat{x}(t_2)) | \Omega \rangle = \frac{1}{W_\lambda [0]} \left. \frac{\delta^2 W_\lambda [f]}{\delta f_1 \delta f_2} \right|_{f \rightarrow 0} = \sum \text{connected Feynman diagrams} \quad (2.138)$$

where $W_\lambda [f]$ is the generating functional and $f_i \equiv f(t_i)$. We found the propagator (to lowest order) is given by,

$$\text{---} \rightarrow G(t_1 - t_2) = \int \frac{dE}{2\pi} \frac{e^{-iE(t_1 - t_2)}}{-E^2 + \omega^2 + i\epsilon}$$

and the vertex is given by

$$\times \rightarrow -i\lambda \int dt$$

$\times \rightarrow -i\lambda \int dt$. These rules are the space time Feynman rules. By Fourier transforming both sides we have the momentum space rules:

$$\begin{aligned} \text{---} & \rightarrow \frac{1}{-E^2 + \omega^2 + i\epsilon} \\ \times & \rightarrow -i\lambda \delta(\sum_i E_i) \end{aligned}$$

2.A Jacobian of the Fourier Transformation

The discrete Fourier transform is given by

$$\tilde{x}_k = \frac{1}{\sqrt{N}} \sum_n e^{-2\pi i n k / N} x_n \quad (2.139)$$

The Jacobian is then the matrix

$$J_{n,k} = \frac{\partial \tilde{x}_k}{\partial x_n} = \frac{1}{\sqrt{N}} e^{-2\pi i n k / N} \quad (2.140)$$

We want to show that the absolute value of the determinate of this matrix is equal to 1.

Note that the determinate any orthonormal matrix is ± 1 since:

$$1 = \det(A^{-1}A) = \det A^T \det A = (\det A)^2 \quad (2.141)$$

thus if we can show that J is orthonormal then $|\det J| = 1$.

A matrix is orthonormal if its rows and columns are orthonormal. We need to show that

$$\langle J_k | J_j \rangle = \frac{1}{N} (e^{2\pi i k / N}, e^{2\pi i 2k / N}, \dots, e^{2\pi i k}) \begin{pmatrix} e^{-2\pi i j / N} \\ e^{-2\pi i 2j / N} \\ \vdots \\ e^{-2\pi i j} \end{pmatrix} \quad (2.142)$$

$$= \frac{1}{N} (e^{2\pi i(k-j)/N} + e^{2\pi i 2(k-j)/N} + \dots e^{2\pi i(k-j)}) \quad (2.143)$$

is equal to 0 for all row and column numbers, $k - j \neq 0$. Let $k - j = m \in \mathbb{Z}$. Then we have,

$$= \frac{1}{N} \sum_{j=1}^N e^{2\pi i j m / N} \quad (2.144)$$

$$= \frac{1}{N} \left(\sum_{j=1}^{N/2m} e^{2\pi i j m / N} + \sum_{j=N/2m+1}^{2N/2m} e^{2\pi i j m / N} + \dots + \sum_{j=(2m-1)N/2m+1}^N e^{2\pi i j m / N} \right) \quad (2.145)$$

$$= \frac{1}{N} \sum_{\ell=0}^{2m-2} \sum_{j=\ell N/2m+1}^{(\ell+1)N/2m} e^{2\pi i j m / N} \quad (2.146)$$

shifting the exponential such that each sum starts at $j = 1$ gives,

$$\sum_{j=\ell N/2m+1}^{(\ell+1)N/2m} e^{2\pi i j m / N} \rightarrow e^{\ell \pi i} \sum_{j=1}^{N/2m} e^{2\pi i j m / N} \quad (2.147)$$

Thus we finally get that the dot product between an arbitrary row and column is given by,

$$= \frac{1}{N} \left(\sum_{j=1}^{N/2m} e^{2\pi i j m / N} \right) \underbrace{\left(\sum_{\ell=0}^{2m-2} e^{\ell \pi i} \right)}_{=0} \quad (2.148)$$

$$= 0 \quad (2.149)$$

There is one loophole. If $k - j = m = 0$ then we can't divide by m . In this case we get,

$$\frac{1}{N} \sum_{j=1}^N e^{2\pi i j m / N} = \frac{1}{N} \sum_{j=1}^N 1 = 1 \quad (2.150)$$

Thus $\langle J_j | J_k \rangle = \delta_{jk}$ and the Jacobian is indeed orthonormal and thus

$$|\det J| = 1 \quad (2.151)$$

as required.

Chapter 3

Quantum Field Theory

3.1 Free Klein Gordon Field

To keep things simple we consider one non-interacting real scalar field (real Klein Gordon theory). We work in $1 + 1$ dimensions.

Since we are working in 1 dimension we have

$$\phi(x, t) \xrightarrow{\text{discretize space}} \boldsymbol{\phi} = \{\phi_1(t), \phi_2(t), \dots, \phi_N(t)\} \quad (3.1)$$

where $\phi_\ell(t) = \phi(\overbrace{\ell\Delta}^{x_\ell}, t)\sqrt{\Delta}$ and the $\sqrt{\Delta}$ is a normalization factor. The equations of motion are

$$\square\phi + m^2\phi = 0 \quad (3.2)$$

To discretize them we need to find the second derivative in terms of finite differences. We have,

$$\frac{\partial^2\phi}{\partial x^2} = \frac{\partial}{\partial x} \lim_{h \rightarrow 0} \frac{\phi(x + h/2) - \phi(x - h/2)}{h} \quad (3.3)$$

$$= \lim_{h \rightarrow 0} \frac{1}{h^2} (\phi(x + h) - 2\phi(x) + \phi(x - h)) \quad (3.4)$$

$$\rightarrow \frac{1}{\Delta^2} (\phi_{\ell+1} - 2\phi_\ell + \phi_{\ell-1}) \quad (3.5)$$

So the discrete equations of motion are,

$$\ddot{\phi}_\ell + \left(\frac{2}{\Delta^2} + m^2 \right) \phi_\ell - \frac{1}{\Delta^2} (\phi_{\ell+1} + \phi_{\ell-1}) = 0 \quad (3.6)$$

Since we are working in one spatial dimension we have $L = \int dx \mathcal{L}$ with

$$\mathcal{L} = \frac{1}{2} (\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2} m^2 \phi^2 \quad (3.7)$$

where our metric is

$$g^{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (3.8)$$

Note that this Lagrangian is equivalent to having N of Harmonic oscillators (no sources or interaction terms) with one at each ℓ value. Discretizing our Lagrangian gives

$$L = \sum_{\ell=1}^N \frac{1}{2} \left(\dot{\phi}_\ell^2 - \left(\frac{\phi_{\ell+1} - \phi_\ell}{\Delta} \right)^2 - m^2 \phi_\ell^2 \right) \quad (3.9)$$

$$= \sum_{\ell=1}^N \frac{1}{2} \dot{\phi}_\ell^2 - \sum_{\ell,m=1}^N \frac{1}{2} \phi_\ell V_{\ell m} \phi_m \quad (3.10)$$

where

$$V \equiv \begin{pmatrix} \ddots & \ddots & 0 & \dots & 0 \\ 0 & -\frac{2}{\Delta^2} - m^2 & \frac{1}{\Delta^2} & 0 & 0 \\ 0 & 0 & \ddots & \ddots & 0 \\ \vdots & \vdots & 0 & \ddots & \ddots \\ 0 & 0 & 0 & 0 & \ddots \end{pmatrix} \quad (3.11)$$

is a coupling matrix between the fields at different spatial points.

Note the following oscillator correspondence:

Klien-Gordan	NR oscillator
t	t
x	label ℓ
ϕ_ℓ	$x_\ell(t)$
$\hat{\phi}_\ell(t)$	$\hat{x}_\ell(t)$
$J(x,t)$	$f_\ell(t)$

(for the time being we use hats to denote operators). We work in the Schrodinger picture:

$$\hat{\phi} = \{ \hat{\phi}_1, \dots, \hat{\phi}_N \} \quad (3.12)$$

and $\lim_{N \rightarrow \infty} \hat{\phi} / \sqrt{\Delta} = \hat{\phi}(x)$. A definite field state is given by $|\phi, t\rangle$:

$$\hat{\phi} |\phi_0, t\rangle = \phi_0 |\phi_0, t\rangle \quad (3.13)$$

we are interested in a transition between two states $\langle \phi_f, T | \phi_i, -T \rangle$. Each state consists of a direct product of states, one at each point in space. Thus finding the bracket between two field configurations is equivalent to a product of brackets. In other words we can write,

$$\langle \phi_f, T | \phi_i, -T \rangle = \langle \phi'_1, T | \phi_1, -T \rangle \langle \phi'_2, T | \phi_2, -T \rangle \dots \langle \phi'_N, T | \phi_N, -T \rangle \quad (3.14)$$

with $\langle \phi'_\ell, T | \phi_\ell, -T \rangle$ being the product of oscillator states at $\Delta\ell$. But we already know what the value of each of these brackets is for a singlet oscillator. Now we are doing the same calculation but for many oscillators. It is given by (c.f. equation 2.32)

$$\langle \phi_f, T | \phi_i, -T \rangle = \mathcal{C} \prod_{\ell=1}^{N-1} \prod_{j=1}^{M-1} \int_{\substack{\phi_{\ell,0}=\phi_{\ell i} \\ \phi_{\ell,M}=\phi_{\ell f}}} d\phi_{\ell,j} \exp \left[i \sum_{n=1}^{M-1} \Delta t L [\phi_n] \right] \quad (3.15)$$

where the label ℓ refers to spatial points and the j denotes to time slices. Furthermore, we have introduced the notation $d\phi_{\ell j} \equiv d\phi_\ell(t_j)$. We now take the limit of $N \rightarrow \infty$ and $M \rightarrow \infty$:

$$\langle \phi_f, T | \phi_i, -T \rangle = \mathcal{C} \int \mathcal{D}\phi \exp \left[i \int_T^T dt \int_{-\infty}^{\infty} dx \mathcal{L} [\phi(x, t)] \right], \quad (3.16)$$

where

$$\int \mathcal{D}\phi \equiv \prod_{\ell=1}^{N-1} \prod_{j=1}^{M-1} d\phi_{\ell,j}. \quad (3.17)$$

The interpretation is similar to before. The amplitude of going from one field configuration to another is equal to the sum over all possible field configurations. Note the extension to multidimensional derivation is completely analogous. For 3 spatial dimensions:

$$\langle \phi_f, T | \phi_i, -T \rangle = \mathcal{C} \int \mathcal{D}\phi \exp \left[i \int_{-T}^T dt \int_{-\infty}^{\infty} d^3x \mathcal{L} [\phi(\mathbf{x}, t)] \right] \quad (3.18)$$

3.2 External Force

Consider some one dimensional external force, $J(x, t)$ which discretized is given by $J_\ell(t)\sqrt{\Delta}$ (one force for each point and since this is just a field it discretizes in the same way as the field ϕ). We take the force to turn on at $-t_0$ and end at t_0 . We want to compute the probability for the system to stay in the ground state, $|0\rangle$. We take the energy of the state to be 0 ($E_0 = 0$). Everything we did for a single oscillator can be done for N oscillators. Each of these oscillators has a force which is dependent on time. We denote the force by ($\mathbf{J} = \{J_1(t), J_2(t), \dots, J_{N-1}(t)\}$) (here $J_i(t)$ is the $f(t)$ for each oscillator).

$$\langle 0, t_0 | 0, -t_0 \rangle = \lim_{\epsilon \rightarrow 0} \prod_i \frac{W[J_i]}{W_0[0]} \equiv \lim_{\epsilon \rightarrow 0} \frac{W_0[\mathbf{J}(t)]}{W_0[0]} \quad (3.19)$$

where (c.f. Eq. 2.55)

$$W_0[\mathbf{J}(t)] = \lim_{T \rightarrow \infty} \prod_{j=1}^N \int_{\substack{\phi_j(-T)= \\ \phi_j(T)}} \mathcal{D}\phi_j \exp \left[i \int_{-T}^T dt \left(\sum_{\ell=1}^N \frac{1}{2} \dot{\phi}_\ell^2 (1 + i\epsilon) - \sum_{\ell,n=1}^N \frac{1}{2} (1 - i\epsilon) \phi_\ell V_{\ell n} \phi_n + \sum_{\ell=1}^N \phi_\ell J_\ell(t) \right) \right] \quad (3.20)$$

here we have added the $1 + i\epsilon$ and $1 - i\epsilon$ factors in an analagous way to our discussion of Non-relativistic QM. While they are present for notational convenience we will just ignore them. More typically we will write

$$\lim_{T \rightarrow \infty} \prod_{j=1}^{N-1} \int_{\substack{\phi_j(-T)= \\ \phi_\ell(T)}} \mathcal{D}\phi_\ell \rightarrow \int \mathcal{D}\phi \quad (3.21)$$

and $dt dx \rightarrow d^2x$ such that

$$W_0[\mathbf{J}(t)] = \int \mathcal{D}\phi \exp \left[i \int d^2x \frac{1}{2} (\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2} m^2 \phi^2 + J(x^\mu) \phi(x^\mu) \right] \quad (3.22)$$

As before then if we can do the Gaussian integral then we are set. The way you do it is to diagonalize the argument so you just a product of individual integrals. Since the term which gives us trouble connects different space points (the derivative) we can diagonalize it using a Fourier transform,

$$W_0[\mathbf{J}(t)] = \mathcal{C} \int \mathcal{D}\tilde{\phi} \exp \left[i \int dt \frac{dk}{2\pi} \frac{1}{2} |\dot{\tilde{\phi}}|^2 - \frac{1}{2} k^2 |\tilde{\phi}|^2 - \frac{1}{2} m^2 |\tilde{\phi}|^2 + \tilde{J}^* \tilde{\phi} \right] \quad (3.23)$$

$$= \mathcal{C} \int \mathcal{D}\tilde{\phi} \exp \left[i \int dt \frac{dk}{2\pi} \frac{1}{2} |\dot{\tilde{\phi}}|^2 - \frac{1}{2} \omega(k)^2 |\tilde{\phi}|^2 + \tilde{J}^* \tilde{\phi} \right] \quad (3.24)$$

where

$$\begin{aligned} \phi(x, t) &= \int \frac{dk}{2\pi} e^{-ikx} \tilde{\phi}(k, t) \quad , \quad J(x) = \int \frac{dk}{2\pi} e^{-ikx} \tilde{J}(k, t) \\ \omega(k)^2 &\equiv k^2 + m^2 \end{aligned} \quad (3.25)$$

and we have changed our integration measure using,

$$\prod_{\ell=1}^{N-1} \int \mathcal{D}\phi_\ell = \prod_{\ell=1}^{N-1} \prod_{m=1}^{M-1} \int d\phi_{\ell m} = \mathcal{J} \prod_{j=1}^{N-1} \prod_{m=1}^{M-1} \int d\tilde{\phi}_{j,m} \quad (3.26)$$

where \mathcal{J} is a Jacobian which we can forget about since it just adds to the overall constant. Thus we can write

$$\prod_{\ell=1}^{N-1} \int \mathcal{D}\phi_\ell \rightarrow \prod_{j=1}^{N-1} \int \mathcal{D}\tilde{\phi}_j \quad (3.27)$$

Now we have (we don't need to worry about the factor of 2π as it goes to the constant in front),

$$W_0[\mathbf{J}(t)] = \mathcal{C} \prod_{k=1}^N \left\{ \int \mathcal{D}\phi \exp \left[i \int_{-\infty}^{\infty} dt \left(\frac{1}{2} |\dot{\phi}_k|^2 - \frac{1}{2} \omega_k^2 |\tilde{\phi}_k|^2 + \tilde{\phi}_k^* \tilde{J}_k \right) \right] \right\} \quad (3.28)$$

$$= \mathcal{C} \prod_{k=1}^N W_0^{SHO} \left[\tilde{J}_k(t), \omega_k \right] \quad (3.29)$$

where W_0^{SHO} is a quantity that we already calculated earlier,

$$W_0^{SHO}[f, \omega] = \exp \left\{ \frac{i}{2} \int dt dt' f(t) G(t-t') f(t') \right\} \quad (3.30)$$

$$= \exp \left[\frac{i}{2} \int \frac{dE}{2\pi} \frac{\tilde{f}(E) \tilde{f}(-E)}{-E^2 + \omega^2 - i\epsilon} \right] \quad (3.31)$$

We can finally write

$$W_0[\mathbf{J}] = W_0[0] \exp \left[\frac{i}{2} \sum_{k=1}^N \int \frac{dE}{2\pi} \frac{\bar{J}_k(E) \bar{J}_k(-E)}{-E^2 + \omega_k^2 - i\epsilon} \right] \quad (3.32)$$

where $\bar{J}_k(E) = \int dt \tilde{J}_k(t) e^{-iEt} = \int dt \sum_{j=1}^N e^{ikj} J_j(t) e^{-iEt}$. We have $x_j = \Delta j$ and we define the Fourier conjugate variable to x as p and it must be given by k/Δ . Taking the continuum limit gives

$$\bar{J}_k(E) = \int dt \left(\int \frac{dx}{\Delta} \right) e^{ikj} \left(J(x, t) \sqrt{\Delta} \right) e^{-iEt} \quad (3.33)$$

$$= \frac{1}{\sqrt{\Delta}} \int dt dx J(x, t) e^{i\Delta j p} e^{-iEt} \quad (3.34)$$

$$= \frac{1}{\sqrt{\Delta}} \int d^2x J(x, t) e^{-i(Et - xp)} \quad (3.35)$$

$$= \frac{1}{\sqrt{\Delta}} \int d^2x J(x, t) e^{-ip_\mu x^\mu} \quad (3.36)$$

$$\equiv \frac{1}{\sqrt{\Delta}} \tilde{J}(p, E) \quad (3.37)$$

now $\sum_k \rightarrow \Delta \int \frac{dp}{2\pi}$ and (Note: We got $\tilde{J}(-p)$ by relating $\tilde{J}(-p)$ to $\tilde{J}^*(p)$).

$$W_0[\mathbf{J}] = W_0[0] \exp \left[\frac{i}{2} \int \frac{dp}{2\pi} \int \frac{dE}{2\pi} \frac{\tilde{J}(p) \tilde{J}(-p)}{-E^2 + p^2 + m^2 - i\epsilon} \right] \quad (3.38)$$

In 3 + 1 dimensions we have

$$W_0[J] = W_0[0] \exp \left[-\frac{i}{2} \int \frac{d^4p}{(2\pi)^4} \frac{\tilde{J}(p) \tilde{J}(-p)}{p^2 - m^2 - i\epsilon} \right] \quad (3.39)$$

we can equally well write this equation as

$$W_0[J] = W_0[0] \exp \left[-\frac{i}{2} \int d^4x d^4x' J(x) G(x-x') J(x') \right] \quad (3.40)$$

where the Green's function is defined by $(\square_x + m^2 - i\epsilon)G(x-x') = i\delta^4(x-x') \Rightarrow$

$$G(x-x') = \int \frac{d^4p}{(2\pi)^4} \frac{e^{-ip \cdot (x-x')}}{p^2 - m^2 + i\epsilon} \quad (3.41)$$

3.3 Interacting Theory

Suppose we turn on an interaction term $H = \int d^3x \frac{\lambda}{4!} \phi^4(\mathbf{x}) \rightarrow \sum_\ell \frac{\lambda}{4!} \phi_\ell^4$. We obtain

$$W_\lambda [J] = \int \mathcal{D}\phi \exp \left[i \int d^4x \left(\bar{\mathcal{L}}_0 - \frac{\lambda \phi^4}{4!} + J(x)\phi \right) \right] \tag{3.42}$$

$$= W_0 [J] - \frac{i\lambda}{4!} \int d^4x \int \mathcal{D}\phi \phi^4(x) \exp \left[i \int d^4x (\bar{\mathcal{L}}_0 + J\phi) \right] + \dots \tag{3.43}$$

$$= W_0 [J] - \frac{i\lambda}{4!} \int d^4x \int \mathcal{D}\phi \left(\frac{1}{i} \frac{\delta}{\delta J(x)} \right)^4 \exp \left[i \int d^4x (\bar{\mathcal{L}}_0 + J\phi) \right] + \dots \tag{3.44}$$

$$= W_0 [J] - \frac{i\lambda}{4!} \int d^4x \left(\frac{1}{i} \frac{\delta}{\delta J(x)} \right)^4 W_0 [J] + \dots \tag{3.45}$$

This equation has the exact same form as for a single harmonic oscillator.

We have

$$W_\lambda [0] = W_0 [0] - \frac{i\lambda}{4!} \int d^4x - 3G(x-x)^2 + \dots \tag{3.46}$$

As before every term is made up of two types of terms. We have

$$\begin{aligned} \times & \rightarrow -i\lambda \int dx \\ x_1 \text{ --- } x_2 & \rightarrow G(x_1 - x_2) \end{aligned}$$

In Quantum Mechanics we had

$$\langle \Omega | \mathcal{T}(\hat{x}(t_1)\hat{x}(t_2)) | \Omega \rangle = \frac{1}{W_\lambda [0]} \frac{\delta^2 W_\lambda [f]}{\delta f(t_1)\delta f(t_2)} \Big|_{f \rightarrow 0} \tag{3.47}$$

In field theory we have **[Q 3: show]**

$$\langle \Omega | \mathcal{T}(\hat{\phi}(x_1)\hat{\phi}(x_2)) | \Omega \rangle = \frac{1}{W_\lambda [0]} \frac{\delta^2 W_\lambda [J]}{\delta J(x_1)\delta J(x_2)} \Big|_{J \rightarrow 0} \tag{3.48}$$

(all we did was replace the quantum mechanics source to the field theory source). One can then develop perturbation theory for fields as we did for the harmonic oscillator. We get

$$\langle \Omega | \mathcal{T}(\hat{\phi}(x_1)\hat{\phi}(x_2)) | \Omega \rangle = \text{_____} + \text{_____} + \text{_____} + \dots$$

3.4 LSZ Reduction

[Q 4: Maybe insert a proof here instead of just quoting the final result.]

The correlation functions are defined as,

$$G^{(n)}(x_1, \dots, x_n) \equiv \langle \Omega | \mathcal{T} \left(\hat{\phi}(x_1) \dots \hat{\phi}(x_n) \right) | \Omega \rangle \quad (3.49)$$

The Fourier transform is

$$\tilde{G}^{(n)}(p_1, \dots, p_n) = \int \left(\prod_{i=1}^n d^4 x_i e^{i p_i x_i} \right) G^{(n)}(x_1, \dots, x_n) \quad (3.50)$$

The LSZ formula says that

$$\tilde{G}^{(n)}(p_1, \dots, p_n) = \prod_{j=1}^n \left(\frac{i\sqrt{Z}}{p_j^2 - m^2 + i\epsilon} \right) (\text{out} \langle \mathbf{p}_3 \dots \mathbf{p}_n | \mathbf{p}_1 \mathbf{p}_2 \rangle_{\text{in}} + \text{permutations}) \quad (3.51)$$

$$+ (\text{terms with } < n \text{ poles}) \quad (3.52)$$

where the product is the product over the external lines. The bottom line is once you know that correlators you can then get all the matrix elements. The scattering problem is reduced to a problem of calculating correlators.

3.5 Statistical Mechanics and Path Integrals

Suppose you have a harmonic oscillator that's in equilibrium with its environment. We want to treat this oscillator quantum mechanically. If you studied quantum statistical mechanics then you know that the fundamental object in statistical mechanics is the partition function which is defined as,

$$Z = \text{tr} \left[e^{-\beta \hat{H}} \right] = \sum_n \langle n | e^{-\beta \hat{H}} | n \rangle \quad (3.53)$$

where $\beta = 1/k_B T$. Let's define $\beta = it_0$ (since β is real we know that t_0 is imaginary). With this we know that

$$Z = \sum_n \langle n | e^{-i\hat{H}t_0} | n \rangle \quad (3.54)$$

$$= \sum_n U(n, t=0 \rightarrow n, t=t_0) \quad (3.55)$$

Where $U(n, t=0 \rightarrow n, t=t_0)$ is the transition amplitude for a state to go to itself after time t_0 . Using our path integral formalism we can write this as

$$\int_{x(0)=x(t_0)} \mathcal{D}x \exp \left[i \int_0^{t_0} dt L \right] \Bigg|_{t_0 \rightarrow -\frac{i}{k_B T}} \quad (3.56)$$

In other words computing the partition function at some temperature is given by a path integral with imaginary time.

Chapter 4

Quantization of Electromagnetism

4.1 Classical Electromagnetism

The electromagnetic (EM) field is

$$A_\mu(x) = (\phi(x), \mathbf{A}(x)) \quad (4.1)$$

with $\mathbf{E} = -\nabla\phi - \dot{\mathbf{A}}$ and $\mathbf{B} = \nabla \times \mathbf{A}$. The field strength tensor is defined as

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \quad (4.2)$$

$$= \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{pmatrix} \quad (4.3)$$

The gauge symmetry is the observation that there is some redundancy in our description. The physics is given by the electric and magnetic fields. One can show that the only transformation you can do that won't change the EM fields is

$$A_\mu \rightarrow A_\mu + \partial_\mu \xi(x) \quad (4.4)$$

where $\xi(x)$ is an arbitrary function. Under this gauge symmetry,

$$F_{\mu\nu} \rightarrow F_{\mu\nu} \quad (4.5)$$

We want to get the equations of motion for this system (Maxwell equations) using the Lagrangian density. One possibility is to make the Lagrangian out of $F_{\mu\nu}$. We want to have a gauge invariant Lagrangian so we use $F_{\mu\nu}$. To make a Lorentz scalar out of $F_{\mu\nu}$ we take the product of two $F_{\mu\nu}$ (Alternatively we could take the trace, but $F_{\mu\nu}$ is traceless). This gives us two possible terms

$$\mathcal{L}_{general} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \epsilon_{\mu\nu\alpha\beta} F^{\mu\nu} F^{\alpha\beta} \quad (4.6)$$

These are the only two non-renormalizable terms. It also turns out that the second term violates parity. Since we know that electromagnetism does respect parity we don't have the second term in EM. Thus

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \quad (4.7)$$

$$= -\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu) \quad (4.8)$$

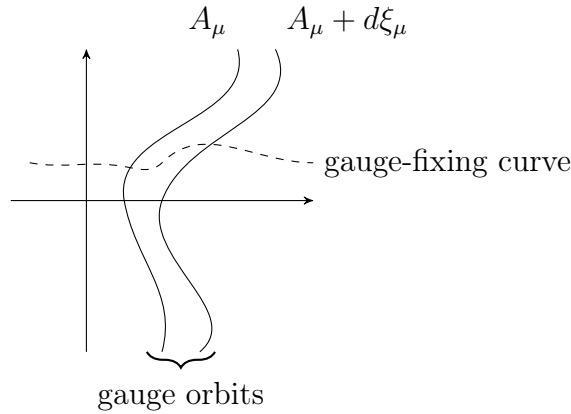
$$= -\frac{1}{2}\partial_\mu A_\nu(\partial^\mu A^\nu - \partial^\nu A^\mu) \quad (4.9)$$

Note that a $m^2 A_\mu A^\mu$ term is not gauge invariant! So we don't have such a term in our Lagrangian. The Euler Lagrange equations give

$$\partial_\mu \frac{\delta \mathcal{L}}{\delta \partial_\mu A_\nu} = 0 \quad (4.10)$$

$$\square A_\nu - \partial_\nu(\partial^\mu A_\mu) = 0 \quad (4.11)$$

Note that we have yet to make a gauge choice. We now fix the gauge to our convenience. Pictorially this is given by



The gauge orbits are lines which show all the possible values of A_μ along some gauge choice. All points along the gauge orbit correspond to physically equivalent systems. All points along the gauge fixing curve correspond to the different values of A_μ with a single gauge. One of the simplest gauge choices is the Lorentz gauge: $\partial_\mu A^\mu = 0$.

The solutions of 4.11 in the Lorentz gauge are plane waves,

$$A_\mu = \int \frac{d^4 k}{(2\pi)^4} \epsilon_\mu e^{-ik \cdot x} \quad (4.12)$$

with $k^2 = 0$ and ϵ_μ is the polarization vector (and also the Fourier transform of the vector field). These are just wave solutions with no mass. To make sure that we are still in the

Lorentz gauge we require,

$$\partial^\mu A_\mu = 0 \quad (4.13)$$

$$\int \frac{d^4 k}{(2\pi)^4} i k^\mu \epsilon_\mu e^{-i(k \cdot x)} = 0 \quad (4.14)$$

$$\Rightarrow k \cdot \epsilon = 0 \quad (4.15)$$

by uniqueness of the Fourier transform. As an example consider a photon moving the z direction:

$$k_\mu = (E, 0, 0, E) \quad (4.16)$$

Our ϵ when dotted into k_μ must give zero. A basis of polarization vectors that obeys the condition above is,

$$\epsilon^{(1)} = \frac{1}{\sqrt{2}} (0, 1, +i, 0) \quad (4.17)$$

$$\epsilon^{(2)} = \frac{1}{\sqrt{2}} (0, 1, -i, 0) \quad (4.18)$$

$$\epsilon^{(3)} = \frac{1}{\sqrt{2}} (1, 0, 0, 1) \quad (4.19)$$

$$(4.20)$$

$\epsilon^{(1)}$ and $\epsilon^{(2)}$ correspond to left and right handed circular polarized waves. The third polarization vector corresponds to a longitudinally polarized wave. However we know that such a wave does not exist! This extra possible state arose because we didn't fully fix the gauge of our field. To see this consider some gauge transformed field \bar{A}_μ . This gauge transformed field also satisfies the Lorentz gauge if the field connecting the two gauge fields obeys,

$$\bar{A}_\mu = A_\mu + \partial_\mu \xi \quad (4.21)$$

$$\Rightarrow \square \xi = 0 \quad (4.22)$$

To get rid of this extra freedom consider the component of the Fourier transform of ξ that matches the wave. We have

$$\xi(x) = \int \frac{d^4 q}{(2\pi)^4} \tilde{\xi}(q) e^{-iq \cdot x} \quad (4.23)$$

and we consider the particular contribution $\tilde{\xi}(k)$ with $k^2 = 0$. We want to know how adding this ξ to A_ν changes the solution. We have,

$$\epsilon'_\nu(k) = \epsilon_\nu(k) + k_\nu \tilde{\xi}(k) \quad (4.24)$$

By choosing $\tilde{\xi}(k) = -\epsilon_\nu^{(3)}$ we can get $\epsilon'^{(3)} = 0$. If one computed the magnetic and electric fields corresponding to this polarization they would turn out to be zero. This is an unphysical solution. With these two conditions we can fully fix the gauge.

4.2 Quantization

To zeroth order we treat A_μ as a set of 4 fields. The most naive generalization of our discussion on scalar fields is

$$W_0 [J_\mu] = \int \mathcal{D}A \exp \left(i \left(\bar{\mathcal{S}}[A] + \int A_\mu(x) J^\mu(x) d^4x \right) \right) \quad (4.25)$$

where $\mathcal{D}A = \mathcal{D}A^0 \mathcal{D}A^1 \mathcal{D}A^2 \mathcal{D}A^3$ and $\mathcal{S} = \int d^4x \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right)$. Note however that we wrote a term $A_\mu(x) J^\mu(x)$ which does not appear gauge invariant. Applying a gauge transformation on the source term gives,

$$\int A^\mu J_\mu d^4x \rightarrow \int A^\mu J_\mu d^4x + \int \partial^\mu \xi J_\mu d^4x \quad (4.26)$$

$$= \int A^\mu J_\mu d^4x - \int \xi \partial^\mu J_\mu d^4x \quad (4.27)$$

If we restrict our attention to sources with no divergence, i.e. that obey $\partial^\mu J_\mu = 0$, then the extra term is in fact gauge invariant.

We are going to want to evaluate this path integrals explicitly. At first sight this seems feasible since the kinetic term is quadratic and our integral appears Gaussian. However it turns out that there are small technical problems which we will discuss.

We take a short detour into how we treat photons with spin s using canonical quantization.

$$A_\mu^s \sim \epsilon_\mu^s e^{-ip \cdot x} \quad (4.28)$$

with $p^2 = 0$ or equivalently, $p_0 = \pm |\mathbf{p}|$. We have a restriction in the Lorentz Gauge on our ϵ given by $\epsilon \cdot p = 0$ and $\epsilon_0 = 0$. We have an added the superscript for the polarization vector for spin, $s = 1, 2$. We found a particular solution. The general solution is a linear combination of the solutions we found:

$$A_\mu = \int \frac{d^3p}{(2\pi)^3} \sum_{s=1}^2 \left(A^s(\mathbf{p}) \epsilon_\mu^s e^{-ip \cdot x} \Big|_{p_0=|\mathbf{p}|} + A^{s*}(\mathbf{p}) \epsilon_\mu^{s*} e^{ip \cdot x} \Big|_{p_0=|\mathbf{p}|} \right) \quad (4.29)$$

This looks identical to the KG field only we now have a spin index s and we have polarization vectors.

One can go about performing the Canonical Quantization method:

$$A^s(\mathbf{p}) \rightarrow \frac{1}{\sqrt{2p_0}} a_{\mathbf{p}}^s \quad , \quad A^{s*}(\mathbf{p}) \rightarrow \frac{1}{\sqrt{2p_0}} a_{\mathbf{p}}^{s\dagger} \quad (4.30)$$

where a, a^\dagger are creation and annihilation operators respectively. We would then define the vacuum, $|0\rangle$ such that

$$a_{\mathbf{p}}^s |0\rangle \quad (4.31)$$

for all \mathbf{p}, s . There is then a one-photon state given by

$$a_{\mathbf{p}}^{s\dagger} |0\rangle = |\mathbf{p}, s\rangle \quad (4.32)$$

4.3 Path Integrals in QED

Recall that

$$W_0[J^\mu] = \int \mathcal{D}A \exp \left[i \underbrace{\int d^4x \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + A_\mu J^\mu \right)}_{F \text{ term}} \right] \quad (4.33)$$

where we restrict our attention to sources such that $\partial_\mu J^\mu = 0$. We can write the exponential argument as

$$F \text{ term} = \int d^4x \left(-\frac{1}{2} \right) \partial_\mu A_\nu (\partial^\mu A^\nu - \partial^\nu A^\mu) \quad (4.34)$$

$$= \int d^4x \frac{1}{2} A_\nu (\square g^{\mu\nu} - \partial^\mu \partial^\nu) A_\mu \quad (4.35)$$

where we have integrated by parts to play with the derivatives. Now (here we write the Fourier Transform of the vector potential as $\tilde{A}(k)$)

$$A_\mu(x) = \int \frac{d^4k}{(2\pi)^4} e^{ik \cdot x} \tilde{A}_\mu(k) \quad (4.36)$$

Thus we have

$$F \text{ term} = \frac{1}{2} \int d^4x \int \frac{d^4k d^4k'}{(2\pi)^4 (2\pi)^4} e^{i(k+k') \cdot x} \tilde{A}_\nu(k') (-k^2 g^{\mu\nu} + k^\mu k^\nu) \tilde{A}_\mu(k) \quad (4.37)$$

$$= -\frac{1}{2} \int \frac{d^4k}{(2\pi)^4} \tilde{A}_\nu(-k) \underbrace{(k^2 g^{\mu\nu} - k^\mu k^\nu)}_{D^{\mu\nu}} \tilde{A}_\mu(k) \quad (4.38)$$

The second term is given by

$$\int d^4x A_\mu J^\mu = \int \frac{d^4k}{(2\pi)^4} \tilde{A}_\mu(k) \tilde{J}^\mu(-k) \quad (4.39)$$

Recall that for a scalar field the source term was,

$$\int \frac{d^4k}{(2\pi)^4} \tilde{\phi}(k) \tilde{J}(k) \quad (4.40)$$

and the kinetic term was

$$\int \frac{d^4k}{(2\pi)^4} \tilde{\phi}(k) (k^2 - m^2 + i\epsilon) \tilde{\phi}(-k) \quad (4.41)$$

and we completed the square by changing variables to

$$\tilde{\phi}'(k) = \tilde{\phi}(k) + (k^2 - m^2 + i\epsilon)^{-1} \tilde{J}(k) \quad (4.42)$$

With this in mind, we define a new \tilde{A} such that

$$\tilde{A}'_\mu = \tilde{A}_\mu + (D^{-1})_{\mu\nu} \tilde{J}^\nu \quad (4.43)$$

and then our Lagrangian is given by,

$$\mathcal{L} = \frac{1}{2} \tilde{A}'_\mu D^{\mu\nu} \tilde{A}'_\nu - \frac{1}{2} \tilde{J}^\mu (D^{-1})_{\mu\nu} \tilde{J}^\nu \quad (4.44)$$

So we have

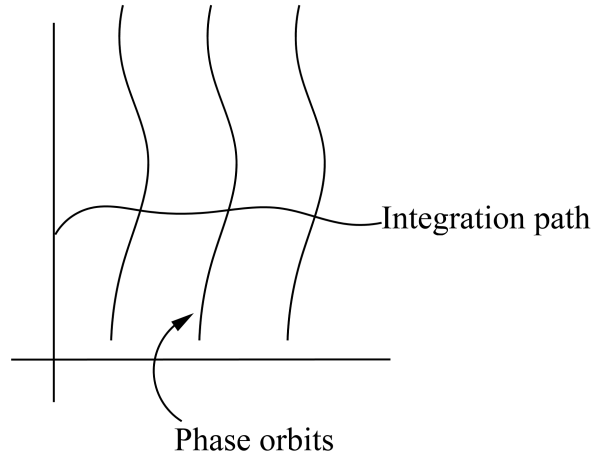
$$W_0[J^\nu] = W_0[0] \exp \left(-\frac{i}{2} \int \tilde{J}^\mu \underbrace{(D^{-1})_{\mu\nu}}_{\text{Green's function}} \tilde{J}^\nu \frac{d^4k}{(2\pi)^4} \right) \quad (4.45)$$

where we have our Gaussian integral included in our definition of $W_0[0]$. Everything seems to have worked flawlessly. The subtlety here is that $D_{\mu\nu}$ is not invertible! To prove this we study the eigenvalues of the D matrix. One such eigenvector is $k_\nu = (k_0, -\mathbf{k})$ with an eigenvalue of zero:

$$(k^2 g^{\mu\nu} - k^\mu k^\nu) k_\nu = 0 \quad (4.46)$$

Since the determinant is the product of the eigenvalues it must be zero, implies that D isn't invertible.

This problem arose because while we are integrating over the gauge fixed field we are also integrating over gauge inequivalent fields. We haven't done anything to set these contributions to zero. We need to find a way such that we integrate only about inequivalent fields. This corresponds to a single gauge-fixing curve (All the A^μ values for a single gauge choice):



The procedure is called the Faddeev-Popov procedure. Suppose we have some field $A_{0,\mu}$. We have a gauge transformed field parameterized by α that is given by,

$$A_{\alpha,\mu} = A_{0,\mu} + \partial_\mu \alpha \quad (4.47)$$

We define a gauge transformation function as $G[A]$. For the Lorentz gauge we have

$$G[A] = \partial_\mu A^\mu = 0 \quad (4.48)$$

We will also later use the Generalized Lorentz Gauge given by

$$G[A] = \partial_\mu A^\mu - \omega(x) = 0 \quad (4.49)$$

In order to ensure we stay on our gauge we introduce a functional delta function (this can be thought of as a product of delta functions, one at each spacetime point):

$$\int \mathcal{D}\alpha \delta(G[A_{\alpha,\mu}]) \quad (4.50)$$

By inserting this into our path integral the path integral will be zero unless $G[A] = 0$. However, we can't insert this into the path integral unless it's equal to 1.

Recall that if $f(x)$ has one zero at x_0 then,

$$\int dx \left| \frac{df(x)}{dx} \right|_{x=x_0} \delta(f(x)) = 1 \quad (4.51)$$

We want to generalize this to instead of having $f(x)$ we have, $\mathbf{g}(\mathbf{a})$ for vectors of arbitrary size. To do this consider the Taylor expansion of \mathbf{g} around its root (we assume it only has one root, \mathbf{a}_0):

$$g_i(\mathbf{a}) = \cancel{g_i(\mathbf{a}_0)} + \sum_j \left. \frac{\partial g_i}{\partial a_j} \right|_{a_0} (a_j - a_{0,j}) + \dots \quad (4.52)$$

We want to insert this into a delta function, $\delta^{(n)}(\mathbf{g}(\mathbf{a}))$. This will only be nonzero near $\mathbf{a} = \mathbf{a}_0$. Thus we have,

$$\delta(\mathbf{g}(\mathbf{a})) = \prod_i \delta(g_i(\mathbf{a})) \quad (4.53)$$

$$= \prod_i \delta\left(\sum_j J_{ij}(a_j - a_{0,j})\right) \quad (4.54)$$

where J_{ij} is the Jacobian matrix defined by $J_{ij} \equiv \left. \frac{\partial g_i}{\partial a_j} \right|_{a_0}$. We have,

$$\delta(\mathbf{g}(\mathbf{a})) = \delta\left(\sum_j J_{1j}(a_j - a_{0,j})\right) \delta\left(\sum_j J_{2j}(a_j - a_{0,j})\right) \dots \quad (4.55)$$

We now use the identity,

$$\delta(\alpha(a - a_0)) = \frac{\delta(a - a_0)}{|\alpha|} \quad (4.56)$$

We choose to isolate each delta function in Eq. 4.55 for a different a_j :

$$\delta(\mathbf{g}(\mathbf{a})) = \frac{\delta(a_1 - a_{0,1})}{|J_{1,1}|} \frac{\delta(a_2 - a_{0,2})}{|J_{2,2}|} \dots \quad (4.57)$$

If we take the Jacobian matrix to be greater than zero then we have the product:

$$(J_{1,1}J_{2,2}\dots)^{-1} = \frac{1}{\det J} \quad (4.58)$$

where we have used the fact that the determinant of J is independent of a unitary transformation. So we finally have,

$$\left(\int \prod_i da_i \right) \delta^{(n)}(\mathbf{g}(\mathbf{a})) \det \left(\frac{\partial g_i}{\partial a_j} \right) = 1 \quad (4.59)$$

where it is understood that the Jacobian matrix is evaluated at the root of \mathbf{g} .

We write the continuum generalization of this equation as,

$$\int \mathcal{D}\alpha(x) \delta(G(A^\alpha)) \det \left(\frac{\delta G(A^\alpha)}{\delta \alpha} \right) = 1 \quad (4.60)$$

We consider the generalized Lorentz gauge: $G(A^\alpha) = \partial_\mu A^\mu + \frac{1}{e} \partial_\mu \alpha(x) - \omega(x)$. In this case it is straight forward to calculate the determinate explicitly. The discretized gauge transformation is (we work here in 1D just for brevity however, its easy to extend the results to four dimensions.

$$G_i(\alpha_i) = \frac{1}{\Delta^2 e} (\alpha_{i+1} - 2\alpha_i + \alpha_{i-1}) + \underbrace{(\partial_\mu A_0^\mu - \omega(x))_i}_{\text{Independent of } \alpha_j} \quad (4.61)$$

Taking the derivative we have,

$$\frac{\partial G_i(\alpha_i)}{\partial \alpha_j} \delta_{ij} = \frac{1}{\Delta^2 e} (\delta_{i+1,j} - 2\delta_{i,j} + \delta_{i-1,j}) \delta_{ij} \quad (4.62)$$

which doesn't have any A dependence so it is already evaluated at the root of G . Thus the determinant has no A or α dependence and we can write:

$$W_0[J^\mu] = \det \left(\frac{\delta G}{\delta \alpha} \right) \int \mathcal{D}\alpha \int \mathcal{D}A \delta(G[A^\alpha]) \exp \left(i \left(\mathcal{S} + \int d^4x A_\mu J^\mu \right) \right) \quad (4.63)$$

However J , $\mathcal{D}A$, and \mathcal{S} are gauge invariant thus we can change our equation to the gauge shifted A_μ^α :

$$W_0[J^\mu] = \det \left(\frac{\delta G}{\delta \alpha} \right) \int \mathcal{D}\alpha \int \mathcal{D}A^\alpha \delta(G[A^\alpha]) \exp \left(i \left(\mathcal{S}[A^\alpha] + \int d^4x A_\mu^\alpha J^\mu \right) \right) \quad (4.64)$$

We now see that the α integral is trivial, it is just the volume of the α space (we include the determinant in what we call Vol):

$$W_0[J^\mu] = Vol \times \int \mathcal{D}A \delta(G[A]) \exp \left(i \left(\mathcal{S}[A] + \int d^4x A_\mu J^\mu \right) \right) \quad (4.65)$$

where we have redefined our integration variable from $A^\alpha \rightarrow A$. We use the Generalized Lorentz Gauge without α as we redefined, $A^\alpha \rightarrow A$. (we ignore the constant in front since as we saw for a scalar field, it cancels away when calculating observables):

$$W_0[J^\mu] = \int \mathcal{D}A \delta(\partial_\mu A^\mu - \omega(x)) \exp\left(i\left(\mathcal{S}[A] + \int d^4x A_\mu J^\mu\right)\right) \quad (4.66)$$

$$= \underbrace{\mathcal{N}(\xi) \int \mathcal{D}\omega e^{-i \int d^4x \frac{\omega^2}{2\xi}}}_1 \int \mathcal{D}A \delta(\partial_\mu A^\mu - \omega(x)) \exp\left(i\left(\mathcal{S}[A] + \int d^4x A_\mu J^\mu\right)\right) \quad (4.67)$$

where ξ is just a constant and we are allowed to through in this integral since we know our W_0 is independent of ω (it is gauge invariant). Thus we have

$$W_0[J^\mu] = N(\xi) \int \mathcal{D}A \exp\left(-i \int d^4x \frac{(\partial_\mu A^\mu)^2}{2\xi} + i \int d^4x (\mathcal{L} + A_\mu J^\mu)\right) \quad (4.68)$$

where we have performed our ω integral (which is trivial due to the delta function). Thus our “gauge-fixing procedure” is equivalent to taking $\mathcal{L} \rightarrow \mathcal{L} - \frac{1}{2\xi}(\partial_\mu A^\mu)^2$. Recall that we had

$$\mathcal{S} = \int \frac{d^4k}{(2\pi)^4} \frac{1}{2} \tilde{A}_\mu(-k) (k^2 g^{\mu\nu} - k^\mu k^\nu) \tilde{A}_\nu(-k) \quad (4.69)$$

Adding this extra term we have (note that we can replace $\frac{1}{2\xi}(\partial_\mu A^\mu)^2 \rightarrow -\frac{1}{2\xi}(A^\mu \partial_\mu \partial_\nu A^\nu)$)

$$\mathcal{S} = \int \frac{d^4k}{(2\pi)^4} \frac{1}{2} \tilde{A}_\mu(-k) \underbrace{\left(k^2 g^{\mu\nu} - k^\mu k^\nu \left(1 - \frac{1}{\xi}\right)\right)}_{D_\xi^{\mu\nu}} \tilde{A}_\nu(k) \quad (4.70)$$

We now have $\det D_\xi^{\mu\nu} \neq 0$. We finally have

$$W_0[J^\mu] = W_0[0] \exp\left(-\frac{1}{2} \int \frac{d^4k}{(2\pi)^4} \tilde{J}^\mu(k) (iD_\xi^{-1})_{\mu\nu} \tilde{J}^\nu(-k)\right) \quad (4.71)$$

where

$$(D_\xi^{-1})_{\mu\nu} \equiv G_{\mu\nu} = -\frac{i}{k^2 + i\epsilon} \left(g_{\mu\nu} - (1 - \xi) \frac{k_\mu k_\nu}{k^2}\right) \quad (4.72)$$

This is known as the photon propagator. To see that this is indeed the inverse of $D_{\mu\nu}$ just multiply it out directly:

$$(D_\xi^{-1})^{\mu\nu} D_{\xi,\nu\rho} = \frac{1}{k^2} \left(g_{\mu\nu} - (1 - \xi) \frac{k_\mu k_\nu}{k^2}\right) \left(k^2 g_{\nu\rho} - k^\nu k_\rho \left(1 - \frac{1}{\xi}\right)\right) \quad (4.73)$$

$$= \frac{1}{k^2} \left(k^2 \delta^\mu_\rho - k^\mu k_\rho \left(1 - \frac{1}{\xi}\right) - (1 - \xi) k^\mu k_\rho + \left(1 - \frac{1}{\xi}\right) (1 - \xi) k^\mu k_\rho\right) \quad (4.74)$$

$$= \delta^\mu_\rho \quad (4.75)$$

Naively one would say that the propagator result depends on ξ . How can this be since we said that ξ is a mathematical convenience? Being more careful, one notices that all the ξ terms look like $\xi(k \cdot \tilde{J})^2$. However these are zero by our choice of source ($\partial_\mu J^\mu = 0 \Rightarrow k_\mu \tilde{J}^\mu = 0$). Any choice of ξ is valid. There are a few popular choices. One popular choice is $\xi = 1$. This is called the Feynman gauge. Another popular gauge is called the Landau gauge ($\xi = 0$). Alternatively, you can keep ξ as a parameter and watch it's dependence drop at the end.

There is also something interesting in the sign of the photon propagator. The spatial components of A_μ propagate with the same sign as the scalar field propagator, however the time-like component has a relative negative sign. This is a consequence of the fact that A_0 doesn't actually propagate since the momentum conjugate to A_0 is zero:

$$\frac{\partial \mathcal{L}}{\partial_0 A^0} = -\frac{1}{4} \frac{\partial}{\partial_0 A^0} F_{\mu\nu} F^{\mu\nu} \quad (4.76)$$

$$= 0 \quad (4.77)$$

4.4 Correlation Functions

Consider the correlation function:

$$\langle 0 | \mathcal{T} \left(\hat{A}_\mu(x_1) \hat{A}_\nu(x_2) \right) | 0 \rangle = \frac{1}{W_0[0]} \frac{1}{i^2} \frac{\delta^2 W_0[J]}{\delta J^\mu(x_1) \delta J^\nu(x_2)} \Big|_{J \rightarrow 0} \quad (4.78)$$

$$= \frac{1}{i} G_{\mu\nu}(x_1 - x_2) \quad (4.79)$$

However we constructed our J^μ such that $\partial_\mu J^\mu = 0$. Having a δJ^μ implies we can “wobble” J^μ in any direction we want but in fact we have some restrictions. We now set out to find what those are

$$A_\mu(x) = \int d^3 k \tilde{A}_\mu(k) e^{-ik \cdot x} \quad (4.80)$$

and

$$\int d^4 x A_\mu J^\mu \rightarrow \int d^4 k \tilde{A}_\mu \tilde{J}^\mu \quad (4.81)$$

Now

$$\tilde{G}(k_1, \dots) = \int (d^4 x_1 d^4 x_2 \dots) (e^{ik_1 x_1} \dots) \langle 0 | \mathcal{T} (A_\mu(x_1) \dots) | 0 \rangle \quad (4.82)$$

$$\propto \left[\frac{\delta}{\delta \tilde{J}^\mu(k)} \frac{\delta}{\delta \tilde{J}^\nu(k)} \dots \right] W_0[\tilde{J}] \Big|_{\tilde{J} \rightarrow 0} \quad (4.83)$$

where \tilde{G} is the Fourier transform of the correlator. This can be done for any k . We take $k^2 = 0$ with $k = (E, 0, 0, E)$.

$$\tilde{A}_\mu(k) = \sum_{s=0}^3 \epsilon_\mu^s \quad (4.84)$$

with a basis

$$\begin{aligned}\epsilon_\mu^0 &= \frac{1}{\sqrt{2}}(1, 0, 0, -1) \\ \epsilon_\mu^1 &= (0, 1, 0, 0) \\ \epsilon_\mu^2 &= (0, 0, 1, 0) \\ \epsilon_\mu^3 &= \frac{1}{\sqrt{2}}(1, 0, 0, 1)\end{aligned}$$

Now

$$\tilde{J}_\mu(k) = \sum J^s \epsilon_\mu^s \quad (4.85)$$

and

$$k_\mu \tilde{J}^\mu = 0 \Rightarrow J^0 = 0 \quad (4.86)$$

so

$$\tilde{A} \cdot \tilde{J} = \sum_{s=1}^3 A^s J^s \epsilon_\mu^s \cdot \epsilon^{s\mu} = A^1 J^1 + A^2 J^2 \quad (4.87)$$

Thus what we can find out is the correlation function between A_1 and A_2 . We can only know

$$\langle 0 | \mathcal{T} (A^1(k) \dots) | 0 \rangle \quad (4.88)$$

$$\langle 0 | \mathcal{T} (A^2(k) \dots) | 0 \rangle \quad (4.89)$$

With this we can write down the LSZ reduction formula for photons:

$$\tilde{G}_\mu(k, \dots) = \frac{i\sqrt{Z_\gamma}}{k^2 + i\epsilon} \sum_{s=1}^2 \epsilon_\mu^s \underbrace{\langle (\mathbf{k}, s) \dots \rangle}_{\text{S matrix}} + \dots \quad (4.90)$$

Chapter 5

Fermions with Path Integrals

Electrons are fermions, thus you can't have more than one electron in a single state. That's why we never thought of electrons as fields until now. On the other hand photons are bosons and they can all be in the same state. That's why we had to think of them as fields right from the start. However the concept of path integrals requires one to think of fermion fields.

5.1 Grassman Numbers

It turns out the correct way to think about fermion fields is through the concept of Grassman numbers, θ, η . They are not like ordinary numbers in many ways. However they still obey rules of addition and multiplication. For Grassman numbers the addition is typical (commutative). However the multiplication of such number is anti commutative. i.e.

$$\eta + \theta = \theta + \eta \quad (5.1)$$

$$\eta\theta = -\theta\eta \quad (5.2)$$

This implies that

$$\theta\theta = -\theta\theta \quad (5.3)$$

$$\Rightarrow \theta^2 = 0 \quad (5.4)$$

Furthermore,

$$f(\theta) = a + b\theta = a + \theta b \quad (5.5)$$

where a and b are ordinary numbers. Note that this is analogous to a Taylor expansion however we have $\theta^2 = 0$ so all higher order terms vanish. These numbers are non-intuitive and one needs to be careful when working with them.

You can define derivatives of Grassman numbers such as

$$\frac{df}{d\theta} = b \quad (5.6)$$

and integration

$$\int d\theta f(\theta) = \int d\theta f(\theta + \eta) \quad (5.7)$$

The definition of the integral implies that (Exercise: use equation 5.5 to derive the first equation. Maxim isn't sure how to derive the second equation)

$$\int d\theta = 0 \quad (5.8)$$

$$\int d\theta\theta = 1 \quad (5.9)$$

This means that we have

$$\int d\theta f(\theta) = b \frac{df}{d\theta} \quad (5.10)$$

One can generalize this discussion to complex Grassman Numbers,

$$\theta^* = \frac{1}{\sqrt{2}}(\theta_1 + i\theta_2), \quad \theta = \frac{1}{\sqrt{2}}(\theta_1 - i\theta_2) \quad (5.11)$$

and

$$\int d\theta\theta = 1 = \int \theta^*\theta \quad (5.12)$$

As an example consider integration over a Gaussian

$$\int d\theta^* d\theta e^{-i\theta^* b\theta} = \int d\theta^* d\theta (1 - \theta^* b\theta) \quad (5.13)$$

$$= \int d\theta^* d\theta (\theta\theta^* b) \quad (5.14)$$

$$= b \quad (5.15)$$

This should be compared to the result for real numbers,

$$\int dx e^{-\frac{1}{2}bx^2} = \sqrt{\frac{2\pi}{b}} \quad (5.16)$$

We now move on to multidimensional integrals. First consider the real variable integral:

$$\int dx_1 dx_2 \dots \exp\left(-\frac{1}{2} \sum_{i,j} x_i A_{ij} x_j\right) \quad (5.17)$$

Suppose that A can be diagonalized by some orthogonal transformation,

$$A = \mathcal{O}^T A_D \mathcal{O} \quad (5.18)$$

which implies that,

$$x^T A x = (\mathcal{O}x)^T A_D (\mathcal{O}x) \quad (5.19)$$

$$= x'^T A_D x' \quad (5.20)$$

where we have defined, $\mathcal{O}x \equiv x'$. The Jacobian for this transformation is,

$$J_{ij} \equiv \frac{\partial x'_i}{\partial x_j} = \mathcal{O}_{ij} \quad (5.21)$$

Thus we have,

$$= \det \mathcal{O} \int dx_1; dx_2; \dots \exp \left(-\frac{1}{2} \sum_i x'_i A_{D,ii} x'_i \right) \quad (5.22)$$

$$= \det \mathcal{O} \prod \int dx_i \exp \left(-\frac{1}{2} x_i A_{D,ii} x_i \right) \quad (5.23)$$

$$= \det \mathcal{O} (2\pi)^{N/2} \left(\prod_i A_{D,ii} \right)^{-1} \quad (5.24)$$

$$= (2\pi)^{N/2} \frac{\det \mathcal{O}}{\det A} \quad (5.25)$$

We now want to find the analogous result for the Grassman integral,

$$\int d^n \theta d^n \theta^* \exp \left(- \sum_{i,j} \theta_i^* B_{ij} \theta_j \right) \quad (5.26)$$

We again want to transform the integration variable and diagonalize B , however we need to be a bit careful as we are working with Grassman numbers. A set of N Grassman variables obey the relations:

$$\theta_1 \theta_2 \dots \theta_N = \frac{1}{N!} \epsilon_{i_1 \dots i_N} \theta_{i_1} \theta_{i_2} \dots \theta_{i_N} \quad (5.27)$$

$$\epsilon_{i_1 i_2 \dots i_N} \theta_1 \theta_2 \dots \theta_N = \theta_{i_1} \theta_{i_2} \dots \theta_{i_N} \quad (5.28)$$

These relations are a consequence of the anticommutating nature of the Grassman variables and are easy to show for $N = 3$ though I can't figure out a general proof yet.

If we consider a unitary transformation on the Grassman variables: $\theta'_i = U_{ij} \theta_j$ then,

$$\prod_i \theta'_i = \frac{1}{N!} \epsilon^{i_1 i_2 \dots i_N} \theta'_{i_1} \theta'_{i_2} \dots \theta'_{i_N} \quad (5.29)$$

$$= \frac{1}{N!} \epsilon^{i_1 i_2 \dots i_N} U_{i_1, i'_1} \theta_{i'_1} U_{i_2, i'_2} \theta_{i'_2} \dots U_{i_N, i'_N} \theta_{i'_N} \quad (5.30)$$

$$= \frac{1}{N!} \epsilon^{i_1 i_2 \dots i_N} U_{i_1, i'_1} U_{i_2, i'_2} U_{i_N, i'_N} \epsilon_{i'_1, i'_2, \dots, i'_N} \left(\prod_j \theta_j \right) \quad (5.31)$$

$$\prod_i \theta'_i = \det U \prod_j \theta_j \quad (5.32)$$

Thus when changing variables we pick up a determinate, analogous to the earlier situation with the real variables. With this we have,

$$\prod_i d\theta_i^* \prod_j d\theta_j = (\det U)^* \det U \prod_i d\theta_i^* \prod_j d\theta_j \quad (5.33)$$

$$= \prod_i d\theta_i^* \prod_j d\theta_j \quad (5.34)$$

and so the integral is,

$$\int d^n\theta d^n\theta^* \exp\left(-\sum_{i,j} \theta_i^* B_{ij} \theta_j\right) = \int d^n\theta' d^n\theta'^* \exp\left(-\sum_{i,j} \theta_i'^* B_{D,ij} \theta_j'\right) \quad (5.35)$$

$$= \int d^n\theta d^n\theta^* \exp\left(-\sum_i \theta_i^* B_{D,ii} \theta_i\right) \quad (5.36)$$

$$= \prod_i B_{D,ii} \quad (5.37)$$

$$= \det B \quad (5.38)$$

5.2 Free Dirac Field

The Dirac field is given as

$$\psi(x) = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \\ \psi_3(x) \\ \psi_4(x) \end{pmatrix} \quad (5.39)$$

To have Fermi Dirac statistics we make $\psi_a(x)$ into a complex Grassman number valued field.

The free Dirac field is given by

$$\mathcal{L}_D = \bar{\psi}(i\rlap{\not{D}} - m)\psi \quad (5.40)$$

with $\bar{\psi} = \psi^\dagger \gamma^0$ and $\rlap{\not{D}} = \partial_\mu \gamma^\mu$.

We want to produce our generating functional. We need to introduce sources. We want something of the form of source multiplied by the Dirac field. We want to add a term for the source that takes the form of a source multiplied by a scalar. A simplest thing one can imagine is

$$\eta = \begin{pmatrix} \eta_1(x) \\ \eta_2(x) \\ \eta_3(x) \\ \eta_4(x) \end{pmatrix} \quad (5.41)$$

where this term transforms in the same way as the Dirac spinor. The η source is also a complex Grassman number valued field. Since we also want a Hermitian Lagrangian our source takes the form

$$\bar{\eta}\psi + \bar{\psi}\eta \quad (5.42)$$

One can treat η and $\bar{\eta}$ as independent as well as $\psi, \bar{\psi}$ as independent. Our generating functional is

$$W_0[\eta, \bar{\eta}] = \int \mathcal{D}\psi \mathcal{D}\bar{\psi} \exp \left[i \int d^4x (\bar{\mathcal{L}}_D + \bar{\eta}\psi + \bar{\psi}\eta) \right] \quad (5.43)$$

where $\mathcal{D}\psi \equiv \prod_{a=1}^4 \mathcal{D}\psi_a = \psi_1\psi_2\psi_3\psi_4$. The procedure is just as before. We now complete the square in an analogous way to what we did earlier.

We have,

$$\int d^4x (...) = \int d^4x \left[\int \frac{d^4k d^4k'}{(2\pi)^4(2\pi)^4} e^{ikx} e^{-ik'x} \left\{ \bar{\psi}(k) (-i\gamma_\mu(ik^\mu) - m) \psi(k) + \bar{\eta}(k)\psi(k) + h.c. \right\} \right] \quad (5.44)$$

$$= \int \frac{d^4k}{(2\pi)^4} \bar{\psi}(k) (\not{k} - m) \psi(k) + \bar{\eta}(k)\psi(k) + h.c. \quad (5.45)$$

Now we shift the fermion fields:

$$\psi \rightarrow \psi - \frac{(\not{k} + m)\eta}{k^2 - m^2 + i\epsilon} \quad (5.46)$$

$$\bar{\psi} \rightarrow \bar{\psi} - \frac{\bar{\eta}(\not{k} + m)}{k^2 - m^2 + i\epsilon} \quad (5.47)$$

With this we have,

$$\bar{\psi}(\not{k} - m)\psi + \bar{\eta}\psi + h.c. \rightarrow \bar{\psi}(\not{k} - m)\psi + \frac{\bar{\eta}(\not{k} + m)(\not{k} - m)(\not{k} + m)\eta}{(k^2 - m^2)^2} \quad (5.48)$$

$$= \bar{\psi}(\not{k} - m)\psi + \frac{\bar{\eta}(\not{k} + m)\eta}{(k^2 - m^2)} \quad (5.49)$$

So our path integral becomes,

$$W_0[\eta, \bar{\eta}] = \int \mathcal{D}\psi \mathcal{D}\bar{\psi} \exp \left[i \int d^4k \bar{\psi}(\not{k} - m)\psi + \frac{\bar{\eta}(\not{k} + m)\eta}{k^2 - m^2} \right] \quad (5.50)$$

$$= W_0[0] \exp \left[i \int d^4k \frac{\bar{\eta}(\not{k} + m)\eta}{k^2 - m^2} \right] \quad (5.51)$$

In real space,

$$W_0[\eta, \bar{\eta}] = W_0[0] \exp \left[- \int d^4x d^4y \bar{\eta}(x) S(x - y) \eta(y) \right] \quad (5.52)$$

where S is the Green's function, given by

$$(i\cancel{\partial}_x - m\mathbb{1}_{4\times 4})S(x-y) = i\delta^4(x-y)\mathbb{1}_{4\times 4} \quad (5.53)$$

where $\mathbb{1}_{4\times 4}$ is the four by four unit matrix. Note that this is really four equations. Solving this equation via Fourier transforms we get

$$S = \int \frac{d^4k}{(2\pi)^4} \frac{i(\cancel{k} + m)e^{-ik\cdot(x-y)}}{k^2 - m^2 + i\epsilon} \quad (5.54)$$

We now move on to functional differentiation. One can show that

$$\langle 0|\mathcal{T}(\hat{\psi}(x_1)\dots\hat{\psi}(y_1)\dots)|0\rangle = \frac{1}{W_0[0]} \left(\frac{1}{i} \frac{\delta}{\delta\bar{\eta}(x_1)} \dots \frac{-1}{i} \frac{\delta}{\delta\eta(y_1)} \dots \right) W_0 \Big|_{\eta, \bar{\eta} \rightarrow 0} \quad (5.55)$$

where $\left(\frac{\delta}{\delta\bar{\eta}}\dots\right)$ that we have one functional derivative for each fermionic field being time ordered. The set of fields can be both Grassman fields and scalar fields. We have

$$\frac{\delta}{\delta\eta^a} \exp(i\bar{\psi}\eta) = \frac{\delta}{\delta\eta^a} (1 + i(\psi^\dagger)^c(\gamma^0)_{cb}\eta^b) \quad (5.56)$$

$$= -i(\psi^\dagger)^c(\gamma^0)_{cb}\delta_a^b \quad (5.57)$$

$$= -i\bar{\psi}_a \quad (5.58)$$

Further note that $\langle 0|\mathcal{T}(\hat{\psi}(x_1)\hat{\psi}(x_2))|0\rangle$ is antisymmetric since the η 's are Grassman numbers.

5.3 QED

Quantum Electrodynamics(QED) describes both spin $\frac{1}{2}$ objects, "matter" (electrons, protons, muons, ...), and a spin-1 object "light" (photons). The most general Lagrangian is given by

$$\mathcal{L} = \bar{\psi}(i\cancel{\partial} - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \mathcal{L}_{int} \quad (5.59)$$

The interaction Lagrangian must be Lorentz invariant, and renormalizable. Renormalizability requires that each term must have dimensionality less than or equal to 4. By our Lagrangian we know that $[\psi] = 3/2$ and $[A^\mu] = 1$. There are very few allowed terms. We cannot have an odd number of ψ terms since then we won't have a Lorentz invariant Lagrangian. The only way to couple them to A 's in a renormalizable way is to have 2 ψ and 1 A . There are two potential bilinears we can use:

$$A_\mu \bar{\psi}\gamma^\mu\psi \quad (5.60)$$

$$A_\mu \bar{\psi}\gamma^\mu\gamma^5\psi \quad (5.61)$$

The second term violates parity. Since we know that electromagnetism conserves Parity we have,

$$\mathcal{L}_{int} = eA_\mu \bar{\psi}\gamma^\mu\psi \quad (5.62)$$

where e is a dimensionless number. It will turn out that e is the electric charge of the quantized ψ fields. One may wonder why the charge has no units. That is a result of using $\hbar = 1$. This can be seen since

$$\alpha \approx \frac{1}{137} = \frac{e^2}{4\pi\hbar c} \xrightarrow{\text{Natural Units}} \frac{e^2}{4\pi} \quad (5.63)$$

and hence $e = \sqrt{4\pi\alpha} \approx 0.3$.

We now derive the same result in a deeper way. The Dirac Lagrangian, \mathcal{L}_D has a “global $U(1)$ ” symmetry:

$$\psi \rightarrow e^{i\alpha}\psi \quad (5.64)$$

i.e. $\mathcal{L}_D \rightarrow \mathcal{L}_D$. There is a Noether current associated with this symmetry given by

$$j^\mu = \bar{\psi}\gamma^\mu\psi \quad (5.65)$$

with $\partial_\mu j^\mu = 0$. Promoting this global symmetry to a local symmetry, “local $U(1)$ ”:

$$\psi \rightarrow e^{i\alpha(x)}\psi \quad (5.66)$$

This is no longer a symmetry since

$$\partial_\mu\psi \rightarrow \partial_\mu(e^{i\alpha}\psi) = e^{i\alpha}\partial_\mu\psi + i\partial_\mu\alpha e^{i\alpha}\psi \quad (5.67)$$

and

$$\mathcal{L}_D \rightarrow \mathcal{L}_D + i(\partial_\mu\alpha)\bar{\psi}\gamma^\mu\psi \quad (5.68)$$

and our Lagrangian is not invariant. The procedure to “fix” this is to introduce a gauge field that has the following transformation:

$$A_\mu \xrightarrow{\text{local } U(1)} A_\mu + \frac{1}{e}\partial_\mu\alpha \quad (5.69)$$

We now introduce another term known as the covariant derivative:

$$D_\mu = \partial_\mu - ieA_\mu \quad (5.70)$$

$$D_\mu\psi \xrightarrow{\text{local } U(1)} e^{i\alpha}D_\mu\psi \quad (5.71)$$

and we end with

$$\bar{\psi}D_\mu\psi \xrightarrow{\text{local } U(1)} \bar{\psi}D_\mu\psi \quad (\text{invariant under local } U(1)) \quad (5.72)$$

Our Dirac Lagrangian becomes

$$\mathcal{L} = \bar{\psi}(i\not{D} - m)\psi = \bar{\psi}(i\not{\partial} - m)\psi + eA_\mu\bar{\psi}\gamma^\mu\psi \quad (5.73)$$

Note that if we had Parity violating $U(1)$ then we would have:

$$\psi \rightarrow e^{i\gamma_5\alpha}\psi, \quad \bar{\psi} \rightarrow \bar{\psi}e^{i\gamma_5\alpha} \quad (5.74)$$

term we could have added a slightly different covariant derivative and gotten the γ^5 term in our final Lagrangian. **[Q 5: Show this!]** Our final QED Lagrangian is given by

$$\boxed{\mathcal{L} = \bar{\psi}(i\not{\partial} - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + eA_\mu\bar{\psi}\gamma^\mu\psi} \quad (5.75)$$

5.4 Three-point function

We get our Feynman rules by looking at different correlation functions. We begin by looking at a 3-pt function ($|\Omega\rangle$ is the messy QFT ground state).

$$\langle \Omega | \mathcal{T} (\bar{\psi}_a(x_1) \psi_b(x_2) A_\mu(x_3)) | \Omega \rangle = \frac{1}{W_e[0]} \frac{1}{i} \frac{\delta}{\delta \eta_a(x_1)} \frac{1}{i} \frac{\delta}{\delta \bar{\eta}_b(x_2)} \frac{1}{i} \frac{\delta}{\delta J^\mu(x_3)} W_e[\eta, \bar{\eta}, J^\mu] \Big|_{\eta \rightarrow 0, J \rightarrow 0} \quad (5.76)$$

where

$$W_e[\eta, \bar{\eta}, J^\mu] = \int \mathcal{D}\psi \mathcal{D}\bar{\psi} \mathcal{D}A \exp \left[i \int d^4x \left(\bar{\mathcal{L}}_D + \overset{-\frac{1}{4}F^2}{\uparrow} \bar{\mathcal{L}}_g + e A_\mu \bar{\psi} \gamma^\mu \psi + \bar{\eta} \psi + \bar{\psi} \eta + J^\mu A_\mu \right) \right] \quad (5.77)$$

We almost know how to calculate this. Without the interaction term we could do it explicitly. We expand in powers of e :

$$W_e[\eta, \bar{\eta}, J^\mu] = W_0 + ie \int d^4x \left(\frac{1}{i} \frac{\delta}{\delta J^\mu(x)} \right) \left(-\frac{1}{i} \frac{\delta}{\delta \eta_c(x)} \right) (\gamma^\mu)_{cd} \left(\frac{1}{i} \frac{\delta}{\delta \bar{\eta}_d(x)} \right) W_0 + \mathcal{O}(e^2) \quad (5.78)$$

where we define

$$W_0 = \exp \left[-\frac{1}{2} \int d^4w d^4z J^\mu(w) G_{\mu\nu}(w-z) J^\nu(z) \right] \exp \left[- \int d^4w d^4z \bar{\eta}(w) S(w-z) \eta(z) \right] \quad (5.79)$$

We now calculate this explicitly:

$$\frac{1}{i} \frac{\delta}{\delta \bar{\eta}_d(x)} \exp[\dots] = i \int d^4z S_{db}(x-z) \eta_b(z) \exp[\dots] \quad (5.80)$$

$$\begin{aligned} \frac{-1}{i} \frac{\delta}{\delta \eta_c(x)} \frac{1}{i} \frac{\delta}{\delta \bar{\eta}_d(x)} \exp[\dots] &= -S_{dc}(0) \exp[\dots] \\ &\quad - \int d^4z S_{db}(x-z) \eta_b(z) \cdot (-1) \int d^4w \bar{\eta}_a(w) S_{ac}(w-x) \exp[\dots] \end{aligned} \quad (5.81)$$

and

$$\frac{1}{i} \frac{\delta}{\delta J^\mu(x)} \exp[\dots] = i \int d^4y J^\mu(y) G_{\mu\nu}(x-y) \exp[\dots] \quad (5.82)$$

which gives,

$$\begin{aligned} W_e[\eta, \bar{\eta}, J^\mu] &= \left\{ 1 - ie \int d^4x d^4y J^\mu(y) G_{\mu\nu}(x-y) \left(S_{dc}(0) \right. \right. \\ &\quad \left. \left. - \int d^4z d^4w S_{db}(w-z) \eta_b(z) \bar{\eta}_a(x) S_{ac}(w-x) \right) \gamma_{cd}^\nu \right\} W_0 \end{aligned} \quad (5.83)$$

To calculate the correlation function to leading order we need to take three derivatives of W_e . This is straight forward and the answer is **[Q 6: Do this calculation!]**

$$\langle \Omega | \mathcal{T} (\bar{\psi}_a(x_1) \psi_b(x_2) A_\mu(x_3)) | \Omega \rangle = ie \int d^4x G_{\mu\nu}(x_3 - x) \left(S_{bc}(x_2 - x) \gamma_{cd}^\nu S_{da}(x - x_1) + S_{ba}(x_2 - x_1) (\gamma^\nu)_{cd} S_{dc}(x - x) \right) \quad (5.84)$$

We develop the position space Feynman rules given by

$$\begin{aligned} \psi_a(x) \longleftarrow \bar{\psi}_b(y) &\rightarrow S_{ab}(x - y) \\ A_\mu(x) \text{ (wavy)} A_\nu(y) &\rightarrow G_{\mu\nu}(x - y) \\ \begin{array}{c} d \\ \nearrow \\ \text{---} \\ \searrow \\ c \end{array} &\rightarrow -ie \int d^4x \gamma_{cd}^\mu \end{aligned}$$

In principle we can now work out every correlation function using these Feynman rules.

The momentum space Feynman rules are

$$\begin{aligned} a \longleftarrow b &\rightarrow \frac{i(\not{p} + m)_{ab}}{p^2 - m^2 + i\epsilon} \\ \begin{array}{c} \mu \\ \text{---} \\ \nu \end{array} &\rightarrow -\frac{i}{k^2 + i\epsilon} \left(g_{\mu\nu} - (1 - \xi) \frac{k_\mu k_\nu}{k^2} \right) \\ \begin{array}{c} d \\ \nearrow \\ \text{---} \\ \searrow \\ c \end{array} &\rightarrow -ie\gamma_{ab}^\mu + \text{momentum conservation} \end{aligned}$$

Note that when the momentum flows opposite to the direction in the fermion propagator we get a negative sign in front of p .

We can now calculate the correlation function,

$$\tilde{G}_{ab\mu}(p_1, p_2, p_3) = \begin{array}{c} a, p_1 \\ \nearrow \\ \text{---} \\ \searrow \\ b, p_2 \end{array} \text{---} p_3 + \text{unconnected diagrams}$$

we ignore unconnected diagrams as they don't contribute to the S matrix. This diagram is

$$\tilde{G}_{ab\mu}(p_1, p_2, p_3) = \frac{i(\not{p}_2 + m)_{bc}}{p_2^2 - m^2 + i\epsilon} (-ie)(\gamma^\mu)_{cd} \frac{i(\not{p}_1 + m)_{da}}{p_1^2 - m^2 + i\epsilon} \frac{(-i)(g_{\nu\mu} - (1 - \xi)\frac{p_{3,\mu}p_{3,\nu}}{p_3^2})}{p_3^2 + i\epsilon} \quad (5.85)$$

We know from our experience with scalars that in order to get the S matrix elements through LSZ reduction we need to know the poles of this expression.

Note that if you try to solve the three equations:

$$p_1 + p_2 = p_3 \quad (5.86)$$

$$p_1^2 = p_2^2 = m^2 \quad (5.87)$$

$$p_3^2 = 0 \quad (5.88)$$

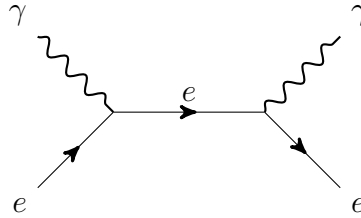
you get a contradiction since squaring the first equation gives

$$p_1 \cdot p_2 = 0 \quad (5.89)$$

and if we work in the center of mass frame we have $p_1 = (m, \mathbf{0}), p_2 = (E, \mathbf{p})$ which would imply

$$mE = 0 \quad (5.90)$$

which is impossible since $m > 0$. Hence it's impossible to have a 3 point process that conserves energy. The first real S matrix process is a four point process:



However in order to see the structure of the S matrix it is sufficient to consider the 3 point function.

Consider an incoming fermion $\xrightarrow{p \rightarrow} \text{blob}$.

$$\tilde{G} \rightarrow (\dots) i \frac{\not{p} + m}{p^2 - m^2 + i\epsilon} \quad (5.91)$$

LSZ reductions for Fermions tells us that

$$\tilde{G} = \frac{i}{p^2 - m^2 + i\epsilon} \sum_{s=1}^2 \langle \dots | \dots(\mathbf{p}, s) \rangle \bar{u}^s(p) + \text{terms with pole in } p^2 \quad (5.92)$$

here the p^2 poles are not important. [Q 7: Isn't this a p^2 pole...?] \tilde{G} is of this form because,

$$|(\mathbf{p}, s)\rangle = a_{\mathbf{p}}^{s\dagger} |0\rangle \quad (5.93)$$

and

$$\psi = \int \frac{d^3k}{(2\pi)^3 \sqrt{2E_{\mathbf{k}}}} \sum_s \left(a_{\mathbf{k}}^s u^s(k) e^{-ik \cdot x} + b_{\mathbf{k}}^{s\dagger} v^s(k) e^{ik \cdot x} \right) \quad (5.94)$$

$$\bar{\psi} = \int \frac{d^3k}{(2\pi)^3 \sqrt{2E_{\mathbf{k}}}} \sum_s \left(a_{\mathbf{k}}^{s\dagger} \bar{u}^s(k) e^{ik \cdot x} + b_{\mathbf{k}}^s \bar{v}^s(k) e^{-ik \cdot x} \right) \quad (5.95)$$

So we cannot directly get the S matrix from the \tilde{G} , but we can use trace technology to simplify our result $\sum_{s=1}^2 u_s(p) \bar{u}_s(p) = \not{p} + m$. Using this result we have,

$$\tilde{G} = (\dots) \frac{i \sum_{s=1}^2 u_s(p) \bar{u}_s(p)}{p^2 - m + i\epsilon} \quad (5.96)$$

Comparing with the LSZ formula we can write,

$$\langle \dots | \dots(\mathbf{p}, s) \rangle = (\dots) u_s(p) \quad (5.97)$$

So for an incoming fermion we see we have a factor of $u(p)$. The derivation for the other an incoming antifermion as well as for the outgoing fermions is similar.

Now consider the full photon terms:

$$\frac{(-i)(g_{\mu\nu} - (1 - \xi)p_\mu p_\nu / p^2) \mathcal{N}^\nu}{p^2 + i\epsilon} \quad (5.98)$$

where \mathcal{N}^ν is just the rest of the object which has no μ dependence. Using LSZ reduction we have for an incoming photon,

$$\tilde{G}_\mu = \frac{i}{p^2 + i\epsilon} \sum_{s=1}^2 \langle \dots | \dots(\mathbf{p}, s) \rangle \epsilon_\mu^{s*} + (\text{no pole at } p^2 = 0) \quad (5.99)$$

where we have used,

$$A_\mu = \int \frac{d^3k}{(2\pi)^3 \sqrt{2E_{\mathbf{k}}}} \sum_s a^s \epsilon_\mu^s e^{ik \cdot x} + a^{s\dagger} \epsilon_\mu^{s*} e^{-ik \cdot x} \quad (5.100)$$

Using the Ward identity (we discuss this identity more later) we know that

$$p_\mu \mathcal{N}^\nu = 0 \quad (5.101)$$

This means that the $(1 - \xi)p_\mu p_\nu / p^2$ term in the incoming and outgoing photon contributions go to zero and we simply have,

$$\frac{-i g_{\mu\nu} \mathcal{N}^\nu}{p^2 + i\epsilon} = -\frac{i}{p^2 + i\epsilon} \sum_s \epsilon_\mu^{s*}(p) \langle \dots | \dots(\mathbf{p}, s) \rangle \quad (5.102)$$

In the fermion case we rearrange the numerator of the propagator using a spin sum. We would like to do the same thing here with $g_{\mu\nu}$ and write it in terms of ϵ_μ 's. To do this

we use a trick that we will prove later on which says that as long as $g_{\mu\nu}$ is going to be multiplied by an amplitude we can make the replacement,

$$g_{\mu\nu} \rightarrow - \sum_s \epsilon_\mu^s \epsilon_\nu^{s*} \quad (5.103)$$

Since the LSZ formula is true for any s ,

$$\mathcal{N}^\nu \epsilon_\mu^s(p) = \langle \dots | \dots(\mathbf{p}, s) \rangle \quad (5.104)$$

Hence we can replace an incoming photon by ϵ_μ^s .

We summarize all the rules for incoming and outgoing particles below

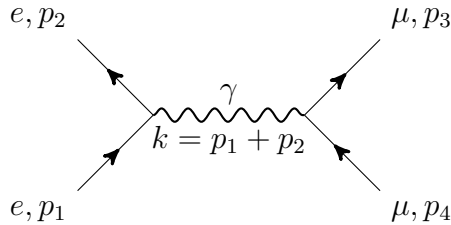
incoming fermion	$\xrightarrow{p \rightarrow} \text{circle with diagonal lines}$	$= u^s(p)$
incoming antifermion	$\xleftarrow{p \rightarrow} \text{circle with diagonal lines}$	$= \bar{v}^s(p)$
outgoing fermion	$\text{circle with diagonal lines} \xrightarrow{p \rightarrow}$	$= \bar{u}^s(p)$
outgoing antifermion	$\text{circle with diagonal lines} \xleftarrow{p \rightarrow}$	$= v^s(p)$
incoming photon	$\xrightarrow{\mu \quad p \rightarrow} \text{circle with diagonal lines}$	$= \epsilon_\mu^s(p)$
outgoing photon	$\text{circle with diagonal lines} \xrightarrow{\mu \quad p \rightarrow}$	$= \epsilon_\mu^{s*}(p)$

5.5 $e^+e^- \rightarrow \mu^+\mu^-$

Now that we have all our Feynman rules we can finally do some calculations. As an example lets consider two electrons turning into two muons. The interaction part of the Lagrangian is

$$\mathcal{L}_{int} = -eA_\nu(\bar{\psi}_e \gamma^\nu \psi_e + \bar{\psi}_\mu \gamma^\nu \psi_\mu) \quad (5.105)$$

The leading order connected diagram is



$$i\mathcal{M}^{s_1, s_2, s_3, s_4} = [\bar{v}_e^{s_2}(p_2)(-ie\gamma^\alpha)u_e^{s_1}(p_1)] \frac{(-i)}{k^2 + i\epsilon} \left(g_{\alpha\beta} - (1 - \xi) \frac{k_\alpha k_\beta}{k^2} \right) \times [\bar{u}_\mu^{s_3}(p_3)(-ie\gamma^\beta)v_\mu^{s_4}(p_4)] \quad (5.106)$$

(note that here μ, e are not variables just labels so we know what mass each u and v corresponds to. Consider the term

$$\frac{k_\alpha}{k^2} [\bar{v}(p_2)\gamma^\alpha u(p_1)] = (-ie)\bar{v}(p_2)\not{k}u(p_1) \quad (5.107)$$

$$= (-ie)\bar{v}(p_2)(\not{p}_1 + \not{p}_2)u(p_1) \quad (5.108)$$

$$= (-ie) (\bar{v}(p_2)m_e u(p_1) + \bar{v}(p_2)(-m_e)u(p_1)) \quad (5.109)$$

$$= 0 \quad (5.110)$$

since $(\not{p} - m)u = 0$, $(\not{p} + m)v = 0$. Thus the amplitude is independent of the $k_\alpha k_\beta$ contribution and hence independent of gauge parameter ξ . The amplitude simplifies to

$$i\mathcal{M}^{s_1 s_2 s_3 s_4} = i\frac{e^2}{s} [\bar{v}_2 \gamma^\alpha u_1] [\bar{u}_3 \gamma_\alpha u_4] \quad (5.111)$$

where $s \equiv (p_1 + p_2)^2$. Typical experiments have polarized beams. In this case we can average over incoming particle spins, and sum over outgoing particle spins.

$$\left(\frac{1}{2}\right)^2 \sum_{s_1, s_2, s_3, s_4} |\mathcal{M}^{s_1, s_2, s_3, s_4}|^2 \equiv \overline{|\mathcal{M}|^2} \quad (5.112)$$

In our case we have

$$\overline{|\mathcal{M}|^2} = \frac{1}{4} \frac{e^4}{s^2} \text{tr} \left[(\not{p}_2 - m_e) \gamma^\alpha (\not{p}_1 + m_e) \gamma^\beta \right] \times \text{tr} \left[(\not{p}_3 + m_\mu) \gamma_\alpha (\not{p}_4 - m_\mu) \gamma_\beta \right] \quad (5.113)$$

$$= \frac{1}{4} \frac{e^4}{s^2} (4) \left(p_2^\alpha p_1^\beta + p_1^\alpha p_2^\beta - g^{\alpha\beta} (p_1 \cdot p_2 + m_e^2) \right) (4) \\ \times \left(p_4^\alpha p_3^\beta + p_3^\alpha p_4^\beta - g^{\alpha\beta} (p_3 \cdot p_4 + m_\mu^2) \right) \quad (5.114)$$

In the high energy regime ($E \gg m_e, m_\mu$) we have

$$\overline{|\mathcal{M}|^2} = \frac{8e^4}{s^2} ((p_1 \cdot p_4)(p_2 \cdot p_e) + (p_1 \cdot p_3)(p_2 \cdot p_4)) \quad (5.115)$$

but,

$$p_1 + p_2 = p_3 + p_4 \quad (5.116)$$

and

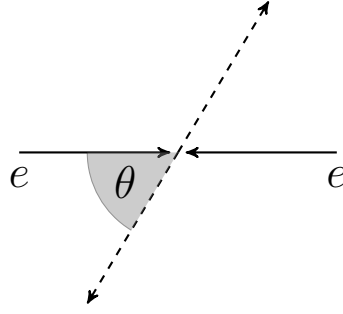
$$(p_1 - p_4)^2 = (p_3 - p_2)^2 \quad (5.117)$$

$$p_1 \cdot p_4 \approx p_3 \cdot p_2 \quad (5.118)$$

since we have in the high energy regime. So after switching the Mandelstam variables the amplitude squared takes the form,

$$\overline{|\mathcal{M}|^2} = \frac{2e^4}{s^2} (t^2 + u^2) \quad (5.119)$$

If we work in the COM frame:



then

$$\begin{aligned} p_1 &= (E, 0, 0, E) \\ p_2 &= (E, 0, 0, -E) \\ p_3 &= (E, E \sin \theta, 0, E \cos \theta) \\ p_4 &= (E, -E \sin \theta, 0, -E \cos \theta) \end{aligned}$$

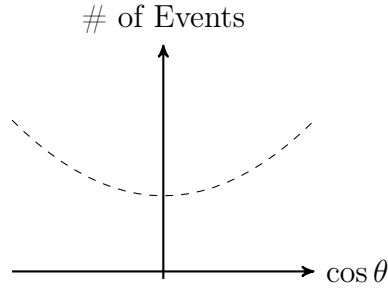
where $s = 4E^2$, $t = -2E^2(1 - \cos \theta)$, $u = -2E^2(1 + \cos \theta)$. We have the simple result

$$\overline{|\mathcal{M}|^2} = e^4(1 + \cos^2 \theta) \quad (5.120)$$

and

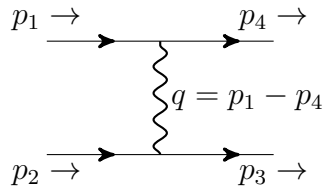
$$\frac{d\sigma}{d\Omega} = \frac{\overline{|\mathcal{M}|^2}}{64\pi^2 s} = \frac{e^4}{32\pi s}(1 + \cos^2 \theta) \quad (5.121)$$

The functional dependence is as follows,



5.6 $e^- \mu^- \rightarrow e^- \mu^-$

We have found the amplitude and cross sections for $e^+e^- \rightarrow \mu^+\mu^-$. We now consider elastic $e^- \mu^-$ scattering but now in the non-relativistic regime,



We can do a similar calculation to last time:

$$i\mathcal{M} = \frac{ie^2}{t} (\bar{u}_4 \gamma^\mu u_1) (\bar{u}_3 \gamma_\mu u_2) \quad (5.122)$$

In the non-relativistic limit we have

$$\begin{aligned} p_1 &= (m_e + \mathcal{O}(\mathbf{p}^2), \mathbf{p}) \\ p_2 &= (m_\mu, \mathbf{0}) \\ p_3 &= (m_\mu + \mathcal{O}(\mathbf{p}^2), \mathbf{q}) \\ p_4 &= (m_e + \mathcal{O}(\mathbf{p}^2), \mathbf{p}') \end{aligned}$$

where $\mathbf{q} = \mathbf{p} - \mathbf{p}'$. Hence we have

$$t = (p_1 - p_4)^2 = -\mathbf{q}^2 + \mathcal{O}(\mathbf{p}^4) \quad (5.123)$$

The Dirac spinor in the non-relativistic limit is

$$u^s(\mathbf{p}) = \sqrt{m} \begin{pmatrix} \xi^s \\ \xi^s \end{pmatrix} + \mathcal{O}(\mathbf{p}) \quad (5.124)$$

where $\xi^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\xi^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. The scalars are:

$$\bar{u}_4 \gamma^0 u_1 = \sqrt{m} (\xi^{s_4 \dagger} \xi^{s_1} + \xi^{s_4 \dagger} \xi^{s_1}) = 2m \xi^{s_4 \dagger} \xi^{s_1} = 2m \delta_{s_4 s_1} \quad (5.125)$$

$$\bar{u}_4 \gamma^i u_1 = \mathcal{O}(\mathbf{p}) \quad (5.126)$$

Thus to leading order we have

$$i\mathcal{M} \approx -ie^2 (2m_e) (2m_\mu) \frac{\delta_{s_4 s_1} \delta_{s_3 s_2}}{\mathbf{q}^2} \quad (5.127)$$

The cross-section is given by,

$$\frac{d\sigma}{d\Omega} = \frac{1}{2E_e 2m_\mu} \frac{E_e}{p_e} \frac{1}{32\pi^2} \frac{2p_e^{CM}}{E_{CM}} \overline{|\mathcal{M}|^2} \quad (5.128)$$

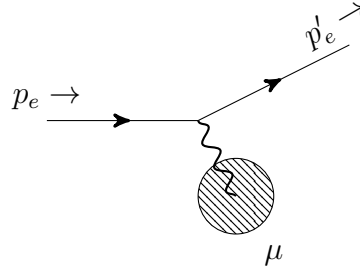
We work in the limit of a stationary muon (though the cross-section is Lorentz invariant). In this case $p_e^{\text{cm}} \approx p_e$ and $E_{\text{cm}} \approx m_\mu$. We have,

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 m_\mu^2} |\mathcal{M}|^2 \quad (5.129)$$

$$= \frac{4\alpha^2}{q^4} m_e^2 \quad (5.130)$$

where the spin averaging is trivial due to the spin delta functions.

In non-relativistic quantum mechanics we'd expect the electron to be moving slowly and the muon to be at rest acting as a source of charge through the potential, $V(r) = \frac{\alpha}{r}$.



In this case the Born approximation tells us that

$$\frac{d\sigma}{d\Omega} = \frac{m_e^2}{4\pi^2} \tilde{V}^2(\mathbf{q}) \quad (5.131)$$

where $\mathbf{q} = \mathbf{p}' - \mathbf{p}$. Comparing the two expressions we identify,

$$\tilde{V}^2(\mathbf{q}) = \frac{\alpha^2 16\pi^2}{\mathbf{q}^4} \quad (5.132)$$

$$\Rightarrow \tilde{V}(\mathbf{q}) = \frac{4\pi\alpha}{\mathbf{q}^2} \quad (5.133)$$

We have ¹

$$V(\mathbf{r}) = \int \frac{d^3q}{(2\pi)^3} e^{-i\mathbf{q}\cdot\mathbf{r}} \frac{4\pi\alpha}{\mathbf{q}^2} \quad (5.134)$$

$$= \frac{\alpha}{\pi} \int d\cos\theta dq e^{iqr\cos\theta} \quad (5.135)$$

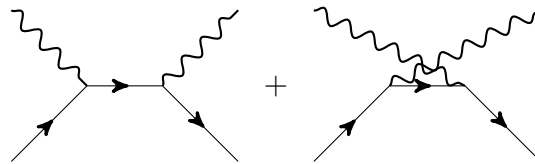
$$= -\frac{2\alpha}{\pi} \int dq \frac{1}{q} \sin(qr) \quad (5.136)$$

$$= -\frac{\alpha}{r} \quad (5.137)$$

where the integral was done by inserting a regulator and taking the regulator to 0 at the end. We see that in the non-relativistic limit QED reproduces classical electromagnetism as expected. Furthermore, this is the first confirmation that the coupling in the Lagrangian, e , really represents an electric charge. Up to now we just assumed the most natural interpretation and we finally see that this is indeed the case.

5.7 Compton Scattering

Consider $e^- \gamma \rightarrow e^- \gamma$. There are two second order vertices:



¹Where there is no ambiguity with four-vectors we write, $r \equiv |\mathbf{r}|$ and $q \equiv |\mathbf{q}|$.

We have the amplitude

$$\mathcal{M}^s = \epsilon_\mu^s(p) \mathcal{M}^\mu \quad (5.138)$$

To get the cross-section we need to average over spins:

$$\sum_s |\mathcal{M}^s|^2 = \left(\sum_s \epsilon_\mu^s \epsilon_\nu^{s*} \right) \mathcal{M}^{\nu*} \quad (5.139)$$

The claim is that we can make the replacement:

$$\sum_s \epsilon_\mu^s \epsilon_\nu^{s*} \rightarrow -g_{\mu\nu} \quad (5.140)$$

To see why this is consider the photon with the momenta, $p = E(1, 0, 0, 1)$. The Ward identity tells us that $p_\mu \mathcal{M}^\mu = 0$. Hence

$$\mathcal{M}^0 = \mathcal{M}^3 \quad (5.141)$$

Since we can choose any basis we wish for the polarization lets consider linear polarization,

$$\begin{aligned} \epsilon^1 &= (0, 1, 0, 0) \\ \epsilon^2 &= (0, 0, 1, 0) \end{aligned}$$

Explicitly we can now calculate

$$\sum_s \epsilon_\mu^s \epsilon_\nu^{s*} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (5.142)$$

This is not quite

$$-g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (5.143)$$

But $\sum_s \epsilon_\mu^s \epsilon_\nu^{s*}$ and $g_{\mu\nu}$ never appear on their own. Instead they come as a product of amplitudes. We have,

$$-g_{\mu\nu} \mathcal{M}^\mu \mathcal{M}^{\nu*} = -|\mathcal{M}^0|^2 + |\mathcal{M}^1|^2 + |\mathcal{M}^2|^2 + |\mathcal{M}^3|^2 = |\mathcal{M}^1|^2 + |\mathcal{M}^2|^2 \quad (5.144)$$

$$\sum_s \epsilon_\mu^s \epsilon_\nu^{s*} \mathcal{M}^\mu \mathcal{M}^{\nu*} = |\mathcal{M}^1|^2 + |\mathcal{M}^2|^2 \quad (5.145)$$

Hence due to the Ward Identity we can always make the substitution,

$$\sum_s \epsilon_\mu^s \epsilon_\nu^{s*} \rightarrow -g_{\mu\nu} \quad (5.146)$$

5.8 Proof of Ward Identity

We finally prove the Ward identity and then discuss the intuition behind it. The defining equation for the vector potential without choosing a gauge is given by

$$\partial_\nu \partial^\nu A_\mu(x) = 0 \quad (5.147)$$

This equation is solved by the an satz,

$$A_\mu(x) = \int \frac{d^4 p}{(2\pi)^4} \epsilon_\mu(p) e^{-ip \cdot x} \quad (5.148)$$

The gauge transformation of the vector potential under which physics is invariant is,

$$A'_\mu = A_\mu + \partial_\mu \alpha \quad (5.149)$$

Plugging in the potential as well as the Fourier transform of $\alpha = \int \frac{d^4 p}{(2\pi)^4} e^{-ip \cdot x} \tilde{\alpha}(p)$ we get

$$\int \frac{d^4 p}{(2\pi)^4} (\epsilon'_\mu - \epsilon_\mu) e^{-ip \cdot x} = \int \frac{d^4 p}{(2\pi)^4} (-ip_\mu) e^{-ip \cdot x} \tilde{\alpha} \quad (5.150)$$

Now multiplying by $e^{-ip' \cdot x}$ and integrating over $d^4 x$ we have

$$\int \frac{d^4 p}{(2\pi)^3} (\epsilon'_\mu - \epsilon_\mu) \delta(p - p') = -i \int \frac{d^4 p}{(2\pi)^4} p_\mu \tilde{\alpha} \delta(p - p') \quad (5.151)$$

$$\epsilon'_\mu(p) = \epsilon_\mu(p) - i\tilde{\alpha}(p)p_\mu \quad (5.152)$$

This is the effect of a gauge transformation on the polarization vector.

Now in general any amplitude with N gauge bosons will take the form,

$$\epsilon^{\mu_1} \epsilon^{\mu_2} \dots \epsilon^{\mu_N} \mathcal{M}_{\mu_1 \mu_2 \dots \mu_N} \quad (5.153)$$

If a theory is gauge invariant then this amplitude must be invariant under any gauge transformation. Applying equation 5.152 to some polarization vector i we have

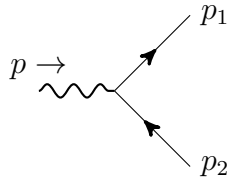
$$\epsilon^{\mu_1} \epsilon^{\mu_2} \dots \epsilon^{\mu_N} \mathcal{M}_{\mu_1 \mu_2 \dots \mu_N} \rightarrow \epsilon^{\mu_1} \epsilon^{\mu_2} \dots \epsilon^{\mu_N} \mathcal{M}_{\mu_1 \mu_2 \dots \mu_N} - i\tilde{\alpha} p^{\mu_i} \mathcal{M}_{\mu_1, \dots, \mu_N} \quad (5.154)$$

In order for the amplitude to be gauge invariant we must have Ward Identity,

$$p^{\mu_i} \mathcal{M}_{\mu_1, \dots, \mu_N} = 0 \quad (5.155)$$

which is just the Ward Identity.

We now try to explore how this relationship holds. Consider the following



where we have \mathbf{p} is off-shell since otherwise it would violate energy conservation. The amplitude is,

$$\mathcal{M} = \epsilon_\nu \underbrace{\bar{u}(p_1)\gamma^\nu v(p_2)}_{\mathcal{M}^\nu} (-ie) \quad (5.156)$$

Then the sum, $p_\nu \mathcal{M}^\nu$, is zero since,

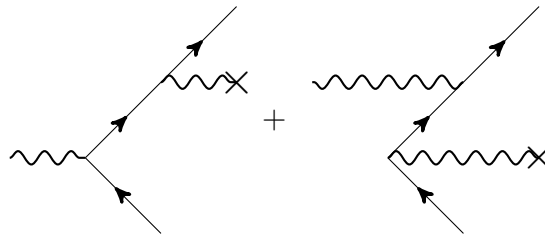
$$p_\nu \mathcal{M}^\nu = \bar{u}(p_1) \not{p} v(p_2) (-ie) \quad (5.157)$$

$$= \bar{u}(p_1) (\not{p}_1 + \not{p}_2) v(p_2) (-ie) \quad (5.158)$$

$$= (-ie) \bar{u}_1 v_2 (m_e - m_e) \bar{u}_1 v_2 \quad (5.159)$$

$$= 0 \quad (5.160)$$

Now if we are considering a more complicated diagram,



where the “ \times ” represents some combination of external lines. Then the Ward identity means that these contributions must cancel. See Peskin chapter 7.4 for details. [\[Q 8: Expand on this section\]](#)

Chapter 6

Loop Diagrams and UV Divergences in QED

6.1 Superficial Divergences

The first thing one thinks about when hearing about loop diagrams is estimating how divergent diagrams are. We use what's known as a "naive power-counting" to find the superficial degree of divergence. For \mathcal{D} greater than or equal to zero we expect $\sim \Lambda^{\mathcal{D}}$ and we expect logarithmic divergence if $\mathcal{D} = 0$. If $\mathcal{D} < 0$ then our diagrams are not superficially divergent. Each loop contributes a factor of $\int \frac{d^4 p}{(2\pi)^4}$ (4 factors of momenta). Fermion propagators have dimension -1 and photon propagators have dimension -2 hence (P_i the number of i propagators)

$$\mathcal{D} = 4L - P_f - 2P_\gamma \tag{6.1}$$

One can relate this expression to the number of external lines by (Exercise)

$$\mathcal{D} = 4 - \frac{3}{2}N_f - N_\gamma \tag{6.2}$$

In reality the power counting is not precise and diagrams may or may not diverge independently of the superficial degree of freedom. Consider the $\mathcal{D} \geq 0$ diagrams in QED:

Diagram	D	Expected # of CT's	Name
	$\mathcal{D} = 4$	vacuum energy \rightarrow unphysical	vacuum polarization
	$\mathcal{D} = 3$	zero!(HW)	
	$\mathcal{D} = 2$	3 counterterms(
	$\mathcal{D} = 1$	zero!(HW)	
	$\mathcal{D} = 0$	1 counterterm	electron self-energy
	$\mathcal{D} = 1$	2 counterterms	
	$\mathcal{D} = 0$	1 counterterm	vertex correction

Our Lagrangian is

$$\mathcal{L} = \bar{\psi}(i\cancel{D} - m_0)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + e_0A_\mu\bar{\psi}\gamma^\mu\psi \quad (6.3)$$

We want to introduce renormalized fields $\psi = \sqrt{Z_2}\psi_r$ and $A_\mu = \sqrt{Z_3}A_\mu^r$. We then have

$$\mathcal{L} = Z_2\bar{\psi}_r(i\cancel{D} - m_0)\psi_r - Z_3\frac{1}{4}F_{\mu\nu}^rF_r^{\mu\nu} + e_0Z_2\sqrt{Z_3}A_\mu^r\bar{\psi}_r\gamma^\mu\psi_r \quad (6.4)$$

We introduce a physical mass and we have $\delta m = m - m_0$:

$$\mathcal{L} = Z_2\bar{\psi}_r(i\cancel{D} - m)\psi_r - Z_3\frac{1}{4}F_{\mu\nu}^rF_r^{\mu\nu} + e_0Z_2\sqrt{Z_3}A_\mu^r\bar{\psi}_r\gamma^\mu\psi_r + \delta mZ_2\bar{\psi}_r\psi_r \quad (6.5)$$

We still need to renormalize the coupling constant. We define the new coupling constant as

$$e = \frac{e_0Z_2\sqrt{Z_3}}{Z_1} \quad (6.6)$$

How we define a coupling is not physical since it's not a number like mass which has a definition on it's own. We just need to require that it won't be divergent. This form turns out to be convenient. With this we have

$$\begin{aligned} \mathcal{L} = & \bar{\psi}_r(i\cancel{D} - m)\psi_r - \frac{1}{4}F_{\mu\nu}^rF_r^{\mu\nu} + eA_\mu^r\bar{\psi}_r\gamma^\mu\psi_r + \\ & (Z_1 - 1)A_\mu^r\bar{\psi}_r\gamma^\mu\psi_r + (Z_2 - 1)\bar{\psi}_r(i\cancel{D} - m)\psi_r - (Z_3 - 1)\frac{1}{4}F_{\mu\nu}^rF_r^{\mu\nu} + \delta mZ_2\bar{\psi}_r\psi_r \end{aligned} \quad (6.7)$$

We define

$$\begin{aligned} \delta_1 & \equiv Z_1 - 1 \\ \delta_2 & \equiv Z_2 - 1 \\ \delta_3 & \equiv Z_3 - 1 \\ \delta_m & \equiv (Z_2 - 1)(m) - \delta mZ_2 \end{aligned} \quad (6.8)$$

Our counterterms are

$$\mathcal{L}_{CT} = \delta_1 A_\mu^r \bar{\psi}_r \gamma^\mu \psi_r + \bar{\psi}_r (i\delta_2 - \delta_m) \psi_r - \frac{\delta_3}{4} F_{\mu\nu}^r F_r^{\mu\nu} \quad (6.9)$$

Note that from here on we drop the r to denote renormalized fields. Naively we expect the number of counter terms to be given by the sum of all the divergences $\rightarrow 3 + 1 + 2 + 1$. However we can only possibly write down 4 terms! We know this theory is renormalizable so the superficial degree of divergence must be overestimating the divergence of this theory. Consider our renormalized propagator

$$\overset{\mu}{\underset{q \rightarrow}{\text{wavy}}} \text{---} \text{---} \text{---} \overset{\nu}{\underset{q \rightarrow}{\text{wavy}}} = \Pi_{\mu\nu}(q) = A(q^2)g_{\mu\nu} + B(q^2)q_\mu q_\nu \quad (6.10)$$

The Ward identity tells us that $q^\mu \Pi_{\mu\nu} = 0 \Rightarrow$

$$Aq^\nu + Bq^2 q^\nu = 0 \quad (6.11)$$

since this must be true for all q^μ we have

$$A = -Bq^2 e \quad (6.12)$$

and hence

$$\overset{\mu}{\underset{q \rightarrow}{\text{wavy}}} \text{---} \text{---} \text{---} \overset{\nu}{\underset{q \rightarrow}{\text{wavy}}} = (g_{\mu\nu} q^2 - q^\mu q^\nu) B(q^2) \quad (6.13)$$

The Ward identity provides a constraint and shows that we only need 1 CT for the propagator. Similarly one can show in the same way that the four photon diagram is in fact finite and does not need a counterterm. So only need the 4 CT's.

6.2 Electron Self-Energy

$$\tilde{G}_{ab}(p) = \int d^4x e^{-ix \cdot p} \langle \Omega | \mathcal{T} (\bar{\psi}_a(x) \psi_b(0)) | \Omega \rangle \quad (6.14)$$

where in general we have $\psi_b(y)$ but we just choose $y = 0$. Using Feynman diagrams we can write

$$\begin{aligned} \tilde{G} = & \text{---} + \text{---} \text{---} \text{---} \text{---} + \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \\ & + \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} + \dots \end{aligned} \quad (6.15)$$

$$= \frac{i}{\not{p} - m_0 + i\epsilon} + \frac{i}{\not{p} - m_0 + i\epsilon} (-i\Sigma(p)) \frac{i}{\not{p} - m_0 + i\epsilon} \quad (6.16)$$

where $\Sigma(p)$ is a matrix that's equal to the sum of all the possible diagrams with two external lines and p is the incoming momentum.

We define 1 particle irreducible (1PI) graphs to be diagrams that you cannot cut into 2 by cutting a single line. We define

$$\hat{\Sigma} = \left(\begin{array}{l} \text{sum of 1PI graphs only,} \\ \text{without external lines} \end{array} \right) = \text{[diagram 1]} + \text{[diagram 2]} + \dots \quad (6.17)$$

we have (for brevity we initially suppress the $+i\epsilon$ in the denominator)

$$\tilde{G} = \frac{i}{\not{p} - m_0} + \frac{i}{\not{p} - m_0} (-i\hat{\Sigma}) \frac{i}{\not{p} - m_0} + \frac{i}{\not{p} - m_0} (-i\hat{\Sigma}) \frac{i}{\not{p} - m_0} (-i\hat{\Sigma}) \frac{i}{\not{p} - m_0} + \dots \quad (6.18)$$

$$= \frac{i}{\not{p} - m_0} \sum_{n=0}^{\infty} \left(\hat{\Sigma} \frac{1}{\not{p} - m_0} \right)^n \quad (6.19)$$

$$= \frac{i}{\not{p} - m_0} \left(\frac{1}{1 - \frac{\hat{\Sigma}}{\not{p} - m_0}} \right) \quad (6.20)$$

$$= \frac{i(\not{p} - m_0 + i\epsilon)}{\not{p} - m_0 + i\epsilon} \left(\frac{1}{\not{p} - m_0 + i\epsilon - \hat{\Sigma}(p)} \right) \quad (6.21)$$

$$= \frac{i}{\not{p} - m_0 - \hat{\Sigma}(p) + i\epsilon} \quad (6.22)$$

where we have used a geometric series for matrices,

$$\sum_{k=0}^{\infty} A^k = (1 - A)^{-1} \quad (6.23)$$

for any matrix A with eigenvalues less than 1.

To better see what this means, we switch to the “proper form” of the propagator (this is straightforward to check by multiplying the $\not{p} - m + i\epsilon$ with its inverse),

$$(\not{p} - m - \Sigma(p) + i\epsilon)^{-1} = \frac{\not{p} + m + \Sigma(p)}{p^2 - (m + \Sigma(p))^2 + i\epsilon} \quad (6.24)$$

Hence the physical mass is shifted to the point where we still have a pole in the propagator,

$$p^2 = (m_0 + \hat{\Sigma}(p))^2 \quad (6.25)$$

or $m \equiv m_0 + \hat{\Sigma}(p^2 = m^2)$. Thus $\hat{\Sigma}$ represents the change in mass from the “bare” mass calculated to all orders in perturbation theory. We can also write from 6.22

$$\tilde{G}^{-1} = \tilde{G}_0^{-1} - \frac{1}{i} \hat{\Sigma}(p) \quad (6.26)$$

so the two-point function contains, apart from the inverse bare propagator, only 1PI contributions.

Note that this expression has a superficial degree of divergence of order 1. The true divergence turns out to be of zeroth order as we will see later due to some terms being odd. To solve this integral and incorporate the divergence we introduce an unphysical mass μ that will regulate the integral. Note that this is not a unique way to regulate. We could have used for example dim-reg.

$$\hat{\Sigma}(p) \rightarrow -e^2 \int \frac{d^4\ell}{(2\pi)^4} \frac{(-2i\ell + 4m_0)}{(\ell^2 - m_0^2 + i\epsilon)((\ell - p)^2 - \mu^2 + i\epsilon)} \quad (6.40)$$

The steps are analogous to our discussion in the Fall of scalar loops diagrams. We introduce Feynman parameters (where $A = ((\ell - p)^2 - \mu^2 + i\epsilon)$ and $B = \ell^2 - m_0^2 + i\epsilon$)

$$\frac{1}{AB} = \int_0^1 \frac{dx}{(Ax + B(1-x))^2} \quad (6.41)$$

Our expression becomes

$$\hat{\Sigma}(p) = -e^2 \int dx \int \frac{d^4\ell}{(2\pi)^4} \frac{(-2i\ell + 4m_0)}{(((\ell - p)^2 - \mu^2 + i\epsilon)x + (\ell^2 - m_0^2 + i\epsilon)(1-x))^2} \quad (6.42)$$

$$= -e^2 \int dx \int \frac{d^4\ell}{(2\pi)^4} \frac{(-2i\ell + 4m_0)}{(\ell^2 - 2\ell px + p^2 x^2 - p^2 x^2 + (p^2 - \mu^2)x + (-m_0^2)(1-x) + i\epsilon)^2} \quad (6.43)$$

$$= -e^2 \int dx \int \frac{d^4\ell}{(2\pi)^4} \frac{(-2i\ell + 4m_0)}{((\ell - px)^2 - \Delta + i\epsilon)^2} \quad (6.44)$$

$$(6.45)$$

where $\Delta \equiv -p^2 x(1-x) - (\mu^2 - m_0^2)x + \mu^2$. Now define $k = \ell - px$:

$$\hat{\Sigma} = -e^2 \int dx \int \frac{d^4k}{(2\pi)^4} \frac{(-2i(\not{k} + \not{p}x) + 4m_0)}{(k^2 - \Delta + i\epsilon)^2} \quad (6.46)$$

$$(6.47)$$

Applying a Wick rotation such that $k_0 \rightarrow ik_0^E$ and $\mathbf{k} \rightarrow \mathbf{k}^E$. We now get ($dk_0 = idk_0^E$ and $k_0 \rightarrow ik_0^E$)

$$\hat{\Sigma} = -e^2 \int dx \int \frac{d^4k_E}{(2\pi)^4} \frac{(2(i\mathbf{k}_E - \not{p}x) + 4m_0)}{(k_E^2 - \Delta + i\epsilon)^2} \quad (6.48)$$

The denominator is even. Thus we can get rid of the top contribution from \mathbf{k}_E^1 :

$$\hat{\Sigma} = -e^2 \int dx \int \frac{d^4k_E}{(2\pi)^4} \frac{-2(\not{p}x) + 4m_0}{(k_E^2 - \Delta + i\epsilon)^2} \quad (6.49)$$

$$(6.50)$$

¹this is where the reduction in the divergence of the diagram occurs

First since our integral is Wick rotated we are no longer near a pole and we can drop the $i\epsilon$. Now taking the integral is a 4D spherical space we have ($S_3 = 2\pi^2$)

$$\hat{\Sigma} = -e^2 \int dx \int \frac{dk_E k_E^3}{8\pi^2} \frac{-2(\not{p}x) + 4m_0}{(k_E^2 - \Delta)^2} \quad (6.51)$$

$$= \frac{-e^2}{4\pi^2} \int dx ((-\not{p}x) + 2m_0) \int dk_E \frac{k_E^3}{(k_E^2 - \Delta)^2} \quad (6.52)$$

$$= \frac{-e^2}{4\pi^2} \int dx ((-\not{p}x) + 2m_0) \frac{1}{2} \left(\frac{-\Lambda^2}{\Lambda^2 - \Delta^2} + \log \frac{\Lambda^2 - \Delta}{\Delta} \right) \quad (6.53)$$

where in the last step we introduced a cutoff. The integral is easy enough to do in *Mathematica*. For simplicity we now assume that $\Lambda^2 \gg \Delta$ then we get the somewhat simpler result,

$$\hat{\Sigma} = -\frac{\alpha}{\pi} \int dx (\not{p}x - 2m_0) \log \left(\frac{\Lambda^2}{\Delta} \right) \quad (6.54)$$

Consider the limit that $\mu \rightarrow 0$. In this case $\Delta \rightarrow m^2(1-x)$ and we have to integrate the logarithm from 0 to 1. This is divergent as logs aren't defined at 0. This causes the electron normalization to diverge. However, the electron normalization itself isn't physical. No observables will depend on μ , however we introduce it for convenience so all the terms in our calculations will be well defined. The physical mass is shifted by

$$m = m_0 + \hat{\Sigma}(m) \quad (6.55)$$

We can use m or m_0 in evaluating the $\hat{\Sigma}$ since the correction will be of order α^2 . To see this consider the structure of $\hat{\Sigma}$:

$$\hat{\Sigma}(\not{p}) = \not{p}f(\not{p}^2) + g(\not{p}^2) \quad (6.56)$$

Setting $\not{p} = m$ and expanding f, p, m we have:

$$\hat{\Sigma}(\not{p} = m) = (m_0 + \delta m^{(1)} + \dots) (f^{(0)} + \alpha f^{(1)} + \dots) + (g^{(0)} + \alpha g^{(1)} + \dots) = 0 \quad (6.57)$$

The zero'th order result is just the propagator evaluated at $\not{p} = m_0$:

$$f^{(0)} = \frac{1}{p^2 - m_0^2 + i\epsilon}, \quad g^{(0)} = -\frac{m_0}{p^2 - m_0^2 + i\epsilon} \quad (6.58)$$

and trivially satisfies the condition. To first order in α we have,

$$\delta m^{(1)} f^{(0)} + m_0 f^{(1)} + g^{(1)} = 0 \quad (6.59)$$

Inside $f^{(1)}$ and $g^{(1)}$ there are contributions proportional to m and m^2 . However, when to this order we drop any additional factors of α which arise from inserting $m_0 + \alpha \delta m^{(1)}$. Thus we can just use $m \approx m_0$ to first order.

We then have the rather simple result,

$$\hat{\Sigma} \approx \frac{\alpha}{\pi} m_0 \int dx (2-x) \log \frac{\Lambda^2}{-m_0^2 x^2} \quad (6.60)$$

This integral can be evaluated in *Mathematica* to be

$$\hat{\Sigma} \approx \frac{3\alpha}{4\pi} \log \frac{\Lambda^2}{m_0^2} \quad (6.61)$$

So we finally have the mass shift,

$$m = m_0 + \hat{\Sigma}(m_0) = m_0 \left(1 + \frac{3\alpha}{4\pi} \log \frac{\Lambda^2}{m_0^2} \right) \quad (6.62)$$

There are two approaches towards renormalization. One interpretation is that it is just mathematical trickery that is used in order to get finite answers. This point of view is fine and it certainly works however there is a more modern interpretation which says that there is some physics in renormalization.

Suppose we are evaluating some loop diagram. Our Feynman rules say to integrate over some loop momentum. If we work in position space we know the relation between momenta and wavelength. Large momenta correspond to short wavelengths. At very high loop momenta it corresponds to fluctuations of the field with short wavelengths.

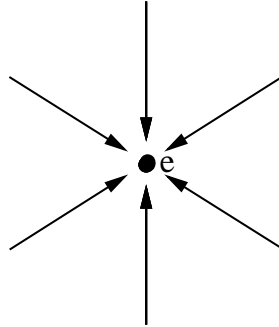
With every theory there is some scale in which we know the theory works. For example for all we know the SM works well until about the energies that we have gotten to so far. However when we do loop integrals we are told to integrate all the way to infinity (all possible virtual particle momenta are allowed). That means that we are extrapolating to energy scales further then we know that the theory works. With this point of view, there probably means that this extrapolation has to fail at some energy. This is analogous to blackbody radiation problem in quantum mechanics. Physicists tried to extrapolate their understanding of blackbody radiation to regimes where they didn't know it held.

The suspicion about these divergences in QFT is that they are there because we are missing some essential pieces of physics that occur at some shorter length scale that we are unable to probe. Putting in a cutoff scale, Λ , is just some crude model for what happens at some short length scale. Another way to model how a theory behaves at small length scales is to switch from field theory to string theory. Depending on the particle high energy theory you must use a different regulator. Different regulators are essentially different toy models for how to model the effects at higher energy scale. The freedom of choice of regulator is a representation of our ignorance of high energy physics.

In particular since we know that gravity cannot be explained by QFTs we believe that the highest Λ can be is at the Planck scale (though it is likely much earlier). Hence Λ is some scale that is of the Planck scale or less. Renormalizability tells us that the details of the high energy physics don't effect the effects at the low energy regime. By definition it means that the dependence of the cutoff can be absorbed into the CT's and won't effect physics quantities. The cutoff scale is taken in some sense to be a real physical energy scale. The interpretation of a formula such as 6.62 is an observer at a low energy will see $m \approx m_0$ however at a large energy there are going to be corrections.

6.3 Classical Renormalization and Regulators

Consider an electron. It has some electric field



Einstein told us that energy is equivalent to mass hence the electric field must change the mass of the particle. This correction is what we calculated using equation 6.62. Classically one can go ahead to calculate this same result.

$$\delta m = \int d^3r \frac{|\mathbf{E}|^2}{8\pi} \quad (6.63)$$

$$= \frac{4\pi}{8\pi} \int_0^\infty r^2 dr \left(\frac{e}{r^2}\right)^2 \quad (6.64)$$

$$\propto e^2 \int_0^\infty \frac{dr}{r^2} \quad (6.65)$$

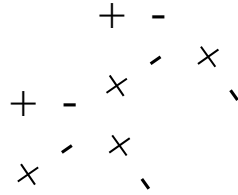
$$\rightarrow \infty \quad (6.66)$$

Classically this energy doesn't converge! The problem occurred since we extrapolated classical EM to a length scale in which it was no longer valid. Classical EM is a low energy effective theory. While it does a good job at describing the physics at low energies (large length scales) it fails at high energies. Alternatively one can introduce a length cutoff (analogous to the energy cutoff, Λ that we are used to in QFT):

$$\delta m = \propto e^2 \int_{r_0}^\infty \frac{dr}{r^2} \quad (6.67)$$

$$\propto \frac{e^2}{r_0} \quad (6.68)$$

This quantity is no longer infinite. One could ask why this length scale was required since to our current understanding the electron truly is a point charge. The problem is QM. In QFT the vacuum is not static, but electron positron pairs pop in and out of the vacuum. The vacuum of QED is filled with these pairs:



These pairs can appear as long as they disappear in a timescale as determined by the uncertainty principle. Normally we don't care about them, since they average to zero (equally likely to be oriented in all direction). However if you have a real electron then they are more likely to orient themselves along the electric field and screen the field. As you approach the electron of order the inverse electron mass these effects become important. The wavelength that such effects will occur is the Compton wavelength

$$\lambda = \frac{1}{m_e} \approx r_c \tag{6.69}$$

is known as the Compton wavelength. We identify r_0 with the Compton wavelength and we get

$$\delta m \propto e^2 m_e \tag{6.70}$$

This is starting to look very similar to our equation that we derived with QFT except with the Logarithmic divergence. Knowing the higher energy theory of QED told us the exact form of the correction. In essence our QED calculation is just an estimate that will be fixed when more physics is added. One can go ahead and renormalize classical EM if one wanted as well.

Consider our correction term in equation 6.62. $\alpha = \frac{1}{137}$, $m_0 \approx 511keV$, $\Lambda \leq M_{Planck} \sim 10^{19}GeV$. This makes the correction

$$\frac{3\alpha}{4\pi} \log \frac{\Lambda^2}{m_0^2} \approx 0.2 \tag{6.71}$$

So even with having this huge cutoff scale the correction is still not large. Note such an argument doesn't work using the ϕ^4 theory. In the end we don't really care since we renormalize our theories anyways.

When picking a regulator it's important to have it respect the symmetry of your problem, since its possible that the high energy theory does respect the theory (of course it is also possible that it won't break the symmetry but you don't want to remove the other possibility). As an example recall our derivation of the photon terms. We were not allowed to write down a mass term:

$$\frac{1}{2} m^2 A_\mu A^\mu \tag{6.72}$$

since we wanted gauge invariance. When finding the photon propagator:

$$\text{Diagram} \propto (g_{\mu\nu} q^2 - q^\mu q^\nu) \Pi(q^2) \tag{6.73}$$

we would in general renormalize the mass and create a mass. However experimentally we know the photon is massless to a high precision. We want to somehow ensure that the photon will remain massless at this high energy scale. We want to introduce a regulator that respects gauge invariance. The cutoff is not a gauge invariant cutoff (this is not obvious at this point). The preferred regulator in QED is through dimensional regularization or “dim reg”.

6.4 Dimensional Regularization

Using dim reg we now consider the integral we found earlier with a hard cutoff

$$\hat{\Sigma}(p) = e^2 \int \frac{d^d \ell}{(2\pi)^d} \frac{\gamma^\mu (\not{\ell} + m_0) \gamma_\mu}{(\ell^2 - m_0^2 + i\epsilon) ((\ell - p)^2 - m_0^2 + i\epsilon)} \quad (6.74)$$

Note that we have changed the dimension of each object in this expression from $4 \rightarrow d$. It's not clear what the γ matrices even mean in more than 4 dimensions. A discussion in Peskin and Schroeder tells us that $\gamma^\mu \gamma_\mu = d$, $\gamma^\mu \gamma^\nu \gamma_\nu = -(d-2)\gamma^\mu$. These expressions are analytic continuations of the expressions of integer d .

$$\hat{\Sigma}(p) = 2e^2 \int_0^1 dx \int \frac{d^d k_E}{d(2\pi)^d} \frac{-(\frac{d}{2} - 1) x \not{p} + \frac{d}{2} m_0}{(k_E^2 + \Delta)^2} \quad (6.75)$$

$$= \frac{2e^2}{(4\pi)^{d/2}} \Gamma(2 - d/2) \int_0^1 dx \left(\frac{d}{2} m_0 - \left(\frac{d}{2} - 1 \right) x \not{p} \right) \left(\frac{1}{\Delta} \right)^{2-d/2} \quad (6.76)$$

In 4 dimensions this expression is infinite because of the Γ term. However in $4 - \epsilon$ dimensions we have

$$\hat{\Sigma}(p) = \frac{e^2}{8\pi^2} \left(\int_0^1 dx (x \not{p} - 2m_0) \left(\frac{2}{\epsilon} - \gamma + \log 4\pi - \log \Delta \right) + \int_0^1 dx (x \not{p} - m_0) + \mathcal{O}(\epsilon) \right) \quad (6.77)$$

where γ is the Euler constant and equal to 0.566....

6.5 Vacuum Polarization

Consider

$$\tilde{G}_{\mu\nu}(q) = \text{diagram with a shaded blob} \quad (6.78)$$

$$= \text{diagram with a loop} + \text{diagram with two loops} + \dots \quad (6.79)$$

$$= -\frac{i}{q^2} \left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \right) + \left(-\frac{i}{q^2} \left(g_{\mu\alpha} - \frac{q_\mu q_\alpha}{q^2} \right) \right) (i\Pi_{\alpha\beta}) \left(-\frac{i}{q^2} \left(g_{\beta\nu} - \frac{q_\beta q_\nu}{q^2} \right) \right) + \dots \quad (6.80)$$

where we use the Landau gauge for the propagator. We define

$$\Pi = \text{○} + \text{○} \text{---} \text{○} + \dots \quad (6.81)$$

and

$$\hat{\Pi} = \sum \text{1PI graphs} \quad (6.82)$$

Note that this is completely analogous to Σ and $\hat{\Sigma}$ from earlier. Then

$$\begin{aligned} \tilde{G}_{\mu\nu} &= -\frac{i}{q^2} \left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \right) + \left(-\frac{i}{q^2} \left(g_{\mu\alpha} - \frac{q_\mu q_\alpha}{q^2} \right) \right) \left(i\hat{\Pi}^{\alpha\beta} \right) \left(-\frac{i}{q^2} \left(g_{\beta\nu} - \frac{q_\beta q_\nu}{q^2} \right) \right) \\ &+ (\dots)_{\mu\alpha} \left(i\hat{\Pi} \right)^{\alpha\beta} (\dots)_{\beta\sigma} \left(i\hat{\Pi} \right)^{\sigma\rho} (\dots)_{\rho\nu} + \dots \end{aligned} \quad (6.83)$$

The Ward identity says that (see equation 6.13)

$$\hat{\Pi}_{\alpha\beta} = q^2 \left(g_{\alpha\beta} - \frac{q_\alpha q_\beta}{q^2} \right) \Pi(q^2) \quad (6.84)$$

which implies

$$\left(g_{\mu\alpha} - \frac{q_\mu q_\alpha}{q^2} \right) \hat{\Pi}^{\alpha\beta} = q^2 \left(g_{\mu\alpha} - \frac{q_\mu q_\alpha}{q^2} \right) \left(g^{\alpha\beta} - \frac{q^\alpha q^\beta}{q^2} \right) \Pi(q^2) \quad (6.85)$$

$$= q^2 \left(g_\mu^\beta - \frac{1}{q^2} (q_\mu q^\beta + q_\mu q^\beta) + \frac{q_\mu q^\beta}{q^2} \right) \Pi(q^2) \quad (6.86)$$

$$= q^2 \left(g_\mu^\beta - \frac{q_\mu q^\beta}{q^2} \right) \Pi(q^2) \quad (6.87)$$

and hence

$$\begin{aligned} \tilde{G}_{\mu\nu} &= -\frac{i}{q^2} \left\{ \left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \right) + \left(g_\mu^\beta - \frac{q_\mu q^\beta}{q^2} \right) \left(g_{\beta\nu} - \frac{q_\beta q_\nu}{q^2} \right) ((-i)(i)\Pi(q^2)) \right. \\ &\quad \left. + (\dots)(\dots)(\dots)(-i\Pi(q^2))^2 + \dots \right\} \end{aligned} \quad (6.88)$$

We define $A_{\alpha\beta} \equiv g_{\alpha\beta} - \frac{q_\alpha q_\beta}{q^2}$. With this we can write as

$$\tilde{G}_{\mu\nu} = \frac{-i}{q^2} \left\{ A_{\mu\nu} + A_\mu^\beta A_{\beta\nu} \Pi(q^2) + A_{\mu\alpha} A^{\alpha\beta} A_{\beta\nu} (\Pi(q^2))^2 + \dots \right\} \quad (6.89)$$

but

$$\left(g_\mu^\beta - \frac{q_\mu q^\beta}{q^2} \right) \left(g_{\beta\nu} - \frac{q_\beta q_\nu}{q^2} \right) = g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \quad (6.90)$$

However this object doesn't obey $q^\mu \hat{\Pi}_{\mu\nu} = 0$. Hence this is not true at $\mathcal{O}(\Lambda^2)$! To solve this integral in a gauge invariant way we use dim reg. We can greatly simplify the amount of work that we need to do by identifying that the final answer must be in the form,

$$\hat{\Pi}^{\mu\nu} = (q^2 g^{\mu\nu} - q^\mu q^\nu) \Pi(q^2) \quad (6.100)$$

Thus we just need to calculate either the $q^\mu q^\nu$ or the $g^{\mu\nu}$ part and just infer the other part. As we will see it will be easier to calculate the $q^\mu q^\nu$ contribution. The denominator is:

$$\frac{1}{\ell^2 - m^2 + i\epsilon} \frac{1}{(\ell + q)^2 - m^2 + i\epsilon} = \int dx \frac{1}{[(\ell + qx)^2 - \Delta + i\epsilon]^2} \quad (6.101)$$

where

$$\Delta \equiv -q^2 x(1-x) + m^2 \quad (6.102)$$

We now shift the integration variable and consider the trace:

$$\text{Tr}[\dots] \rightarrow \text{Tr} [(-\not{q} + \not{\ell} + \not{q}x + m) \gamma_\nu (-\not{\ell} - \not{q}x + m) \gamma_\mu] \quad (6.103)$$

$$= -\text{Tr} [(\not{\ell} + \not{q}(x-1)) \gamma_\nu (\not{\ell} + \not{q}x) \gamma_\mu] + m^2 \text{Tr} [\gamma_\mu \gamma_\nu] \quad (6.104)$$

$$= -\text{Tr} [\not{\ell} \gamma_\nu \not{\ell} \gamma_\mu] - \text{Tr} [\not{q}(x-1) \gamma_\nu \not{q}x \gamma_\mu] + m^2 \text{Tr} [\gamma_\mu \gamma_\nu] + \text{linear in } \ell \quad (6.105)$$

$$= 4(2\ell^\mu \ell^\nu - \ell^2 g^{\mu\nu}) + 4x(1-x)(2q^\mu q^\nu - q^2 g^{\mu\nu}) \quad (6.106)$$

Δ only depends on q^2 so there is only one term that has a $q^\mu q^\nu$ contribution. We work out this term and infer the value of the rest of the terms. We need the integral,

$$\int \frac{d^4 \ell}{(2\pi)^4} \frac{1}{[\ell^2 - \Delta]^2} = \frac{i}{(4\pi)^2} (1 + \epsilon \log 4\pi) \left(\frac{2}{\epsilon} - \gamma \right) (1 - \epsilon \log \Delta) \quad (6.107)$$

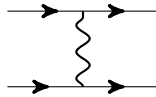
$$= \frac{i}{16\pi^2} \left(\frac{2}{\epsilon} - \log \frac{\Delta}{4\pi} - \gamma \right) \quad (6.108)$$

Thus

$$\hat{\Pi}_{\mu\nu} = -q^\mu q^\nu \left(-\frac{e^2}{4\pi^2} \int dx 2x(1-x) \left(\frac{2}{\epsilon} - \log \frac{\Delta}{4\pi} - \gamma \right) \right) + (\dots) g^{\mu\nu} \quad (6.109)$$

$$\Rightarrow \Pi(q^2) = -\frac{2\alpha}{\pi} \int dx (1-x) \left(\frac{2}{\epsilon} - \log \frac{\Delta}{4\pi} - \gamma \right) \quad (6.110)$$

Recall that earlier we showed that for an electron scattering off a muon in the non-relativistic limit given by



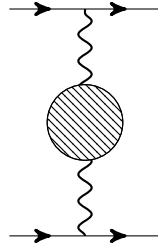
the amplitude went as $i\mathcal{M} \sim \frac{e^2}{q^2}$. The Born approximation tells us that

$$\langle \mathbf{p} | iT | \mathbf{p} \rangle \sim \tilde{V}(\mathbf{p} - \mathbf{p}') \quad (6.111)$$

We had

$$V = \frac{e^2}{r} \rightarrow \tilde{V} \sim \frac{e^2}{|\mathbf{q}|^2} \quad (6.112)$$

what we really need to do is



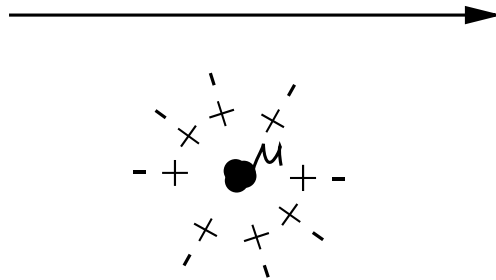
(we need to have the renormalized propagator for the photon). This results in $i\mathcal{M} \sim \frac{e_0^2 Z_3}{|\mathbf{q}|^2}$ using instead of the bare charge we have $e = e_0 \sqrt{Z_3}$. At higher energies we need to include the following correction as well:

$$i\mathcal{M} \sim \frac{e_0^2 Z_3}{|\mathbf{q}|^2} \left(\frac{1 - \hat{\Pi}(0)}{1 - \hat{\Pi}(-\mathbf{q}^2)} \right) \quad (6.113)$$

Going through the calculation one can show

$$V(r) = \frac{\alpha}{r} \left(1 + \frac{\alpha}{4\sqrt{\pi}} \frac{e^{-2mr}}{(mr)^{3/2}} + \dots \right) \quad (6.114)$$

The interaction gets stronger than the Coulomb expectation as the charges get close. This goes back to why this effect is called vacuum polarization. Imagine the muon:



Hence at low energy what you see is the screened potential. This is the effect of having $e^2 \rightarrow e_0^2 Z_3$. As you start throwing more energetic electrons toward the muon, then you get the electron very close to the muon. You can get rid of some of the screening effect and the stronger the field will be. Your interaction with the muon will not be determined by e^2 but more by something that's more close to the bare charge. Eventually when you are very close to the center of the muon you will be able to measure Z_3 . This is the concept behind the running coupling constant. This effect is observed and well tested.

6.6 Vertex Function

Consider the vertex function which represents the following diagram

$$\tilde{G}_\mu(p, p') = \text{Diagram: A shaded circle vertex with two incoming fermion lines (momenta } p \text{ and } p') \text{ and one outgoing photon line (momentum } q \equiv p' - p \text{).}$$

in the limit of $p^2 \rightarrow m_e^2, p'^2 \rightarrow m_e^2$ but q has to stay off shell by energy momentum conservation. Now recall that

$$\frac{i}{\not{p}' - m + i\epsilon} = \frac{i(\not{p}' + m)}{p'^2 - m^2 + i\epsilon} = \frac{i \sum_{s=1}^2 u^s(p') \bar{u}^s(p')}{p'^2 - m^2 + i\epsilon} \quad (6.115)$$

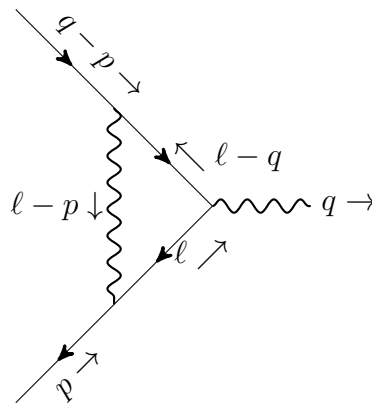
so we have a

$$\tilde{G}_\mu = \frac{\sum u^s(p') \bar{u}^s(p')}{p'^2 - m^2 + i\epsilon} \Gamma^\nu(p, p') \frac{\sum u^{s'}(p) \bar{u}^{s'}(p)}{p^2 - m^2 + i\epsilon} D_{\mu\nu}(q) \quad (6.116)$$

The interesting object is Γ^ν . To leading order we have just a single vertex:

$$\text{Diagram: A vertex with two incoming fermion lines and one outgoing photon line.} \rightarrow \Gamma^\nu = -ie\gamma^\nu$$

(in the terminology we will introduce shortly, $F_1 = 1, F_2 = 0$). Next to leading order we have the diagram:



$$\Gamma^\nu = (-ie)^3 \int \frac{d^d \ell}{(2\pi)^d} \gamma^\rho \frac{i(\not{\ell} + \not{q} + m)}{(\ell + q)^2 - m^2 + i\epsilon} \gamma^\nu \frac{i(\not{\ell} + m)}{\ell^2 - m^2 + i\epsilon} \gamma^\sigma \frac{(-ig_{\rho\sigma})}{(\ell - p)^2 + i\epsilon} \quad (6.117)$$

We won't solve this explicitly however we consider the Lorentz structure. The only nontrivial scalar in this problem is q^2 . From the fact it's a four vector we already know that we will have something of the form,

$$\Gamma^\nu = \gamma^\nu A(q^2) + (p + p')^\nu B(q^2) + (p - p')^\nu C(q^2) \quad (6.118)$$

In general B and C can be matrices as well however since $mu(p) = \not{p}u(p)$ we can write the expressions in terms of ordinary numbers. From the Ward identity we know that

$$q^\mu \tilde{G}_\mu = 0 \quad (6.119)$$

and so,

$$\bar{u}(p') q_\nu \Gamma^\nu u(p) = 0 \quad (6.120)$$

$$\bar{u}(p') \left(\underbrace{\not{q}}_{=\bar{u}(p)(\not{p}-\not{p}')u(p')=0} A(q^2) + \overbrace{q \cdot (p + p')}^{=p^2-p'^2=0} B(q^2) + \underbrace{q \cdot (p - p')}_{q^2 \neq 0} C(q^2) \right) u(p) = 0 \quad (6.121)$$

where we have used the fact that $q \equiv p' - p$ and hence we have $C = 0$. Now we use the Gordon identity:

$$\bar{u}(p') \gamma^\mu u(p) = \bar{u}(p') \left(\frac{(p' + p)^\mu}{2m} + \frac{i\sigma^{\mu\nu} q_\nu}{2m} \right) u(p) \quad (6.122)$$

where $\sigma^{\mu\nu} = \frac{i}{2} [\gamma^\mu, \gamma^\nu]$.

We quickly prove this identity below. Consider

$$-p^\mu + i\sigma^{\mu\nu} p_\nu = -p^\mu - \frac{1}{2} (\gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu) p_\nu \quad (6.123)$$

$$= -p^\mu - \frac{1}{2} (2\gamma^\mu \not{p} - 2g^{\mu\nu} p_\nu) \quad (6.124)$$

$$= -p^\mu - (\gamma^\mu \not{p} - p^\mu) \quad (6.125)$$

$$= -\gamma^\mu \not{p} \quad (6.126)$$

Furthermore,

$$p'^\mu + i\sigma^{\mu\nu} p'_\nu = p'^\mu + \frac{1}{2} p'_\nu (\gamma^\nu \gamma^\mu - \gamma^\mu \gamma^\nu) \quad (6.127)$$

$$= p'^\mu + \frac{1}{2} (2\not{p}' \gamma^\mu - 2p'^\mu) \quad (6.128)$$

$$= \not{p}' \gamma^\mu \quad (6.129)$$

Subtracting these two equations gives

$$(p' + p)^\mu + i\sigma^{\mu\nu} (p' - p)_\nu = \not{p}' \gamma^\mu + \gamma^\mu \not{p} \quad (6.130)$$

$$\bar{u}_{p'} \{ (p' + p)^\mu + i\sigma^{\mu\nu} (p' - p)_\nu \} u_p = \bar{u}_{p'} m \gamma^\mu u_p + \bar{u}_{p'} \gamma^\mu m u_p \quad (6.131)$$

$$\bar{u}(p') \gamma^\mu u(p) = -\bar{u}_{p'} \left(\frac{(p' + p)^\mu}{2m} + i\sigma^{\mu\nu} \frac{(p' - p)^\mu}{2m} \right) u_p \quad (6.132)$$

Using this identity we see that

$$\Gamma^\nu(p, p') = (-ie) \left[\gamma^\nu F_1(q^2) + \frac{i\sigma^{\mu\nu} q_\mu}{2m} F_2(q^2) \right] \quad (6.133)$$

where the $F_i(q^2)$ are known as form factors. To lowest order we have no loop and $F_1(q^2) = 1$ and $F_2(q^2) = 0$. To second order they are given by [Q 9: Show this]

$$F_1 = \frac{e^2}{(4\pi)^{d/2}} \int_0^1 dx dy dz \delta(x+y+z-1) \left\{ \frac{\Gamma(2-d/2)}{\Delta^{2-d/2}} \frac{(2-\epsilon)^2}{2} + \frac{\Gamma(3-d/2)}{\Delta^{3-d/2}} (q^2 [2(1-x)(1-y) - \epsilon xy] + m^2 [2(1-4z+z^2) - \epsilon(1-z)^2]) \right\} \quad (6.134)$$

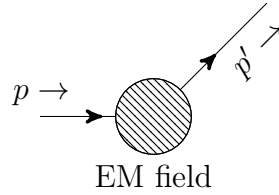
where $\Delta = (1+z)^2 m^2 - xyq^2 + z\mu^2$ and $\epsilon = 4-d$ as well as

$$F_2 = \frac{\alpha}{2\pi} \int_0^1 dx dy dz \delta(x+y+z-1) \frac{2m^2 z(1-z)}{\Delta} \Big|_{\mu^2 \rightarrow 0} \quad (6.135)$$

where $F_2(q^2 \rightarrow 0) = \frac{\alpha}{2\pi}$. Note that the UV divergences are completely hidden inside F_1 [Q 10: Is this true to all orders?]

6.6.1 Physics of F_2

Consider the semi classical picture of a fermion in an EM field



$$\mathcal{L}_{int} = e \bar{A}_\mu \bar{\psi} \gamma^\mu \psi \quad (6.136)$$

To leading order we have

$$\langle \mathbf{p}', s' | i\mathcal{T} | \mathbf{p}, s \rangle = \bar{u}^{s'}(p') (-ie\gamma^\mu) u^s(p) \tilde{A}_\mu(p-p') \quad (6.137)$$

and the Feynman rule is,

The Feynman rule diagram shows a fermion line with a wavy photon line attached, resulting in the vertex factor $-ie\gamma^\mu \tilde{A}_\mu(q)$.

In non relativistic electron scattering we have (for a scalar potential, $A_\mu = (\phi(\mathbf{x}), \mathbf{0}) \Rightarrow \tilde{A}_\mu = (\tilde{\phi}(\mathbf{q}), \mathbf{0})$). Recall that in the Weyl basis $u^s = \sqrt{2m} \begin{pmatrix} \xi^s \\ \xi^s \end{pmatrix}$

$$\langle \mathbf{p}', s' | iT | \mathbf{p}, s \rangle = -ie \bar{u}(p') \gamma^0 u(p) \tilde{\phi}(\mathbf{q}) = -ie(2m) \delta_{ss'} \tilde{\phi}(\mathbf{q}) \quad (6.138)$$

If we have a vector potential $A_\mu = (0, \mathbf{A})$ with $\mathbf{B} = \nabla \times \mathbf{A}$ then

$$\langle \mathbf{p}', s' | iT | \mathbf{p}, s \rangle = -ie \bar{u}^{s'}(p') \gamma^i u^s(p) \tilde{A}_i(\mathbf{q}) \quad (6.139)$$

$$= -i(2m) \frac{e}{m} (\xi^{s'\dagger} \sigma^i \xi^s) \tilde{B}_i(\mathbf{q}) \quad (6.140)$$

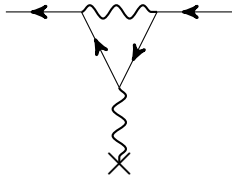
[Q 11: I don't see how we got from \mathbf{A} to \mathbf{B} .] (recall that $\gamma^i = \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix}$).

Now in the quantum mechanics Born approximation we have,

$$\langle \mathbf{p}', s' | iT | \mathbf{p}, s \rangle \sim \tilde{V}(\mathbf{q}) \quad (6.141)$$

This equation shows that the potential is given by $V = \frac{e}{m} \hat{\mathbf{S}} \cdot \mathbf{B} = \hat{\boldsymbol{\mu}} \cdot \mathbf{B}$. From non-relativistic QM we know that $\boldsymbol{\mu} = g \frac{e}{2m} \mathbf{S}$, where g is the Landé g factor. Hence to leading order it was found that $g = 2$. See Peskin and Schroeder pg 187-188 for more details.

Now consider the next to leading order expression



$$= -ie \bar{u}(p') \left(\gamma^\mu F_1(0) + \frac{i\sigma^{\mu\nu} q_\nu}{2m} F_2(0) \right) u(p) \tilde{A}_\mu(q)$$

The scalar field is $V = eF_1(0) \cdot \phi(\mathbf{x})$. With the vector field one can show that you get the same structure we found before, $V \propto \mathbf{S} \cdot \mathbf{B}$ with a different g factor

$$g = 2 + 2F_2(0) = 2 + \frac{\alpha}{\pi} \quad (6.142)$$

This was the first radiative correction that was predicted and confirmed by experiment.

Chapter 7

Renormalized Perturbation Theory

7.1 ϕ^4 Theory

Recall ϕ^4 theory:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m_0^2\phi^2 - \frac{1}{4!}\lambda_0\phi^4 \quad (7.1)$$

where

$$\tilde{G}(p) = \frac{iZ}{p^2 - m^2 + i\epsilon} + (\text{no poles}) \quad (7.2)$$

Define new renormalized fields as $\tilde{\phi}_r = Z^{-1/2}\phi$ which implies that $\tilde{G}_r(p) = \frac{i}{p^2 - m^2 + i\epsilon} + \dots$ and we have

$$\mathcal{L} = \frac{Z}{2}(\partial_\mu\phi_r)^2 - \frac{Z}{2}m_0^2\phi_r^2 - \frac{Z^2}{4!}\lambda_0\phi_r^4 \quad (7.3)$$

Experimentally we want to measure the mass, m and the coupling, λ . λ can be measured from $\phi\phi \rightarrow \phi\phi$ at $\mathbf{p} \rightarrow 0$. More precisely we can measure

$$i\mathcal{M}_4(s = 4m^2, u = t = 0) = \lambda \quad (7.4)$$

Notice that when we do this measurement we don't know about what's a loop diagram, what's tree level. This term includes all diagrams. We want to rewrite our Lagrangian in terms of these physical parameters:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi_r)^2 - \frac{1}{2}m^2\phi_r^2 - \frac{1}{4!}\lambda\phi_r^4 + \frac{1}{2}\delta_z(\partial_\mu\phi_r)^2 - \frac{1}{2}\delta_m\phi_r^2 - \frac{1}{4!}\delta_\lambda\phi_r^4 \quad (7.5)$$

where $\delta_z = Z - 1$, $\delta_m = Zm_0^2 - m^2$, $\delta_\lambda = Z^2\lambda_0 - \lambda$. Our new Feynman rules are

$$\begin{aligned}
\text{---} &= \frac{i}{p^2 - m^2 + i\epsilon} \\
\text{---} \times \text{---} &= -i\lambda \\
\text{---} \otimes \text{---} &= i(p^2\delta_z - \delta m) \\
\text{---} \times \otimes \text{---} &= -i\delta_\lambda
\end{aligned}$$

we impose our renormalization conditions:

$$\left(\tilde{G}_r^2(p^2)\right)^{-1} \Big|_{p^2=m^2} = 0 \quad (7.6)$$

$$\frac{d}{dp^2} \left(\tilde{G}_r^2(p^2)\right)^{-1} \Big|_{p^2=m^2} = -i \quad (7.7)$$

At first order these conditions are trivially satisfied. At higher orders these conditions give us our values of δ_z, δ_m .

7.2 QED

The QED Lagrangian in terms of the renormalized parameters is,

$$\begin{aligned}
\mathcal{L} = & \bar{\psi}_r(i\cancel{\partial} - m)\psi_r - e\bar{\psi}_r\gamma^\mu\psi_r A_{r,\mu} - \frac{1}{4}(F_{r,\mu\nu})^2 \\
& + \bar{\psi}_r(i\delta_2\cancel{\partial} - \delta_m)\psi_r - e\delta_1\bar{\psi}_1\gamma^\mu\psi_r A_{r,\mu} - \frac{1}{4}\delta_3(F_{r,\mu\nu})^2
\end{aligned} \quad (7.8)$$

The original theory was gauge invariant so we better maintain that gauge invariance. The counter-terms must also be gauge invariant. The only way that we happen is if $\delta_1 = \delta_2$. To see this consider part of QED Lagrangian (the rest is trivially gauge invariant on its own),

$$\Delta\mathcal{L} = \delta_2\bar{\psi}i\cancel{\partial}\psi + ie\delta_1\bar{\psi}_1\gamma^\mu\psi A_\mu \quad (7.9)$$

Now consider a gauge shift $A_\mu \rightarrow A_\mu - \frac{1}{e}\partial_\mu\alpha(x)$, $\psi \rightarrow e^{-i\alpha}$:

$$\delta\mathcal{L} = -\delta_2\bar{\psi}i\gamma^\mu i\partial_\mu\alpha\psi - i\delta_1\bar{\psi}_1\gamma^\mu\psi\partial_\mu\alpha \quad (7.10)$$

$$= (\delta_2 - \delta_1)\bar{\psi}\gamma^\mu\psi\partial_\mu\alpha \quad (7.11)$$

In order for this to be gauge invariant we require $\delta_1 = \delta_2$. This highly non-trivial result implies that the renormalization of the QED coupling is equal to the renormalization of the fermion wavefunction renormalization! This implies that $Z_1 = Z_2$ and the renormalization couplings is simplify given by (see Eq. 6.6),

$$e = e_0 \sqrt{Z_3} \quad (7.12)$$

We have the Feynman rules

$$\begin{aligned} \longrightarrow & \longrightarrow & \longrightarrow & \frac{i}{\not{p}-m+i\epsilon} \\ \text{~~~~~} & \text{~~~~~} & \longrightarrow & \frac{-ig_{\mu\nu}}{q^2+i\epsilon} \\ \longrightarrow \otimes \longrightarrow & \longrightarrow & \longrightarrow & i(\not{p}\delta_2 - \delta_m) \\ \text{~~~~~} \otimes \text{~~~~~} & \text{~~~~~} & \longrightarrow & -i(g^{\mu\nu}q^2 - q^\mu q^\nu)\delta_3 \\ \begin{array}{l} \nearrow \\ \searrow \end{array} & \begin{array}{l} \nearrow \\ \searrow \end{array} & \begin{array}{l} \nearrow \\ \searrow \end{array} & -ie\gamma^\mu \end{aligned}$$

The renormalization conditions are

$$\begin{aligned} \left(\begin{array}{c} \longrightarrow \\ \bullet \\ \longrightarrow \end{array} \right)^{-1} \Big|_{\not{p}=m} &= 0 \\ \frac{d}{d\not{p}} \left(\begin{array}{c} \longrightarrow \\ \bullet \\ \longrightarrow \end{array} \right)^{-1} \Big|_{\not{p}=m} &= -i \\ \frac{d}{dq^2} \left(\begin{array}{c} \text{~~~~~} \\ \bullet \\ \text{~~~~~} \end{array} \right)^{-1} \Big|_{q^2=0} &= -ig_{\mu\nu} \\ \begin{array}{l} \nearrow \\ \searrow \end{array} \bullet \begin{array}{l} \nearrow \\ \searrow \end{array} &= -ie\gamma_\mu \end{aligned}$$

The first two conditions provide us with δ_z, δ_m , the third conditions gives δ_3 and the last condition gives δ_1 .

We now consider the two point function at 1-loop. We have already derived this expression (the amputated version):

$$\text{~~~~~} \bigcirc \text{~~~~~} = i (g^{\mu\nu}q^2 - q^\mu q^\nu) \Pi(q^2) \quad (7.13)$$

We have

$$\text{wavy line with shaded circle} = \text{wavy line} + \text{wavy line with loop} + \text{wavy line with cross} + \dots = -\frac{ig_{\mu\nu}}{q^2 + i\epsilon} \frac{1}{1 - \hat{\Pi}(q^2)} \quad (7.14)$$

where $\hat{\Pi} = \sum 1\text{PI diagrams}$. Our condition is that $\hat{\Pi}(q^2 = 0) = 0$. We have (the amputated version)

$$\text{wavy line with loop} + \text{wavy line with cross} = i(g^{\mu\nu}q^2 - q^\mu q^\nu) (\Pi_{loop}(q^2) + \delta_3) \quad (7.15)$$

and hence we know that

$$\delta_3 = -\Pi_{loop}(q^2 = 0) \quad (7.16)$$

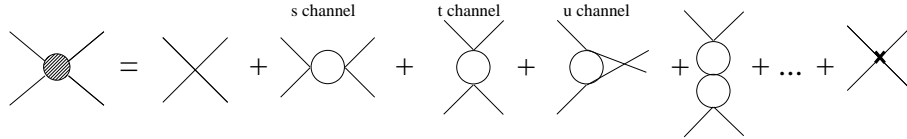
(calculated last lecture)

Chapter 8

Running Couplings and Renormalization Group

8.1 “Large Logs”

Consider $\lambda\phi^4$ theory. The phenomena happens in almost all QFT’s however it is convenient and easier to work with ϕ^4 theory. We consider $2 \rightarrow 2$ scattering. We will work at a COM energy of $\sqrt{s} = \mu \gg m$. We will basically be computing Green’s function like this:



The first diagram is trivial and given by λ . Based on dimensional analysis we know that the second diagram is

$$\Gamma = c \frac{\lambda^2}{16\pi^2} \left(\log \left(\frac{\Lambda}{\sqrt{s}} \right) + \text{non-log } \Lambda\text{-ind terms} \right) + \delta_\lambda \quad (8.1)$$

δ_λ is given when $\Gamma = 0$. When $s = 4m^2 \Rightarrow \delta_\lambda = -c \frac{\lambda^2}{16\pi^2} (\log \frac{\Lambda}{m} + \dots)$. Hence Γ is given by

$$\Gamma = c \frac{\lambda^2}{16\pi^2} \left(\log \left(\frac{\Lambda}{\sqrt{s}} - \log \frac{\Lambda}{m} \right) + \text{non-log } \Lambda\text{-ind terms} \right) \quad (8.2)$$

$$= c \frac{\lambda^2}{16\pi^2} \left(\log \left(\frac{m}{\sqrt{s}} \right) + \text{non-log } \Lambda\text{-ind terms} \right) \quad (8.3)$$

$$(8.4)$$

Going to higher order terms we have higher order powers of these logs:

$$\Gamma = c \frac{\lambda^2}{16\pi^2} \left(\log \frac{m}{\sqrt{s}} + \text{non-log } \Lambda\text{-ind terms} \right) + c' \frac{\lambda^3}{(16\pi^2)^2} \left(\log \frac{m}{\sqrt{s}} \right)^2 + \dots \quad (8.5)$$

Our perturbation theory is not a series in λ but it's a series in $\lambda \log \frac{\sqrt{s}}{m}$. This is an issue since you lose control of your calculation at sufficiently high energies. This issue can be avoided by organizing your perturbation theory in a better way.

We define

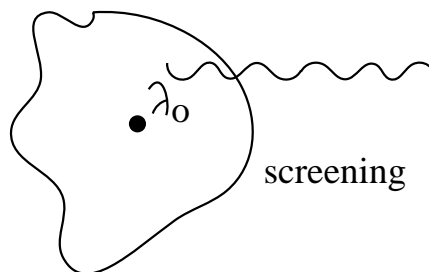
$$\tilde{\lambda} \equiv i \left[\text{diagram of a shaded circle with four external lines} \right] \Big|_{\sqrt{s}=\mu}$$

We impose a different renormalization condition given by $\delta_\lambda : \Gamma = 0$ when $s = \mu^2 \Rightarrow \delta_\lambda = -c \frac{\lambda^2}{16\pi^2} \left(\log \frac{\Lambda}{\mu} \right)$ and we have

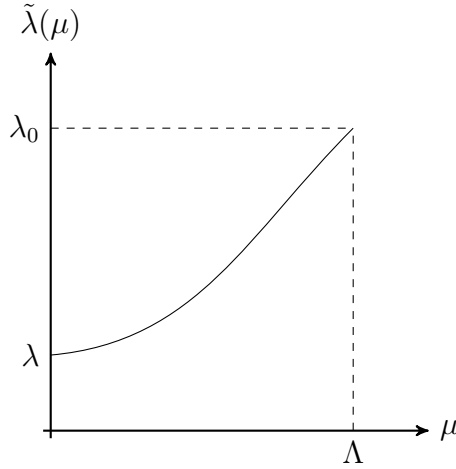
$$\Gamma = c \frac{\lambda^2}{16\pi^2} \left(\log \frac{\mu}{\sqrt{s}} + \text{non-log } \Lambda\text{-ind terms} \right) \tag{8.6}$$

Instead of on shell renormalization condition you can impose your renormalization condition at any scale you want. In the end your results do not depend on the scale that you renormalize however that's only true for the sum of the series. Since we always truncate our series, the error made by that truncation depends on the scale that you chose for your renormalization scheme. Our $\tilde{\lambda}(\mu)$ is called the running coupling.

Type of PT	Coupling	Notes
Bare PT	$\lambda_0 \rightarrow$ “bare coupling”	doesn't include screening (scattering at $\sqrt{s} \approx \Lambda$)
Renormalized PT (on-shell)	$\lambda \rightarrow$ “physical coupling”	includes screening (scattering at $\sqrt{s} \approx 2m$)
Off-shell Renormalized PT	$\tilde{\lambda}(\mu) \rightarrow$ “running coupling”	partial screening $2m < \sqrt{s} < \Lambda$



The “Renormalization Group evolution curve” is shown below



Now consider the Lagrangian:

$$\mathcal{L} = \frac{1}{2} (\partial_\mu \phi)^2 - \frac{1}{2} \tilde{m}(\mu)^2 \phi^2 - \frac{1}{4!} \tilde{\lambda}(\mu) \phi^4 + \frac{1}{2} \delta_Z (\partial_\mu \phi)^2 - \frac{1}{2} \delta_m \phi^2 - \frac{1}{4!} \delta_\lambda \phi^4 \quad (8.7)$$

where as before we have

$$\delta_Z = Z - 1, \delta_m = Z m_0^2 - \tilde{m}^2, \delta_\lambda = Z^2 \lambda_0 - \tilde{\lambda}$$

Before we had a certain scale and that was m . now our scale is μ instead. Before we go ahead and start calculating. We try to think what our δ 's will depend on :

$$\delta_i = \delta_i(\tilde{m}, \tilde{\lambda}, \Lambda, \mu) \quad (8.8)$$

We can invert our formulas for δ_i to get

$$\tilde{\lambda} = \tilde{\lambda}(m_0, \lambda_0, \mu, \Lambda) \quad (8.9)$$

$$\tilde{m} = \tilde{m}(m_0, \lambda_0, \mu, \Lambda) \quad (8.10)$$

$$\tilde{Z} = Z(m_0, \lambda_0, \mu, \Lambda) \quad (8.11)$$

We fix m_0, λ_0, Λ (UV physics). We define the Beta functions as

$$\beta \equiv \mu \frac{\partial \tilde{\lambda}}{\partial \mu} \Big|_{m_0, \lambda_0, \Lambda \text{ fixed}}, \quad \beta_m \equiv \mu \frac{\partial \tilde{m}}{\partial \mu}, \quad \gamma \equiv \frac{1}{2Z} \mu \frac{\partial Z}{\partial \mu} \Big|_{m_0, \lambda_0, \Lambda \text{ fixed}}$$

where for historical reasons the last expression has a different name (the γ function). Note that if you use a different regulator you would keep that fixed instead of Λ .

To 1 loop in ϕ^4 theory one can show that (assuming $\mu \gg m$)

$$\beta(\tilde{\lambda}) \equiv \mu \frac{\partial \tilde{\lambda}}{d \log \mu} = \frac{3\tilde{\lambda}^2}{16\pi^2} \quad (8.12)$$

This is called the RG equation (RGE). $\tilde{\lambda}(\Lambda) \equiv \lambda_0 \Rightarrow$

$$\int_{\tilde{\lambda}(\mu)}^{\lambda_0} \frac{d\tilde{\lambda}}{\tilde{\lambda}^2} = \frac{3}{16\pi^2} \int_{\mu=\mu}^{\Lambda} d\log \mu \quad (8.13)$$

$$\frac{1}{\tilde{\lambda}(\mu)} - \frac{1}{\lambda_0} = \frac{3}{16\pi^2} \log \frac{\Lambda}{\mu} \quad (8.14)$$

$$\frac{1}{\tilde{\lambda}(\mu)} = \frac{1 + \frac{3\lambda_0}{16\pi^2} \log \frac{\Lambda}{\mu}}{\lambda_0} \quad (8.15)$$

$$\tilde{\lambda}(\mu) = \frac{\lambda_0}{1 + \frac{3\lambda_0}{16\pi^2} \log \frac{\Lambda}{\mu}} \quad (8.16)$$

$$(8.17)$$

This is the plot we showed earlier.

We don't need to use λ_0 we could use the non-relativistic coupling to estimate the coupling at a different scale: $\tilde{\lambda}(\mu) = \lambda_{NR} \Rightarrow \tilde{\lambda}(\mu) = \frac{\lambda_{NR}}{1 + \frac{3\lambda_{NR}}{16\pi^2} \log \frac{\mu}{\mu_0}}$. The large log will no longer be in the PT. Note as $\mu \rightarrow 0$, $\lambda \rightarrow 0$. Hence at arbitrarily low energies the particle is free. These kinds of theories are called *IR-free* theories. It turns out that in practice the running coupling does not run to zero. Instead when the coupling gets close to m , it begins to level off. We will not show this. This turns out to be due to particles having a nonzero mass.

Note further that the coupling blows up at

$$\log \frac{\Lambda}{\mu} = -\frac{16\pi^2}{3\lambda_0} \quad (8.18)$$

$$\Lambda_L = \mu \exp\left(-\frac{16\pi^2}{3\lambda(\lambda_0)e}\right) \quad (8.19)$$

This is called the “Landau pole”. The coupling becomes large at some scale. At this point perturbation theory no longer makes sense. The theory is strongly coupled.

We now develop the formalism required to calculate these β functions. Consider the following

$$\langle \Omega | \mathcal{T}(\phi_r(x_1) \dots \phi_r(x_n)) | \Omega \rangle = G_r^{(n)}(x_1, \dots, x_n; \tilde{\lambda}, \tilde{m}, \overbrace{\Lambda, \mu}^{\text{via CT}}) \quad (8.20)$$

we also have

$$\langle \Omega | \mathcal{T}(\phi(x_1) \dots \phi(x_n)) | \Omega \rangle = G_b^{(n)}(x_1, \dots, x_n; \lambda_0, m_0, \Lambda) \quad (8.21)$$

$\phi_r = Z^{-1/2} \phi$, $G_b^{(n)} = Z^{n/2} G_r^{(n)}$. But $G_b^{(n)}$ is clearly independent of μ (the unrenormalized

propagator doesn't depend on what renormalization scale you use). Hence we know that

$$\mu \frac{d}{d\mu} G_b^{(n)} = 0 \quad (8.22)$$

$$\mu \frac{d}{d\mu} (Z^{n/2} G_r^{(n)}) = 0 \quad (8.23)$$

$$\mu \left(Z^{n/2} \frac{\partial \tilde{\lambda}}{\partial \mu} \frac{\partial}{\partial \tilde{\lambda}} + Z^{n/2} \frac{\partial \tilde{m}}{\partial \mu} \frac{\partial}{\partial \tilde{m}} + Z^{n/2} \frac{\partial}{\partial \mu} + \frac{n}{2Z} Z^{n/2} \frac{\partial Z}{\partial \mu} \right) G_r^{(n)} = 0 \quad (8.24)$$

$$\left(\beta_\lambda \frac{\partial}{\partial \tilde{\lambda}} + \beta_m \frac{\partial}{\partial \tilde{m}} + \frac{\partial}{\partial \mu} + n\gamma \right) G_r^{(n)} = 0 \quad (8.25)$$

$$(8.26)$$

where in the last step we used the definitions of β and γ functions as well as divided by $Z^{n/2}$.

This is called the Callan-Symanzik equation. It applies very generally and not just to $\lambda\phi^4$. One can show that to one loop one doesn't need to worry about the β_m . Conceptually it is the same as the other functions however it is more convenient to ignore this term. In other words we take $\tilde{m} \rightarrow 0$.

We consider 1-loop calculations: $G_r^{(2)}$ and $G_r^{(4)}$.

$$G_r^{(2)} = \overset{p \rightarrow}{\text{---}} + \underbrace{\overset{p \rightarrow}{\text{---}} \bigcirc \overset{p \rightarrow}{\text{---}}}_{0, \text{ by construction}} + \overset{p \rightarrow}{\text{---}} \overset{\delta_Z}{\text{---}} \overset{p \rightarrow}{\text{---}} + \mathcal{O}(\lambda^2) \quad (8.27)$$

Our one loop diagram gives the CT. To first order we have,

$$\Pi(p^2) = \lambda \int \frac{d^4 \ell}{(2\pi)^4} \frac{1}{\ell^2 + i\epsilon} = \lambda N \Lambda^2 \quad (8.28)$$

where N is a number. Note that this integral is independent of p . Our renormalization condition is $\Pi(\mu^2) + \delta_Z = 0$, which implies that $\delta_Z = -\Pi(\mu^2)$. To first order we have

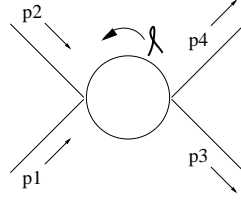
$$G_r^{(2)} = \frac{i}{p^2 + i\epsilon} \quad (8.29)$$

which is independent of $\tilde{\lambda}$, \tilde{m} , and μ . Thus the Callan-Symanzik equation says that in $\lambda\phi^4$ the Gamma function is zero to second order. i.e. $\gamma = 0 + \mathcal{O}(\lambda^2)$. Recall that our four point function was given by

$$\tilde{G}_r^{(4)} = \text{---} \bigcirc \text{---} = \text{---} \times \text{---} + \overset{\text{s channel}}{\text{---} \bigcirc \text{---}} + \overset{\text{t channel}}{\text{---} \bigcirc \text{---}} + \overset{\text{u channel}}{\text{---} \bigcirc \text{---}} + \text{---} \bigcirc \text{---} + \dots + \text{---} \times \text{---} \quad (8.30)$$

$$= \left(\prod_{i=1}^4 \frac{i}{p_i^2 + i\epsilon} \right) \left(-i\tilde{\lambda} + (-i\tilde{\lambda})^2 (V(s) + V(t) + V(u)) - i\delta_Z \right) + \mathcal{O}(\tilde{\lambda}^3) \quad (8.31)$$

Consider the second loop:



Its contribution to the vertex is

$$V(s) = \frac{1}{2} \int \frac{d^d \ell}{(2\pi)^d} \frac{1}{\ell^2} \frac{1}{(\ell + p)^2} \Big|_{p=p_1+p_2} \quad (8.32)$$

$$= \frac{1}{2} \int \frac{d^d \ell}{(2\pi)^d} \int dx \frac{1}{(x(\ell^2 + 2p\ell + p^2) + (1-x)\ell^2)^2} \quad (8.33)$$

$$= \frac{1}{2} \int \frac{d^d \ell}{(2\pi)^d} \int dx \frac{1}{(\ell^2 + 2\ell p x + p^2 x^2 + \Delta)^2} \quad (8.34)$$

where $\Delta \equiv -p^2 x^2 + p^2 x$.

$$= \frac{1}{2} \int \frac{d^d \ell}{(2\pi)^d} \int dx \frac{1}{((\ell + px)^2 + \Delta)^2} \quad (8.35)$$

$$= \frac{1}{2} \int dx \int \frac{d^d \ell}{(2\pi)^d} \frac{1}{(\ell^2 + \Delta)^2} \quad (8.36)$$

Using the known integral we have

$$= \frac{1}{2} \int_0^1 dx i \frac{\Gamma(2 - d/2)}{(4\pi)^{d/2} \Gamma(2)} (-\Delta)^{d/2-2} \quad (8.37)$$

We now take $d = 4 - \epsilon$:

$$= \frac{1}{2} i \int dx \frac{\Gamma(\epsilon)}{(4\pi)^2 \Gamma(2)} (-\Delta)^{-\epsilon} \quad (8.38)$$

$$= \frac{i}{32\pi^2} \int dx \left(\frac{1}{\epsilon} - \gamma_E + \frac{1}{12} (6\gamma_E^2 + \pi^2) \epsilon \right) (1 - \epsilon \log(p^2(x^2 - x))) \quad (8.39)$$

$$= \frac{i}{32\pi^2} \int dx \left(\frac{1}{\epsilon} - \log(p^2(x^2 - x)) - \gamma_E \right) \quad (8.40)$$

$$= \frac{i}{32\pi^2} \left(\frac{1}{\epsilon} + 2 - \log(p^2) - \gamma_E \right) \quad (8.41)$$

or using the Mandelstam variable $s \equiv p_1 + p_2 = p$ we can finally write

$$i\mathcal{M} = \frac{i}{32\pi^2} \left(\frac{1}{\epsilon} - \log s + 2 - \gamma_E \right) \quad (8.42)$$

or (the two constant are equivalent...)

$$= -\frac{i}{32\pi^2} \left(\frac{1}{\epsilon} - \log s + \gamma_E \log 4\pi \right) \quad (8.43)$$

We set the renormalization condition given by $s = \mu^2 \Rightarrow (\cdot: \text{we have massless particles in the CM frame})$ $t = \frac{\mu^2}{2}(1 - \cos \theta)$, $u = \frac{\mu^2}{2}(1 + \cos \theta)$. Our renormalization condition gives us (using crossing symmetry)

$$\delta_Z = -i\tilde{\lambda}^2 (V(s) + V(t) + V(u)) \quad (8.44)$$

$$= \frac{\tilde{\lambda}^2}{32\pi^2} \left(\frac{3}{\epsilon} - 3 \log \mu^2 + 3\gamma_E \log 4\pi \right) \quad (8.45)$$

We have

$$\left(\mu \frac{\partial}{\partial \mu} + \beta \frac{\partial}{\partial \tilde{\lambda}} + n\gamma \right) \tilde{G}_r^{(4)} = 0 \quad (8.46)$$

At order $\tilde{\lambda}$ we have $0 = 0$. At $\mathcal{O}(\tilde{\lambda}^2)$ we have μ dependence:

$$\mu \frac{\partial}{\partial \mu} \tilde{G}_r^{(4)} = \mu \frac{\partial}{\partial \mu} (-i\delta_\lambda) = \frac{3i\tilde{\lambda}^2}{16\pi^2} \quad (8.47)$$

and we have

$$\beta \frac{\partial \tilde{G}_r^{(4)}}{\partial \tilde{\lambda}} = -i\beta + \mathcal{O}(\tilde{\lambda}) \quad (8.48)$$

To order $\tilde{\lambda}^2$ the gamma term still doesn't contribute since $\mathcal{O}(\gamma) = \tilde{\lambda}$, $\mathcal{O}(G_r) = \tilde{\lambda}^2$. Using our equation relating the β functions we have

$$\beta = \frac{3\tilde{\lambda}^2}{16\pi^2} \quad (8.49)$$

8.2 QED β -function

We begin by considering the two photon Green's function

$$G_{\mu\nu}^{(2\gamma)}(p) = \int d^4x e^{-ix \cdot p} \langle \Omega | T(A_\mu(x) A_\nu(0)) | \Omega \rangle \quad (8.50)$$

$$= \text{wavy} + \text{wavy} \overset{\curvearrowright}{\text{---}} \text{wavy} + \text{wavy} \times \text{wavy} + \mathcal{O}(e^4) \quad (8.51)$$

$$= i \left(g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right) \frac{1}{p^2 + i\epsilon} \left(1 + e^2 f(p, \epsilon) \overset{\text{convention}}{\text{---}} \delta_3(e, \epsilon, \mu) \right) \quad (8.52)$$

Here we also set $m_e = 0$ (or equivalently $\mu \gg m_e$). Thus we have $\beta_m = 0$. The Callan-Symanzik (CS) equation is given by (this is exactly the same as before with renaming the variables)

$$\left[\mu \frac{\partial}{\partial \mu} + \beta \frac{\partial}{\partial e} + 2\gamma_A \right] G_{\mu\nu}^{(2\gamma)} = 0 \quad (8.53)$$

We need to compute δ_3 in order to know our μ dependence.

$$\text{wavy line with blob} \stackrel{q \rightarrow}{=} \frac{i \left(g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right)}{q^2} \frac{1}{1 - \hat{\Pi}(q^2)} \quad (8.54)$$

Our on-shell renormalization condition was $\hat{\Pi}(q^2 = 0) = 0$. We now abandon this condition and have the condition $\hat{\Pi}(q^2 = \mu^2) = 0$. At one loop we have

$$\text{loop diagram} + \times = \Pi_{loop}(q^2) - \delta_3 \quad (8.55)$$

and hence we have $\delta_3 = \Pi_{loop}(\mu^2)$. We found earlier that

$$\Pi_{loop}(q^2) = \frac{-8e^2}{(4\pi)^{d/2}} \underbrace{\int_0^1 dx (1-x)x}_{1/6} \frac{\Gamma(2-d/2)}{\Delta^{2-d/2}} \quad (8.56)$$

where $\Delta = \overbrace{m_e^2}^0 - x(1-x)q^2$. Expanding the denominator we have

$$\Pi_{loop}(q^2) = -\frac{4}{3} \frac{e^2}{16\pi^2} \left(\frac{1}{\epsilon} - \log \mu^2 + \left(\begin{array}{c} \mu\text{-independent} \\ \text{constants} \end{array} \right) \right) \quad (8.57)$$

where we have used

$$\Delta^\epsilon = e^{\log \Delta^\epsilon} = e^{\epsilon \log \Delta} = 1 + \epsilon \log \Delta + \dots \quad (8.58)$$

Now $\beta = \mathcal{O}(e^3)$ (we'll see why soon). If we are consistent to $\mathcal{O}(e^3)$ we have the CS equation gives:

$$\mu \frac{\partial}{\partial \mu} (-\mathcal{P}_{\mu\nu} \delta_3) + 2\gamma_A \mathcal{P}_{\mu\nu} = 0 \quad (8.59)$$

where we have defined $\mathcal{P}_{\mu\nu} = i \left(g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right) \frac{1}{p^2 + i\epsilon}$. Hence we have

$$\gamma_A = \frac{1}{2} \frac{\partial}{\partial \mu} \delta_3 = \frac{e^2}{12\pi^2} \quad (8.60)$$

Hence

$$\langle \Omega | \bar{\psi}(x) \psi(0) | \Omega \rangle \Rightarrow \gamma_\psi = \frac{1}{2} \mu \frac{\partial}{\partial \mu} \delta_2 = \frac{e^2}{16\pi^2} \quad (8.61)$$

This function is somewhat abstract and doesn't have a physical meaning. However we now go on to find the β function which is physical:

$$\langle \Omega | T(\bar{\psi}(x) \psi(y) A_\mu(0)) | \Omega \rangle = \text{diagrams} \quad (8.62)$$

The CS equation is given by

$$\left[\mu \frac{\partial}{\partial \mu} + \beta \frac{\partial}{\partial e} + \gamma_A + 2\gamma_\psi \right] G^{(3)} = 0 \quad (8.63)$$

To lowest order we have

$$G^{(3)} = \text{diagram} = \frac{i}{\not{p}'} (-ie\gamma^\nu) \frac{i}{\not{p}} D_{\mu\nu}(q) \equiv (-ie)F_\mu(p, p') \quad (8.64)$$

The counterterms are

$$\text{diagram 1} + \text{diagram 2} + \text{diagram 3} + \text{diagram 4} = -ieF_\mu + \frac{1}{\not{p}}(i\cancel{p}\delta_2)F_\mu + \dots \quad (8.65)$$

$$= -ie(\delta_1 - 2\delta_2 - \delta_3)F_\mu \quad (8.66)$$

so we have

$$0 = (-ie)\mu \frac{\partial}{\partial \mu} (\delta_1 - 2\delta_2 - \delta_3)F_\mu + \beta(-i)F_\mu + (2\gamma_\psi + \gamma_A)(-ieF_\mu) \quad (8.67)$$

$$\beta = e \left[-\mu \frac{\partial}{\partial \mu} (\delta_1 - 2\delta_2 - \delta_3) - 2\gamma_\psi - \gamma_A \right] \quad (8.68)$$

$$\beta = e\mu \frac{\partial}{\partial \mu} \underbrace{(-\delta_1 + \delta_2 + \frac{1}{2}\delta_3)}_{=0} \quad (8.69)$$

where $\delta_1 = \delta_2$ is zero by gauge invariance.

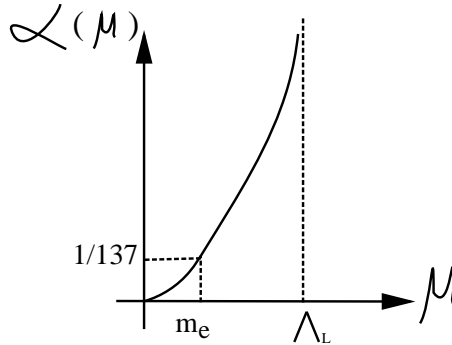
Thus we have

$$\beta = \frac{e\mu}{2} \delta \frac{\partial}{\partial \mu} \delta_3 = \frac{e^3}{12\pi^2} \quad (8.70)$$

Recall that

$$\alpha(\mu) = \frac{\alpha(\mu_0)}{1 - \frac{2\alpha(\mu_0)}{3\pi} \log \frac{\mu}{\mu_0}} \quad (8.71)$$

The coupling goes as



where $\Lambda_L = m_e \exp\left(\frac{3\pi}{2\alpha_{NR}}\right) \gg m_{Planck}$

Chapter 9

Non-Abelian Gauge Theories (“Yang-Mills”)

9.1 Looking Closely at Abelian Gauge Theories

QED is an “Abelian Gauge Theory”. You start with a Dirac Lagrangian:

$$\mathcal{L} = i\bar{\psi}\gamma^\mu\partial_\mu\psi - m\bar{\psi}\psi \quad (9.1)$$

This is the only thing you can write down that is renormalizable, Lorentz invariant, and only has a fermion field. Once you impose these requirements you see an accidental “global $U(1)$ ” symmetry:

$$\psi \rightarrow e^{i\alpha}\psi, \quad \bar{\psi} \rightarrow e^{-i\alpha}\bar{\psi}$$

and hence $\mathcal{L} \rightarrow \mathcal{L}$. What if we promote the global symmetry to a local symmetry? Then we have “local $U(1)$ ” (this is also known as “gauge $U(1)$ ”).

$$\psi \rightarrow e^{i\alpha(x)}\psi, \quad \bar{\psi} \rightarrow e^{-i\alpha(x)}\bar{\psi}$$

\Rightarrow

$$\psi_\mu\psi \rightarrow e^{i\alpha}\psi + i\partial_\mu\alpha e^{i\alpha}\psi \quad (9.2)$$

and hence $\mathcal{L}' \neq \mathcal{L}$. We can fix this by letting $\partial_\mu \rightarrow D_\mu$ where the covariant derivative obeys

$$D_\mu\psi \rightarrow e^{i\alpha}D_\mu\psi \quad (9.3)$$

This holds if

$$D_\mu = \partial_\mu - ieA_\mu \quad (9.4)$$

where $A_\mu \rightarrow A_\mu + \frac{1}{e}\partial_\mu\alpha$. This procedure is unique. To turn this into a real field we need to add kinetic terms for A . These terms should be quadratic in $\partial^\mu A_\mu$ and require gauge invariance. The only way this can be done is by taking $F_{\mu\nu}F^{\mu\nu}$ or $\epsilon_{\mu\nu\alpha\beta}F^{\mu\nu}F^{\alpha\beta}$. The second term violates parity so we are left with the first term. We get our Lagrangian and adding in a normalization we have

$$\mathcal{L} = i\bar{\psi}\gamma^\mu D_\mu\psi - m\bar{\psi}\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \quad (9.5)$$

Here it was easy to find $F_{\mu\nu}$. However in different gauge theories it would be harder to construct. We now consider a useful way to build such terms. Note that phase factors commute with the covariant derivative (by construction). In other words

$$D_\mu(e^{i\alpha} \dots) = e^{i\alpha} D_\mu(\dots) \quad (9.6)$$

with that in mind consider the commutator:

$$[D_\mu, D_\nu] \psi \rightarrow e^{i\alpha} [D_\mu, D_\nu] \psi \quad (9.7)$$

$$\{(\partial_\mu - ieA_\mu)(\partial_\nu - ieA_\nu) - (\partial_\nu - ieA_\nu)(\partial_\mu - ieA_\mu)\} \psi \rightarrow e^{i\alpha} [D_\mu, D_\nu] \psi \quad (9.8)$$

$$-ie(\partial_\mu A_\nu - \partial_\nu A_\mu) \psi \rightarrow e^{i\alpha} [D_\mu, D_\nu] \psi \quad (9.9)$$

$$-ieF_{\mu\nu} \psi \rightarrow e^{i\alpha} [D_\mu, D_\nu] \psi \quad (9.10)$$

$$(9.11)$$

but $F_{\mu\nu}$ doesn't transform under gauge symmetry. So we know that $F_{\mu\nu} \psi \rightarrow e^{i\alpha} F_{\mu\nu} \psi$. Thus we have

$$-e^{i\alpha} ieF_{\mu\nu} \psi = e^{i\alpha} [D_\mu, D_\nu] \psi \quad (9.12)$$

Thus we can finally write

$$[D_\mu, D_\nu] \psi = -ieF_{\mu\nu} \psi \quad (9.13)$$

9.2 $SU(2)$

We now generalize $U(1) \rightarrow SU(2)$. We put two fields with the same mass in a “doublet”:

$$\Psi(x) \equiv \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \end{pmatrix} \quad (9.14)$$

$$\mathcal{L} = \sum_{a=1}^2 (i\bar{\psi}_a \gamma^\mu \partial_\mu \psi_a - m\bar{\psi}_a \psi_a) \quad (9.15)$$

$$= i\bar{\Psi} \gamma^\mu \partial_\mu \Psi - m\bar{\Psi} \Psi \quad (9.16)$$

where $\bar{\Psi} \equiv \begin{pmatrix} \bar{\psi}_1 \\ \bar{\psi}_2 \end{pmatrix}$. This Lagrangian has an accidental global symmetry given by rotation between the two doublets,

$$\Psi \rightarrow V \Psi \quad (9.17)$$

where V is a 2×2 unitary matrix. The possible V matrices form a group known as $U(2)$. We can write V as

$$V = \exp \left(i\alpha_0 + i \sum_{k=1}^3 \alpha_k \frac{\sigma_k}{2} \right) \quad (9.18)$$

where $\{1, \frac{\sigma_k}{2}\}$ are the generators of $U(2)$ algebra. We can split this symmetry into two parts: $U(2) = U(1) \times SU(2)$. $SU(2)$ is defined in terms of its algebra:

$$\left[\frac{\sigma_i}{2}, \frac{\sigma_j}{2}\right] = i\epsilon_{ijk} \frac{\sigma_k}{2} \quad (9.19)$$

where ϵ_{ijk} are the structure constants of $SU(2)$.

We first consider the first identity generator. If we demand a local symmetry out of this generator we would get back QED for two different fields. Now consider the case of $\det V = 1$ (an $SU(2)$ symmetry). The identity operator does not obey this condition since it is not traceless. Thus we only have the Pauli matrices as our generators and we have

$$V = \exp\left(i \sum_{k=1}^3 \alpha_k \frac{\sigma_k}{2}\right) \quad (9.20)$$

We promote this symmetry to a local (gauge) $SU(2)$ symmetry.

$$V(x) = \exp\left(i\alpha_k(x) \frac{\sigma_k}{2}\right) \quad (9.21)$$

where

$$\Psi \rightarrow V\Psi \quad , \quad \bar{\Psi} \rightarrow V^\dagger \bar{\Psi}$$

but

$$\partial_\mu \Psi \rightarrow V\partial_\mu \Psi + (\partial_\mu V)\Psi \quad (9.22)$$

and hence \mathcal{L} is not invariant under this transformation. We need to introduce a covariant derivative. We define

$$D_\mu \Psi \rightarrow VD_\mu \Psi \quad (9.23)$$

We have

$$\mathbb{1}_{2 \times 2} \partial_\mu - ig\tilde{A}_\mu \quad (9.24)$$

where \tilde{A}_μ is some matrix.

$$(\partial_\mu - ig\tilde{A}_\mu)\Psi \rightarrow V\partial_\mu \Psi + (\partial_\mu V)\Psi - ig\tilde{A}'_\mu V\Psi \quad (9.25)$$

To find our covariant derivative. We enforce the requirement in equation 9.23. In other words we set this equation to $V(\partial_\mu - ig\tilde{A}_\mu)\Psi$:

$$\tilde{A}'V = V\tilde{A} - \frac{i}{g}(\partial_\mu V) \quad (9.26)$$

$$\tilde{A}' = V\tilde{A}V^\dagger - \frac{i}{g}(\partial_\mu V)V^\dagger \quad (9.27)$$

For linear order in α we can write $V = 1 + i \sum_{k=1}^3 \alpha_k \frac{\sigma_k}{2}$. It is sufficient for us to restrict ourselves to Hermitian, traceless 2×2 matrices. In this case we can write

$$\tilde{A}_\mu = \sum_{a=1}^3 A_\mu^a \frac{\sigma_a}{2} \quad (9.28)$$

With this one can show that our requirement for the covariant derivatives finally says

$$A_\mu^a \rightarrow A_\mu^{a'} = A_\mu^a + \frac{1}{g} \partial_\mu \alpha^a + \epsilon^{abc} A_\mu^b \alpha^c \quad (9.29)$$

Note that our vector boson transformation looks identical to the $U(1)$ transformation with an extra part. All of this can be applied equally well to $U(1)$ however since the group is so simple it is fairly trivial.

We write our Lagrangian as

$$\mathcal{L} = i\bar{\Psi} \gamma^\mu D_\mu \Psi - m\bar{\Psi} \Psi \quad (9.30)$$

$$= \mathcal{L}_D[\psi_1] + \mathcal{L}_D[\psi_2] + g \sum_{a=1}^3 A_\mu^a \left(\sum_{i,j} \bar{\psi}_i \left(\frac{\sigma_a}{2} \right)_{ij} \gamma^\mu \psi_j \right) \quad (9.31)$$

where $\mathcal{L}_D[\psi] = i\psi(\not{\partial} - m)\psi$. We can try to guess our Feynman rules. This is a bit premature since we haven't quantized our gauge fields. However let's give it a shot. We have the interactions

$$\begin{array}{c} \mu, a \\ \text{wavy line} \\ \nearrow \quad \searrow \\ i \quad \quad j \end{array} \rightarrow -ig\gamma^\mu \left(\frac{\sigma_a}{2} \right)_{ij}$$

For example for $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ we have different fermion interaction.

We have yet to make these gauge fields dynamical. We need to add a kinetic term for these fields. To do this we use the trick we learned from QED:

$$[D_\mu, D_\nu] \Psi = \left\{ \left(\partial_\mu - igA_\mu^a \frac{\sigma^a}{2} \right) \left(\partial_\nu - igA_\nu^a \frac{\sigma^a}{2} \right) - \left(\partial_\nu - igA_\nu^a \frac{\sigma^a}{2} \right) \left(\partial_\mu - igA_\mu^a \frac{\sigma^a}{2} \right) \right\} \Psi \quad (9.32)$$

$$= -ig \left(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a - igA_\mu^a A_\nu^b \left[\frac{\sigma^a}{2}, \frac{\sigma^b}{2} \right] \right) \Psi \quad (9.33)$$

$$= -ig \underbrace{\left(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g\epsilon^{abc} gA_\mu^b A_\nu^c \right)}_{F_{\mu\nu}^a} \frac{\sigma^a}{2} \Psi \quad (9.34)$$

$$(9.35)$$

We call this $F_{\mu\nu}^a$ our field strength. This object is antisymmetric, $F_{\mu\nu}^a = -F_{\nu\mu}^a$. We define $-ig\tilde{F}_{\mu\nu}\Psi \equiv [D_\mu, D_\nu]\Psi$ or $\tilde{F}_{\mu\nu} = F_{\mu\nu}^k \frac{\sigma^k}{2}$. We have

$$-ig\tilde{F}_{\mu\nu}\Psi \rightarrow -ig\tilde{F}_{\mu\nu}V\Psi \quad (9.36)$$

and

$$\tilde{F}_{\mu\nu} \rightarrow V\tilde{F}_{\mu\nu}V^\dagger \quad (9.37)$$

We have two options of terms that conserve parity to add into our Lagrangian $\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu}$ or $\tilde{F}_{\mu\mu}\tilde{F}^{\nu\nu}$. However $\tilde{F}_{\mu\nu} = 0$. We’re not done since $\tilde{F}_{\mu\nu}$ is still a matrix. The simplest thing we can do is take the trace of this (we can take the determinate but that’s a mess):

$$\text{tr} \left(\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu} \right) \tag{9.38}$$

The trace is gauge invariant:

$$\text{tr} \left(\tilde{F}\tilde{F} \right) \rightarrow \text{tr} \left(V\tilde{F}V^\dagger V\tilde{F}V^\dagger \right) \tag{9.39}$$

$$= \text{tr} \left(V^\dagger V\tilde{F}\tilde{F} \right) \tag{9.40}$$

$$= \text{tr} \left(\tilde{F}\tilde{F} \right) \tag{9.41}$$

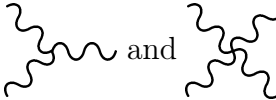
Lastly we need to normalize this term. In terms of the non-matrix fields we have

$$\mathcal{L} = -\frac{1}{2}\text{tr} \left(\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu} \right) \tag{9.42}$$

$$= -\frac{1}{2} \sum_{k,j} F_{\mu\nu}^k F^{j,\mu\nu} \overbrace{\text{tr} \left(\frac{\sigma_k}{2} \frac{\sigma^j}{2} \right)}^{\frac{1}{2}\delta^{k,j}} \tag{9.43}$$

$$= -\frac{1}{4} \sum_{k=1}^3 F_{\mu\nu}^k F^{k,\mu\nu} \tag{9.44}$$

so this normalization gives the same normalization we had in QED. This is the kinetic term of 3 “photons” + cubic and quartic terms in A (due to the extra $g\epsilon^{ijk}A_\mu^j A_\nu^k$ addition). These are self interactions of the gauge field! In QED these vertices did not exist at tree level. However this is a property of non-Abelian gauge fields. We have diagrams such as

 and even without any other particles. What’s important to note is that

the $g\epsilon^{ijk}A_\mu^j A_\nu^k$ where not optional, they were required by gauge invariance.

As an example, consider $\psi\bar{\psi} \rightarrow VV$ where V is a gauge boson. This an analogue to $e^+e^- \rightarrow \gamma\gamma$. This corresponds to

$$\left(\text{[diagram 1]} + \text{[diagram 2]} \right) + \text{[diagram 3]}$$

([diagram 1] + photon crossing) + [diagram 3]

Note that the last diagram is fundamental to non-Abelian gauge theories and is required in order for the Ward Identity to work.

9.3 General Recipe

We now discuss gauge theories more generally. To construct the theory you need to

- Choose a symmetry group (any Lie group, G). In our example this was $SU(2)$.
- Choose a representation $r(G)$. In our example we chose the fundamental representation ($\sim \square$, in Young Tableaux notation).
- Introduce $n \equiv \dim(r(G))$ fields - “n-tuplet”. In our example we had $n = 2\psi$ fields. These fields are called “matter fields” (as opposed to gauge fields). These n fields must be in the same representation of the Lorentz group. For Yang-Mills theories these have to be spin 0 or spin 1/2. In our example we have spin 1/2.
- Consider a global symmetry

$$\psi \rightarrow V\psi \quad (9.45)$$

where $V \in r(G)$. We build the most general Lagrangian that is invariant under this transformation:

$$\mathcal{L} = i\bar{\psi}\gamma^\mu\partial_\mu\psi - \bar{\psi}\psi \quad (9.46)$$

- Promote the global symmetry to local. In other words we rotate our ψ such that $\psi \rightarrow V(x)\psi$ with

$$V(x) = e^{i\sum_i^{N_g}\alpha_i(x)T_i} \quad (9.47)$$

where N_g is the number of generators and the T_i are generators of the algebra G in the representation $r(G)$. The number of generators is independent of the representation chosen. The number, N_g is equal to the dimension of the ad-joint representation ($\dim(\text{Adj}(G))$). Where for the ad-joint representation we have

$$(T^i)_{jk} = f_{ijk} \quad (9.48)$$

The generators are given in terms of the structure constants by $[T_i, T_j] = f_{ijk}T_k$, where f_{ijk} are the structure constants of G . These T_i 's are $n \times n$ matrices.

To promote to a local symmetry we need to require

$$\partial_\mu \rightarrow D_\mu = \partial_\mu - ig\tilde{A}_\mu \quad (9.49)$$

where \tilde{A}_μ always has the same form and given by

$$\tilde{A}_\mu = \sum_{i=1}^{N_g} A_\mu^i(x)T^i \quad (9.50)$$

and the T^i are the generators in the representation $r(G)$. We introduce one new number that is our gauge coupling. Note that in order to enforce gauge invariance you are forced to have the same coupling for each A_μ . This is the case for simple groups (groups that do not have any generators that commute with all other generators). For non-simple groups we are allowed to have such coupling differences.

For example you can split the couplings for $U(2)$ through $U(2) = \overbrace{U(1)}^g \times \underbrace{SU(2)}_{g'}$.

We have the gauge field transformation

$$\tilde{A}_\mu \rightarrow V \tilde{A}_\mu V^\dagger - \frac{i}{g} (\partial_\mu V) V^\dagger \quad (9.51)$$

It's a simple exercise to show that

$$A_\mu^i \rightarrow A_\mu^i + \frac{1}{g} \partial_\mu \alpha^i + f^{ijk} A_\mu^j \alpha^k \quad (9.52)$$

Note that there is no dependence on $r(G)$. Further note that you can have any multiple of matter fields. However there are a single set of gauge fields that are fixed by G .

One last comment is often people say “gauge fields belong in the adjoint representation, $Adj(G)$ ” (The more correct way to say it is transforms as the adjoint representation). For global transformations ($\alpha = \text{const}$) we have

$$A_\mu^i = A_\mu^i + f^{ijk} A_\mu^j \alpha^k \quad (9.53)$$

$$= A_\mu^i + (T^i)^{jk} A_\mu^j \alpha^k \quad (9.54)$$

so A_μ^i transforms through adjoint representation. However this is only for a global transformation and is strictly wrong in general due to the $\partial_\alpha \alpha^i$ term.

- The Field Strength Tensor is defined as

$$ig \tilde{F}_{\mu\nu} \psi [D_\mu, D_\nu] \psi \quad (9.55)$$

where D_μ are $n \times n$ matrices where n is the dimension of the representation. Further

$$\tilde{F}_{\mu\nu} = \sum_{i=1}^{N_g} F_{\mu\nu}^i T^i \quad (9.56)$$

where $F_{\mu\nu}^i = \partial_\mu A_\nu^i - \partial_\nu A_\mu^i + g f^{ijk} A_\mu^j A_\nu^k$. This is independent of $r(G)$. The transformation law is given by

$$\tilde{F}_{\mu\nu} \rightarrow V \tilde{F}_{\mu\nu} V^\dagger \quad (9.57)$$

- The Gauge Field Lagrangian is given by

$$\mathcal{L} = -\frac{1}{2} \text{tr} \left(\tilde{F}_{\mu\nu} F^{\mu\nu} \right) \quad (9.58)$$

assuming that the generators are normalized as $\text{tr} (T^i T^j) = \frac{1}{2} \delta^{ij}$.

Now the Lagrangian of these fields is of the form,

$$\mathcal{L} = \bar{\psi}_A D_\mu^A \gamma \psi_A + \bar{\psi}_B i D_\mu^B \gamma^\mu \psi + \dots \quad (9.64)$$

where the covariant derivatives are,

$$D_\mu^A = \partial_\mu - g_A W_\mu^a T_a \quad (9.65)$$

$$D_\mu^B = \partial_\mu - g_B W_\mu^a T_a \quad (9.66)$$

where here we assume different gauge couplings. But the first derivative implies that the W_μ^a field transforms as,

$$W_\mu^a \rightarrow W_\mu^a + f^{abc} W_{\mu,b} \theta_c + \frac{1}{g_A} \partial_\mu \theta^a \quad (9.67)$$

and the second derivative implies that

$$W_\mu^a \rightarrow W_\mu^a + f^{abc} W_{\mu,b} \theta_c + \frac{1}{g_B} \partial_\mu \theta^a \quad (9.68)$$

However W_μ^a are the same fields in either case. Thus we must have $g_A = g_B$.

In the above discussion we assumed that the fields were in the same representation of the gauge group. This discussion fails if the fields are in different representations. In an Abelian group since there are no commutation relations, every field can be in a different representation of the Lie group. Thus every Abelian field can have different gauge couplings.

9.5 Quantum Chromodynamics

Quantum Chromodynamics (QCD) is Yang-Mills theory with an $SU(3)$ gauge group. For $SU(N)$ the number of generators is $N_g = N^2 - 1$. In this case since N is 3 we have 8 generators which in turn gives 8 gauge fields A_μ^a ($a = 1, 2, \dots, 8$). We call these fields gluons. There is one coupling constant g . They are given by structure constants f^{abc} .

We want to write the matter fields that are charged under this gauge group. The fundamental matter fields are the quarks. We only consider QED so we call these spin - $\frac{1}{2}$ Dirac fermions (in reality due to the weak interaction we should split these into Weyl fermions). The quarks are $q_f = (u, d, s, c, b, t)$ ($f = 1, \dots, 6$) we call this the “flavor” of the quarks. The spin of the particles are the same. The electric charges split into 3 sets of doublets. Their could be a symmetry in between these sets. However it would not exact since they have different masses. We take one of the quarks for simplicity (they all have the same interactions). These transform as the fundamental representation of $SU(3)$. The transformations act on vectors given by U . $\dim(\square) = N$ (Young Tableaux notation).

$$U = \begin{pmatrix} U_1 \\ U_2 \\ U_3 \end{pmatrix} \quad (9.69)$$

where $1, 2, 3 \rightarrow r, g, b$ (“colour”). The Lagrangian is given by

$$\mathcal{L} = \sum_f i\bar{q}_f \gamma^\mu D_\mu q_f - m\bar{q}_f q_f - \frac{1}{4} \sum_{a=1}^9 (F_{\mu\nu}^a F^{a,\mu\nu}) \quad (9.70)$$

where $D_\mu = \partial_\mu - ig \sum_{a=1}^8 A_\mu^a T^a$ and T^a are the generators of $SU(3)$ in the fundamental representation, the “Gell-Mann matrices”.

9.6 Quantization of Yang Mills

We have

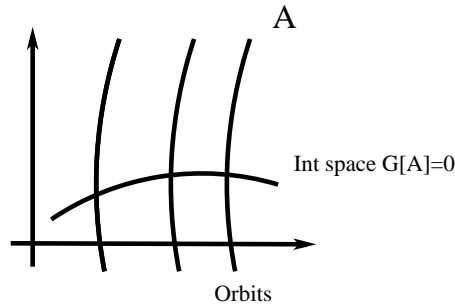
$$W_0 = \int \mathcal{D}A \exp \left[i \int d^4x \left(-\frac{1}{4} \sum_{a=1}^{N_g} F_{\mu\nu}^a F^{a,\mu\nu} \right) \right] \quad (9.71)$$

where $\mathcal{D}A = \prod_{a=1}^{N_g} \prod_{\mu=0}^3 \prod_x dA_x^{a,\mu}$. We tried this in QED and we ran into a problem. The procedure was to go to Fourier space and we got a Green’s function which we needed to invert and it was not invertible. We traced that problem back to the fact that this integral is redundant (we were dealing with all the gauge fields). We required the Faddeev-Popov procedure. The idea is the same here. The gauge freedom gives

$$A_\mu^a \xrightarrow{\alpha(x)} A_\mu^a + \frac{1}{g} \partial_\mu \alpha^a + f^{abc} A_\mu^b \alpha^c \quad (9.72)$$

There is a huge amount of redundancy in this integral which we need to eliminate.

The procedure is as follows. You choose some functional of the gauge field and you choose this to be zero, $G[A] = 0$. Geometrically:



We have a mathematical identity:

$$1 = \int \mathcal{D}\alpha \delta(G[A^\alpha]) \det \left(\frac{\delta G[A^\alpha]}{\delta \alpha} \right) \quad (9.73)$$

We have

$$W_0 = \int \mathcal{D}A^\alpha \mathcal{D}\alpha \exp \left[i \int d^4x (\mathcal{L}[A^\alpha]) \right] \delta(G[A^\alpha]) \det \left(\frac{\delta G[A^\alpha]}{\delta \alpha} \right) \quad (9.74)$$

where we shifted our integration measure to A^α . Furthermore, the Lagrangian is by definition gauge invariant so we have $\mathcal{L}[A^\alpha] = \mathcal{L}[A]$. We now relabel our integration variable.

$$W_0 = \int \mathcal{D}A \mathcal{D}\alpha \exp \left[i \int d^4x (\mathcal{L}[A]) \right] \delta(G[A]) \det \left(\frac{\delta G[A^\alpha]}{\delta \alpha} \right) \Big|_{A^\alpha \rightarrow A} \quad (9.75)$$

Now it's clear that $\int \mathcal{D}\alpha = \mathcal{N}$ is just a constant. So we can forget about it

$$W_0 = \int \mathcal{D}A \exp \left[i \int d^4x (\mathcal{L}[A]) \right] \delta(G[A]) \det \left(\frac{\delta G[A^\alpha]}{\delta \alpha} \right) \Big|_{A^\alpha \rightarrow A} \quad (9.76)$$

For $U(1)$ we had $G[A] = \partial_\mu A^\mu - \omega(x)$ and $G[A^\alpha] = \partial_\mu (A^\mu + \frac{1}{e} \partial^\mu \alpha) - \omega(x)$. Clearly $\frac{\delta G}{\delta \alpha} = \text{const}$ (A independent).

For non-Abelian gauge theories we have

$$G[A] = \prod_{a=1}^{N_g} (\partial_\mu A^{a,\mu} - \omega^a(x)) \quad (9.77)$$

and

$$G[A^\alpha] = \prod_a \left(\partial_\mu \left(A^{a,\mu} + \frac{1}{g} \partial^\mu \alpha^a + f^{abc} A^{b,\mu} \alpha_c \right) - \omega_a(x) \right) \quad (9.78)$$

So we have

$$\frac{\delta G}{\delta \alpha^c} = \frac{\delta_c^a}{g} \partial_\mu \partial^\mu + f^{abc} \partial_\mu A^{b,\mu} \quad (9.79)$$

The definition of the first term will become clear soon. In order to make sense of this expression we go back to the discrete version of our integrals. $x \rightarrow x_i$, $\alpha(x) \rightarrow \alpha_i$, $A_\mu^a(x) \rightarrow A_{\mu,i}^a$.

$$G^a[A^\alpha(x)] = \partial_\mu \left(A^{a,\mu} + \frac{1}{g} \partial^\mu \alpha_a + f^{abc} A^{b,\mu} \alpha_c \right) - \omega^a(x) \quad (9.80)$$

$$\begin{aligned} &\rightarrow \frac{A_{i+1}^{a,\mu} - A_i^{a,\mu}}{\Delta} + \frac{1}{g} \frac{1}{\Delta^2} (\alpha_{i+1}^a + \alpha_{i-1}^a - 2\alpha_i^a) + f^{abc} \frac{A_{i+1}^{b,\mu} - A_i^{b,\mu}}{\Delta} \alpha_c \\ &\quad + f^{abc} A^{b,\mu} \frac{\alpha_{i+1}^c - \alpha_i^c}{\Delta} - \omega_i^a \end{aligned} \quad (9.81)$$

So we have

$$\frac{\delta G^a}{\delta \alpha^c} \rightarrow \frac{dG_i^a}{d\alpha_j^c} \quad (9.82)$$

$$= \frac{1}{g\Delta^2} (\delta_c^a) (\delta_{i+1,j} + \delta_{i-1,j} - 2\delta_{i,j}) + \dots \quad (9.83)$$

$$\equiv M_{c,j}^{a,i} \quad (9.84)$$

The next step was to convert everything in the path integral into a term in the Lagrangian. The delta function becomes a gauge fixing term. In QED we had an extra

$$\int \mathcal{D}\omega e^{-\int d^4x \omega^2 / 2\xi} \tag{9.85}$$

and we have

$$\mathcal{L}_{\text{gauge fix.}} = \frac{1}{2\xi} \sum_a (\partial_\mu A^{a,\mu})^2 \tag{9.86}$$

which gives

$$W = \int \mathcal{D}A \exp \left[i \int d^4x \left(-\frac{1}{4} F_{\mu\nu}^a F^{a,\mu\nu} + \frac{1}{2\xi} (\partial_\mu A^{a,\mu})^2 \right) \right] \tag{9.87}$$

The last question is how to convert the determinant into a term into the Lagrangian. This is done by something which are known as *ghosts*. These in some sense behave like ordinary fields. They take part into the Feynman rules but just cannot propagate.

Ghosts are essentially just way to exponentiate the determinate term so we will be able

to perform a Taylor expansion. We want to rewrite the determinate: $\det \overbrace{[\partial_\mu \partial^\mu \delta^{ab} - g f^{abc} \partial_\mu A^{c,\mu}]}^{M_{j,b}^{i,a}}$ in an exponential form. Faddeev and Popov showed that you can write

$$\det [\partial_\mu \partial^\mu \delta^{ab} - g f^{abc} \partial_\mu A^{c,\mu}] = \int \mathcal{D}c \mathcal{D}\bar{c} \exp \left(i \int d^4x \bar{c}^a (\partial_\mu \partial^\mu \delta^{ab} - g f^{abc} \partial_\mu A^{c,\mu}) c^b \right) \tag{9.88}$$

The new ghost fields, c, \bar{c} are Grassman variables. Recall that for two Grassman variables:

$$\int d\theta d\bar{\theta} e^{\bar{\theta} A \theta} = A \tag{9.89}$$

Earlier we showed that,

$$\int \left(\prod_i d\theta_i d\bar{\theta}_i \right) e^{\sum_{i,j} \bar{\theta}_i A_{i,j} \theta_j} = \det A \tag{9.90}$$

So any determinate you wish can be written as a product of integrals over Grassman numbers. The $c(x)$ are “ghost fields”. They are anti-commuting however unlike Dirac fields they don’t have an extra index and are spin-0 particles. Anti-commuting spin-0 fields violate spin-statistics! On the other hand spin-statistics is only for real asymptotic particles. So far we have not introduced a source for these ghost fields and adding this source will violate spin-statistics theorem. In other words these cannot be external particles (there are no S matrix elements for these particles).

Adding in these ghost fields we have

$$\begin{aligned}
 W &= \int \mathcal{D}A \mathcal{D}c \mathcal{D}\bar{c} \exp \left[i \int d^4x \left(-\frac{1}{4} F_{\mu\nu}^a F^{a,\mu\nu} + \frac{1}{2\xi} (\partial_\mu A^{a,\mu})^2 + \bar{c}^a \partial_\mu (D^\mu)^{ab} c_b \right) \right] \quad (9.91) \\
 &= \int \mathcal{D}A \mathcal{D}c \mathcal{D}\bar{c} \exp \left[i \int d^4x \left(-\frac{1}{4} F_{\mu\nu}^a F^{a,\mu\nu} + \frac{1}{2\xi} (\partial_\mu A^{a,\mu})^2 - (\partial_\mu \bar{c}_a) (D^\mu)^{ab} c_b \right) \right] \quad (9.92)
 \end{aligned}$$

where we have just integrated by parts to which the derivative term. Furthermore D^μ is just the just regular covariant derivative since we have

$$\partial_\mu \partial^\mu \delta^{ab} - g f^{abc} \partial_\mu A^{c,\mu} = \partial_\mu (\partial^\mu \delta^{ab} - g f^{abc} A^{c,\mu}) \quad (9.93)$$

$$= \partial_\mu \left(\partial^\mu \delta^{ab} - g (T_{Adj}^c)^{ab} A^{c,\mu} \right) \quad (9.94)$$

The Feynman rules for these ghosts are,

$$\begin{array}{c} \mu, a \\ \text{---} \end{array} \begin{array}{c} \xrightarrow{k} \\ \text{---} \end{array} \begin{array}{c} \nu, b \\ \text{---} \end{array} = \delta^a_b \left(\frac{-i}{k^2 + i\epsilon} \right) \left(g_{\mu\nu} - (1 - \xi) \frac{k_\mu k_\nu}{k^2} \right)$$

$$\begin{array}{c} \mu, a \\ \text{---} \end{array} \begin{array}{c} \xrightarrow{k} \\ \text{---} \end{array} \begin{array}{c} \nu, b \\ \text{---} \end{array} = \delta^a_b \left(\frac{-i}{k^2 + i\epsilon} \right)$$

$$\begin{array}{c} a \\ \text{---} \end{array} \begin{array}{c} \xleftarrow{p} \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \text{---} \end{array} \begin{array}{c} \xrightarrow{p} \\ \text{---} \end{array} \begin{array}{c} b \\ \text{---} \end{array} = g f^{abc} p^\mu$$

Notice that ghosts can only show up in pairs. With this in mind and the fact that they cannot show up in external states, they only appear in loop diagrams. Thus in tree level you can just forget about ghosts.

9.7 Unitary and Ghosts

The fact that ghosts can only show up in loops is not obviously consistent with the unitarity of the S matrix. This topic is a bit subtle. We will not do many calculations however we will go over the logic and series of statements that describe the situation.

The statement of unitarity is the statement that (this is the same as saying that probability is correctly normalized)

$$S^\dagger S = 1 \quad (9.95)$$

gn where S is the S matrix. We split S into the part that's associated with no collision and the interested part which is due to the interactions, $S = 1 + iT$ (see for example [Peskin and Schroeder(1995)] pg. 104).

$$S = 1 + iT \tag{9.96}$$

Plugging this into 9.95 we have

$$\begin{aligned} (1 + iT)(1 - iT^\dagger) &= 1 \\ 1 + i(T - T^\dagger) + T^\dagger T &= 1 \\ \Rightarrow -i(T - T^\dagger) &= T^\dagger T \end{aligned} \tag{9.97}$$

This is further discussed in [Peskin and Schroeder(1995)]. What this says is that (for ϕ^4)

$$\text{Diagram} + \dots = \int d^3p/E_1 d^3p/E_2 (\text{Diagram})^*$$

This is known as the optical theorem. Unitarity becomes more intricate for non-Abelian theories. We denote V as the following amplitude: $(\psi_i \bar{\psi}_j \rightarrow V_a V_b, \text{ two fermions into two gauge bosons})$. The square of this corresponds to $\psi_i \bar{\psi}_j \rightarrow \psi_k \bar{\psi}_\ell$.

$$\text{Diagram} = \text{Im}(\text{Diagram}) + \text{Diagram}$$

This is equivalent to

$$\int \frac{d^3k_1}{E_1} \frac{d^3k_2}{E_2} (V_{s_1, s_2}(1, 2 \rightarrow k_1 k_2) V_{s_1, s_2}^*(k_1, k_2 \rightarrow 3, 4)) = \sum_{s_1 s_2} \mathcal{M}^{\mu\nu} \epsilon_\mu^{s_1}(k_1) \epsilon_\nu^{s_2}(k_2) \epsilon_\rho^{s_1*} \epsilon_\sigma^{s_2*} \mathcal{M}^{*\rho\sigma} \tag{9.98}$$

Recall that in QED we had $\sum_{s=1}^2 \epsilon_\mu^s(k) \epsilon_\nu^{s*}(k) = -g_{\mu\nu} + A k_\mu k_\nu$ where we relied on the Ward identity to simplify our results.

This was all relatively simple however things get complicated in non-Abelian gauge theories. It turns out that for 2 or more external gauge bosons this replacement is not allowed (we don't have $\sum \epsilon_\mu \epsilon_\nu^* \rightarrow -g_{\mu\nu}$). This is because when you have two gauge bosons you can make one boson unphysical. If you make one boson longitudinal then Ward identity for the other boson will not work. What always holds is

$$\sum_{s=0}^3 \epsilon_\mu^s \epsilon_\nu^{*,s} = -g_{\mu\nu} \tag{9.99}$$

however this requires summing over all the polarizations. You can work this out and we’ll basically be working this out on the homework.

This now sounds like a problem. Since on the left hand side of the unitarity relation we have ϵ ’s (we have external gauge bosons) and on the right hand side we don’t have any such terms. The term with ghosts fixes this problem. This is quite neat. The ghosts are in a sense there to preserve unitarity. The formal proof of this is known as the BRST theorem.

We now go back to tree level. We just said that at tree level we don’t worry about ghosts. However consider $\psi_i \bar{\psi}_j \rightarrow V_a V_b$. We have

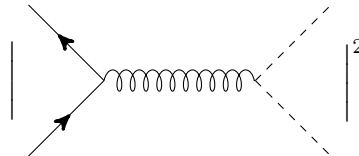
$$\mathcal{M} = \mathcal{M}^{\mu\nu} \epsilon_\mu^{s_1}(p_1) \epsilon_\nu^{s_2} \quad (9.100)$$

In QED we would have

$$\sum_{s_1, s_2} |\mathcal{M}|^2 \stackrel{QED}{=} g_{\mu\alpha} g_{\nu\beta} \mathcal{M}^{\mu\nu} \mathcal{M}^{*,\alpha\beta} \quad (9.101)$$

However in non-Abelian theories we can’t do that. We have two options

1. We can have explicit ϵ ’s. If $p = (E, 0, 0, E)$ then $\epsilon = \frac{1}{\sqrt{2}}(0, 1, \pm i, 0)$. You can just compute $\sum_{s_1, s_2} |\mathcal{M}^{s_1, s_2}|^2$ without using the spin sum formula. We will do this on the homework. This always works and you never need to mention ghosts.
2. There is a second way to deal with this that is more convenient but a little bit weird. This is to remember that we still have $\sum_{s=0}^3 \epsilon_\mu^s \epsilon_\nu^{*,s} = -g_{\mu\nu}$ (notice our summing limits). What you can do is use $\sum \epsilon_\mu \epsilon_\nu^* \rightarrow -g_{\mu\nu}$ but then you always need to include an extra diagram:



In other words our tactic is to add an extra diagram with the external gauge bosons are replaced with external ghosts.

9.8 Renormalized Perturbation Theory

This just means that we have counter terms. We start with the Yang-Mills Lagrangian. We assume that the fermion is massless. This makes the calculations for the β functions a lot easier but this is not necessary.

$$\mathcal{L} = \bar{\psi}^i i \not{D}^{i,j} \psi^j - \frac{1}{4} F_{\mu\nu}^a F^{a,\mu\nu} + \frac{1}{2\xi} (\partial_\mu A^{a,\mu})^2 - (\partial_\mu \bar{c}^a)(D^\mu)^{ab} c^b + \mathcal{L}_{CT} \quad (9.102)$$

where

$$\mathcal{L}_{CT} = \delta_2 \bar{\psi} i \not{\partial} \psi - \delta_1 g \bar{\psi} \gamma^\mu A_\mu^a T^a \psi - \frac{1}{4} \delta_3 (\partial_\mu A_\nu^a - \partial_\nu A_\mu^a)^2 + (4 \text{ more CT's}) \quad (9.103)$$

In the case of QED we did not have the last 4 terms since ghosts didn't exist. In this case gauge invariance no longer implies that $\delta_1 = \delta_2$. This is because now for example $(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a)$ is not even gauge invariant since the gauge invariance is only saved by the ghost CT's.

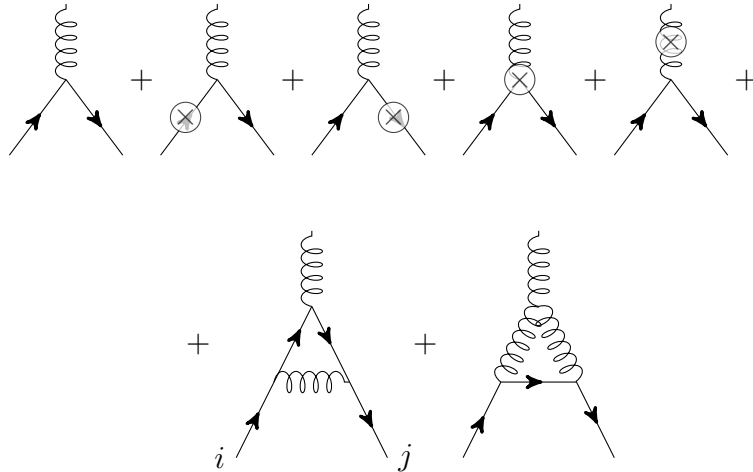
What we are really interested in is the β function. To do this we need to use the CS equations. We need to compute

$$\begin{aligned} \langle AA \rangle &\xrightarrow{gave} \gamma_A \\ \langle \bar{\psi} \psi \rangle &\xrightarrow{gave} \gamma_\psi \\ \langle \bar{\psi} \psi A \rangle &\xrightarrow{gave} \beta = M \frac{\partial e(M)}{\partial M} \end{aligned}$$

The formula we had in QED for the β function was

$$\beta = eM \frac{\partial}{\partial M} \left(-\delta_1 + \delta_2 + \frac{1}{2} \delta_3 \right) \quad (9.104)$$

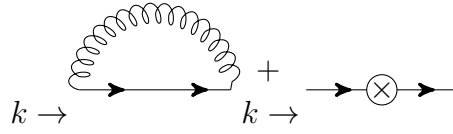
If you look back at that procedure. Nothing in it used the Feynman rules of the theory in anyway. We just used the structure of counterterms. In non-Abelian Yang Mills we can do the exact same thing. Just like in QED the three point function will be



This calculation is historically significant and lead to a Nobel prize. The counterterms have the following conditions

$$\delta_2 : \Sigma_2(k = M) = 0 \quad (9.105)$$

We evaluate the following



The first diagram gives,

$$i\mathcal{M} = (ig)^2 \int \frac{d^d \ell}{(2\pi)^d} \gamma^\mu (T^a)_{ij} \frac{i\delta_{jk}}{\ell + \not{k} + i\epsilon} \gamma^\nu (T^b)_{k\ell} \overbrace{\frac{(-ig_{\mu\nu})}{k^2 + i\epsilon}}^{\text{Feynman gauge}} \delta_{ab} \quad (9.106)$$

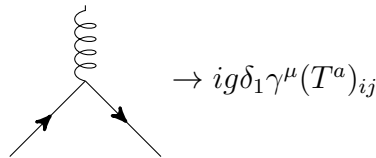
$$= (ig)^2 \underbrace{\sum_a (T^a T^a)_{i\ell}}_{C_2(r_f)\delta_{i\ell}} \underbrace{\int \frac{d^d \ell}{(2\pi)^d} \gamma^\mu \frac{1}{\ell + \not{k}} \gamma_\mu \frac{1}{k^2}}_{\text{same as QED}} \quad (9.107)$$

$$= \frac{ig^2}{(4\pi)^{d/2}} \frac{\Gamma(2 - d/2)}{(k^2)^{2-d/2}} \not{k} C_2(r_f) \delta_{i\ell} \quad (9.108)$$

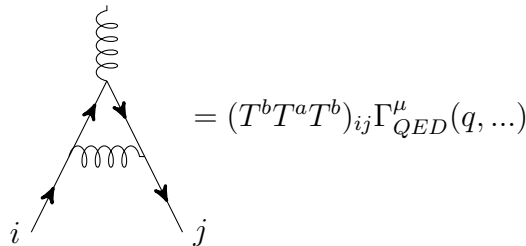
This gives (using our renormalization condition for Σ_2)

$$\delta_2 = -\frac{g^2}{(4\pi)^{d/2}} \frac{\Gamma(2 - d/2)}{(M^2)^{2-d/2}} C_2(r_f) \quad (9.109)$$

In this calculation we got lucky. We were able to factor our answer into a QED contribution and a group theory contribution. This will not happen again. We now calculate δ_1 . For this we use



Our renormalization condition is to set $F_1(q^2 = M^2) = 0$. We need to calculate



we have the matrix coefficient:

$$T^b T^b T^a + T^b [T^a, T^b] = C_2(r_f) T^a + T^b f^{abc} T^c \quad (9.110)$$

Structure constants are antisymmetric so we can write this as

$$T^b T^b T^a + T^b [T^a, T^b] = C_2(r_f) T^a + \frac{1}{2} f^{abc} (T^b T^c + T^b T^c) \quad (9.111)$$

$$= C_2(r_f) T^a + \frac{1}{2} f^{abc} (T^b T^c - T^b T^c) \quad (9.112)$$

$$= C_2(r_f) T^a + \frac{1}{2} f^{abc} [T^b, T^c] \quad (9.113)$$

$$= C_2(r_f) T^a + \frac{1}{2} f^{abc} f^{bce} T^e \quad (9.114)$$

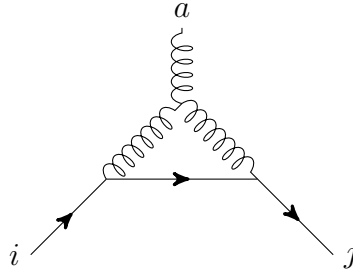
$$= C_2(r_f) T^a - \frac{1}{2} \underbrace{f^{bac} f^{bce}}_{(T_{Adj}^b)_{ac} (T_{Adj}^b)_{ce}} T^e \quad (9.115)$$

$$= C_2(r_f) T^a - \frac{1}{2} (T_{Adj}^b T_{Adj}^b)_{ae} T^e \quad (9.116)$$

$$= C_2(r_f) T^a - \frac{1}{2} C_2(Adj) \delta_{ae} T^e \quad (9.117)$$

$$(9.118)$$

We now discuss the group structure of another diagram contribution:



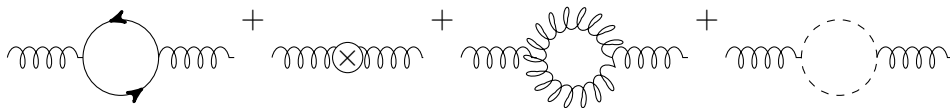
This has the group theory contribution of

$$(T^b T^c)_{ij} f^{abc} = \frac{1}{2} f^{abc} [T^b, T^c] = \frac{1}{2} \underbrace{f^{abc} f^{bce}}_{-C_2(Adj) \delta_{ae}} T^e \quad (9.119)$$

The full calculation gives

$$\delta_1 = \frac{-g^2}{(4\pi)^{d/2}} \frac{\Gamma(2 - d/2)}{(M^2)^{2-d/2}} (C_2(r_f) + C_2(Adj)) \quad (9.120)$$

There is one more set of diagrams to calculate which is



We now calculate the group factors. The fermionic loop has a group theory factor of,

$$(T^a)_{ij}(T^b)_{ji} = \text{tr}(T^a T^b) = C(r_f)\delta^{ab} \quad (9.121)$$

while for the gluon loop we have,

$$f^{acd}f^{bcd} = C_2(Adj)\delta_{ab} \quad (9.122)$$

A full calculation of the last counterterm gives

$$\delta_3 = \frac{g^2}{(4\pi)^{d/2}} \frac{\Gamma(2-d/2)}{(M^2)^{2-d/2}} \left(\frac{5}{3}C_2(Adj) - \frac{4}{3} \sum_{r_f} C(r_f) \right) \quad (9.123)$$

We have $(M^2)^\epsilon = e^{\epsilon \log M^2} = 1 + \epsilon \log M^2 + \dots$ (and a similar expansion for 4π). Using these expressions we have

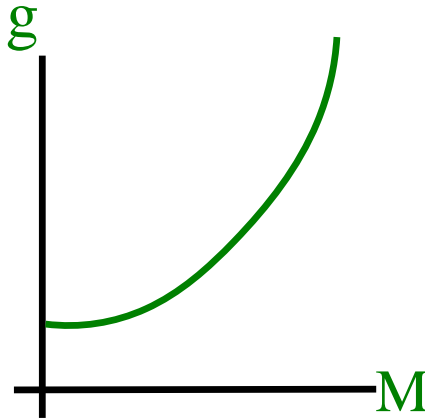
$$\frac{1}{(4\pi)^{d/2}} \frac{\Gamma(2-d/2)}{(M^2)^{2-d/2}} \Big|_{d=4-\epsilon} = \frac{1}{16\pi^2} \left(\frac{1}{\epsilon} + \gamma_E - \log M^2 - \log(4\pi) + \mathcal{O}(\epsilon) \right) \quad (9.124)$$

Using $M \frac{\partial}{\partial M} \log M^2 = 2$, Our expression for the β function is

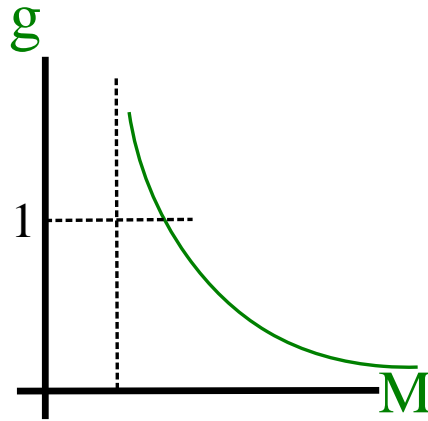
$$\beta(g) = -\frac{g^3}{16\pi^2} \left[\frac{11}{3}C_2(Adj) - \frac{4}{3} \sum_f C(r_f) \right] \quad (9.125)$$

We now interpret this result. The first interesting thing to notice is that this is non zero if there are no fermions at all. What’s even more interesting is that $\beta < 0$. This is completely different from the ϕ^4 case and the *QED* case where the β function was positive. This is interesting since we know that β controls how the coupling changes with scale that we are studying the system at.

We had the graph in QED of

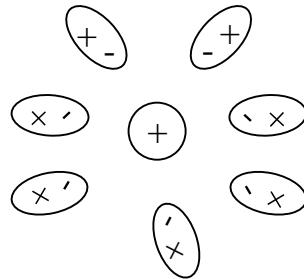


The reason this worked in this way was because the β function was positive. However in this case we will have the opposite effect. Instead we have



This is called an asymptotically free theorem. However at low energies this theory becomes non-perturbative. You cannot take a zero momentum scattering (or equivalently take the quantum mechanics limit). QCD of course has fermions however it doesn't have enough fermions to change the sign of the β function. In QCD the energy that we have non perturbative results is about 1GeV or below.

In QED we had screening:



In non-Abelian Yang Mills we have “anti-screening” due to gauge fields. To understand where this anti-screening comes from we consider a $SU(2)$ gauge theory. We have

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^a F^{a,\mu\nu} + gA_{\mu}^a j^{a,\mu} + \mathcal{L}_{\text{matter}} \tag{9.126}$$

where $j^{a,\mu}$ is the current (it's easy to show that our earlier form of the Lagrangian is equivalent to this with the appropriate conserved current) and $F_{\mu\nu}^a = \partial_{\mu}A_{\nu}^a - \partial_{\nu}A_{\mu}^a + g\epsilon^{abc}A_{\mu}^b A_{\nu}^c$.

We consider the equations of motion for the gauge field,

$$\frac{\delta\mathcal{L}}{\delta A_{\mu}^a} = \partial_{\nu} \frac{\delta\mathcal{L}}{\delta\partial_{\nu}A_{\mu}^a} \tag{9.127}$$

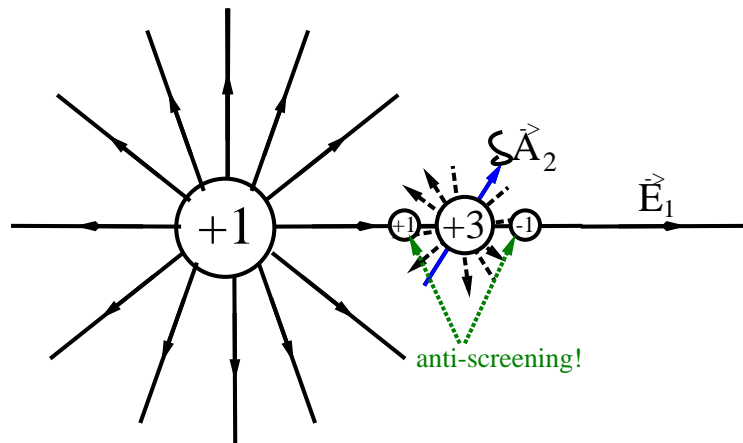
Recall that in QED we had $E_i = F_{0i}$. We can extend this idea to a non-Abelian gauge theories and write, $E_i^a = F_{0i}^a$. This is often called the “chromo-electric field”. Note this is a bit of a misnomer in this case since we are dealing with $SU(2)$ but people often use this term for all non-Abelian gauge theories. The equations of motion are

$$\partial_i E_i^a = g\rho^a - g\epsilon^{abc} A_i^b E_i^c \tag{9.128}$$

where $\rho^a \equiv j^{a,0}$ is called (in analogy with QED) the charge density. This is the analogue to Gauss’s law.

Consider a point-like charge, $\rho^a = \delta^3(\mathbf{x})\delta^{a,1}$. Note that we needed to specify a direction in the group space for the charge. Notice that Gauss’s law is a non-linear equation. The QED classical solution is $\mathbf{E}^1 = \frac{g}{4\pi r^2}\hat{r}$, $\mathbf{E}^2 = \mathbf{E}^3 = 0$. You can choose a gauge such that $\mathbf{A}^a = 0$ and $A_0^a \neq 0$. If we do this then (since $\epsilon^{aab} = 0$) the difficult term disappears. So we can recycle the QED solution.

This was a classical computation. However the β function arose from a quantum solution (it came from loop corrections which necessarily imply the use of quantum mechanics). These non-linear terms inevitably come in at the quantum level. In quantum mechanics we can’t just set $\mathbf{A} = 0$. In other words, $\delta\mathbf{A} \neq 0$. We have



Consider $\delta\mathbf{A}^2$. The sources are $\delta\mathbf{E}^3$. So we have

$$\nabla \cdot \mathbf{E}^3 = g\delta\mathbf{A}^2 \cdot \mathbf{E}^\perp \tag{9.129}$$

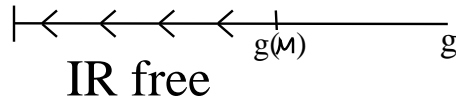
So we have

$$\nabla \cdot \delta\mathbf{E}^1 = -ig\delta\mathbf{A}^2 \cdot \delta\mathbf{E}^3 \tag{9.130}$$

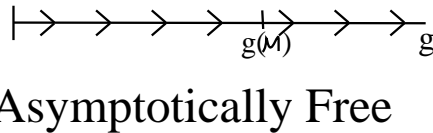
This is demonstrated above.

9.9 RG flows (evolution)

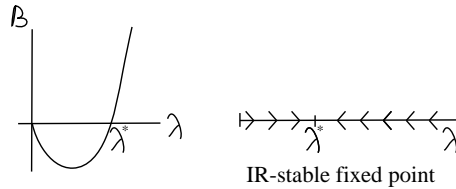
So far to display our running couplings we have shown plots of g as a function of the renormalization scale. There is an alternative way to represent this.



Such theories are called IR free. These include $\lambda\phi^4$, QED, Yang-Mills when $\beta > 0$.
 Alternatively we can have UV or asymptotically free theories:



This includes theories such as Yang-Mills with no/little matter ($\beta < 0$). The alternative, which we didn't see in our calculation but is possible is called "fixed-point". This occurs when $\beta = 0$, but $\lambda \neq 0$. You have an interacting theory but the coupling doesn't run. For example



The theory is "attracted" to some stable coupling. We have

$$\beta \sim a\lambda + b\lambda^2 + \dots \tag{9.131}$$

Another thing that's possible is so called Conformal Field Theories. These have $\beta = 0$ for any λ . For this to be the case you require some symmetry to make it this way. This is also known as scale invariance.

Chapter 10

Non-perturbative results

10.1 Kallen-Lehmann Representation

The pole structure present in correlation functions has a very particular structure. This can already be seen in the two-point function where we can make powerful non-perturbative observations. To this end consider the 2-point function:

$$\langle 0 | \phi(x) \phi^\dagger(y) | 0 \rangle \quad (10.1)$$

Recall that for a free-field this is given by,

$$\Delta_+(x-y; \mu^2) \equiv \langle \phi(x) \phi(y)^\dagger \rangle = \int \frac{d^4 p}{(2\pi)^4} e^{-ip \cdot (x-y)} \theta(p^0) 2\pi \delta(p^2 - \mu^2) \quad (10.2)$$

More generally we can insert in a complete set of states and use the fact that the vacuum is translationally invariant to write:

$$\langle \phi(x) \phi(y)^\dagger \rangle = \sum_n \langle 0 | \phi(x) | n \rangle \langle n | \phi(y)^\dagger | 0 \rangle \quad (10.3)$$

$$= \sum_n e^{-ip_n \cdot (x-y)} |\langle 0 | \phi(0) | n \rangle|^2 \quad (10.4)$$

where the sum here runs over all continuous and discrete indices (in the simplest case of ϕ^4 theory where internal states cannot carry any other indices this is just an integral over momenta of the possible internal states. More generally this can be spin, flavor, etc. of all the possible internal states.).

We can pretty this up a bit by inserting in a delta function,

$$\langle \phi(x) \phi(y)^\dagger \rangle = \sum_n \left[\int d^4 p \delta^{(4)}(p - p_n) \right] e^{-ip_n \cdot (x-y)} |\langle 0 | \phi(0) | n \rangle|^2 \quad (10.5)$$

Swapping around the sums and enforcing the delta function relation onto the p_n we find,

$$\langle \phi(x) \phi(y)^\dagger \rangle = \int d^4 p e^{-ip \cdot (x-y)} \sum_n \delta^{(4)}(p - p_n) |\langle 0 | \phi(0) | n \rangle|^2 \quad (10.6)$$

The sum over n is a scalar function and, in terms of the external states, can only depend on the four-vector, p^μ . Furthermore, if the external state has negative energy this should be equal to zero. Thus we can write,

$$\langle \phi(x)\phi(y)^\dagger \rangle = \int d^4p e^{-ip \cdot (x-y)} \frac{1}{(2\pi)^3} \theta(p^0) \rho(p^2) \quad (10.7)$$

where the added factor of $(2\pi)^3$ is conventional. ρ is known as the *spectral function*. Since ρ is equal to a sum over positive definite quantities, it is real and positive. Furthermore, introducing an additional delta function we can write,

$$\langle \phi(x)\phi(y)^\dagger \rangle = \int d\mu^2 \rho(\mu^2) \int \frac{d^4p}{(2\pi)^4} 2\pi \delta(p^2 - \mu^2) e^{-ip \cdot (x-y)} \theta(p^0) \quad (10.8)$$

This is exactly the free-field two-point function! Finally we have,

$$\langle \phi(x)\phi(y)^\dagger \rangle = \int d\mu^2 \rho(\mu^2) \Delta_+(x-y, \mu^2) \quad (10.9)$$

Now consider,

$$\langle \phi(y)^\dagger \phi(x) \rangle = \sum_n e^{-ip_n \cdot (y-x)} |\langle 0 | \phi(0)^\dagger | n \rangle|^2 \quad (10.10)$$

$$= \int d^4p e^{-p_n \cdot (y-x)} \sum_n \delta^{(4)}(p - p_n) |\langle 0 | \phi(0)^\dagger | n \rangle|^2 \quad (10.11)$$

$$= \int \frac{d^4p}{(2\pi)^3} e^{-p_n \cdot (y-x)} \bar{\rho}(p^2) \theta(p^0) \quad (10.12)$$

where $\bar{\rho}$ is the same as ρ with $\phi \rightarrow \phi^\dagger$. We conclude that we get the same result as above with $x \leftrightarrow y$ and $\rho \rightarrow \bar{\rho}$:

$$\langle \phi(y)^\dagger \phi(x) \rangle = \int d\mu^2 \bar{\rho}(\mu^2) \Delta_+(y-x, \mu^2) \quad (10.13)$$

To understand the relation between ρ and $\bar{\rho}$ consider the expectation value of the commutator,

$$\langle [\phi(x), \phi(y)^\dagger] \rangle = \int_0^\infty d\mu^2 (\rho(\mu^2) \Delta_+(x-y; \mu^2) - \bar{\rho}(\mu^2) \Delta_+(y-x; \mu^2)) \quad (10.14)$$

Now assume that $x-y$ is space-like. In this case $\Delta_+(z; \mu^2) \propto \frac{1}{\sqrt{z^2}} K_1(\mu\sqrt{z^2}) = \Delta_+(-z; \mu^2)$. Furthermore, causality demands that the commutator vanishes. Hence we require,

$$0 = \int_0^\infty d\mu^2 \Delta_+(x-y; \mu^2) (\rho(\mu^2) - \bar{\rho}(\mu^2)) \quad (10.15)$$

This can only be satisfied if $\bar{\rho} = \rho$. We could also have observed this directly from CPT invariance.

We are finally in a position to compute the 2-point function we are actually interested in, which corresponds to a particle travelling from x to y ,

$$\langle \mathcal{T} \phi(x) \phi(y)^\dagger \rangle = \theta(x^0 - y^0) \langle \phi(x) \phi(y)^\dagger \rangle + \theta(y^0 - x^0) \langle \phi(y)^\dagger \phi(x) \rangle \quad (10.16)$$

$$= \int d\mu^2 \rho(\mu^2) (\Delta_+(x - y; \mu^2) \theta(x^0 - y^0) + \Delta_+(y - x; \mu^2) \theta(y^0 - x^0)) \quad (10.17)$$

$$\equiv \int d\mu^2 \rho(\mu^2) \Delta_F(x - y; \mu^2) \quad (10.18)$$

where we have introduced the usual Feynman propagator for a free field. In momentum space this relation takes the form,

$$\int d^4x e^{-ip \cdot (x-y)} \langle \mathcal{T} \phi(x) \phi(y)^\dagger \rangle = \int d\mu^2 \rho(\mu^2) \frac{i}{p^2 - \mu^2 + i\epsilon} \quad (10.19)$$

Now lets consider the case where ϕ is a conventionally normalized (not renormalized) field. In this case we have the equal-time commutation relations,

$$[\partial_t \phi(\mathbf{x}, t), \phi^\dagger(\mathbf{y}, t)] = i\delta^{(3)}(\mathbf{x} - \mathbf{y}) \quad (10.20)$$

$$[\phi(\mathbf{x}, t), \partial_t \phi^\dagger(\mathbf{y}, t)] = -i\delta^{(3)}(\mathbf{x} - \mathbf{y}) \quad (10.21)$$

Taking the derivative of equation 10.13 and using the relation,

$$\left. \frac{\partial \Delta_+(x - y)}{\partial x^0} \right|_{x^0 \rightarrow y^0} = i\delta^{(3)}(\mathbf{x} - \mathbf{y}) \quad (10.22)$$

gives a unitarity condition on the spectral density,

$$\int_0^\infty \rho(\mu^2) d\mu^2 = 1 \quad (10.23)$$

We now make some general remarks,

- The full propagator must have a pole at $p^2 = m^2$. This requires $\rho(\mu^2)$ to take the form,

$$\rho(\mu^2) = Z\delta(\mu^2 - m^2) + \sigma(\mu^2) \quad (10.24)$$

where Z is independent of μ and σ is a known function without a pole at $\mu^2 = m^2$. The value of Z is the wavefunction renormalization of the field.

- Positivity of ρ requires that $\sigma(\mu^2) \geq 0$. Furthermore, unitarity of σ gives the condition:

$$1 = Z + \int_0^\infty \sigma(\mu^2) d\mu^2 \quad (10.25)$$

Since $\sigma \geq 0$, we conclude that $Z \leq 1$.

- Above we see that the propagator can't vanish faster than $1/p^2$ for $|p^2| \rightarrow \infty$. Indeed adding in higher order derivatives into the Lagrangian could induce a faster fall-off the propagator but this general argument shows that such a theory must violate either positivity of the states or causality.

Chapter 11

Broken Symmetries

11.1 SSB of Discrete Global Symmetries

Classical Analysis

The symmetries we have discussed so far have been exact. There are two types of symmetry breaking. Explicit symmetry breaking is fairly trivial and less interesting. Spontaneous symmetry breaking (SSB) plays a key role and nature and can be very subtle.

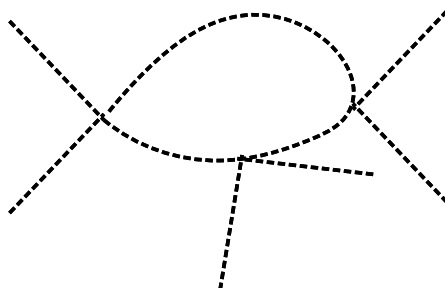
A field transformation is $\phi \rightarrow \Lambda\phi$ (where ϕ is a scalar or a fermion, $\Lambda \in \text{Rep}(G)$, and G can be a Lie or discrete group. Λ is x -independent, global or a function of x and local. It is a symmetry if

$$\mathcal{L} \rightarrow \mathcal{L} + \partial_\mu \mathcal{J}^\mu \quad (11.1)$$

As an example consider a local discrete symmetry. We take the Lagrangian to be given by

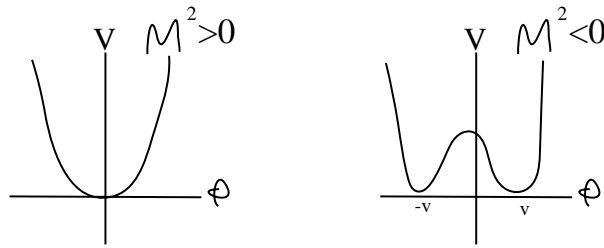
$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - V(\phi) \quad (11.2)$$

with $V(\phi) = \frac{\mu^2}{2}\phi^2 + \frac{\lambda}{4}\phi^4$. This theory is invariant under $\phi \rightarrow \phi' = -\phi$. This is a Z_2 symmetry. This theory cannot produce an odd number of external legs. This is clear diagrammatically:



An odd number of external legs is forbidden due to the Z_2 symmetry. This symmetry can be broken in two different ways.

1. This can be done explicitly by adding the term $\Delta V = \epsilon\phi^3$ to the Lagrangian. You can then do perturbation theory in ϵ . To order ϵ^0 Z_2 is a good symmetry.
2. Alternatively this can be broken spontaneously. We denoted the quadratic term as $\frac{\mu^2}{2}\phi^2$. However we don't really know the sign of μ^2 . The way to tell the sign of terms is by the requirement that the Hamiltonian ($\frac{1}{2}(\dot{\phi}^2 + (\nabla\phi)^2) + V(\phi)$) is bounded from below. To be bounded from below we must have $\lambda > 0$. However we can have $\mu^2 > 0$ or $\mu^2 < 0$. In either case we have



The first case is what we've done before. Consider $\mu^2 < 0$. Classically our system will settle down in the lowest energy state. $\frac{\delta V}{\delta \phi} = 0 \Rightarrow v \equiv \frac{-\mu^2}{\lambda}$. You don't know which point the system will fall to. If we lived in a universe where this was present and the field had enough time to evolve and settle at a lowest energy state. By measuring this field we could find whether our field will be in either v or $-v$. So even though the Lagrangian is symmetric about this symmetry, to an observer it will appear the system is not symmetric. Once you specify which configuration your system is in you have spontaneously broken the symmetry.

Quantization

In the normal case we quantize the field by finding the classical equations of motion:

$$\square\phi + \mu^2\phi = -\lambda\phi^3 \quad (11.3)$$

We first solve the system in the case of $\lambda = 0$:

$$\phi \sim \alpha_{\mathbf{k}}e^{-ik \cdot x} + \alpha_{\mathbf{k}}^*e^{ik \cdot x} \quad (11.4)$$

where $k^2 = \mu^2$. We then replace $\alpha_{\mathbf{k}} \rightarrow \hat{a}_{\mathbf{k}}$, $\hat{a}_{\mathbf{k}}^* = \hat{a}_{\mathbf{k}}^\dagger$. The vacuum is given by $\hat{a}_{\mathbf{k}}|0\rangle = 0$ for all \mathbf{k} . Hence we have

$$\langle 0|\hat{\phi}|0\rangle = 0 \quad (11.5)$$

It turns out that this approach doesn't really work when $\mu^2 < 0$. The reason is very simple. If you try to just solve this classically with $\mu^2 < 0$ then formally the solution still works but now $k^2 < 0$ which means that at least something in k must be imaginary. In particular it means that k^0 is imaginary, $k^0 = i\bar{k}$. Hence the solutions aren't oscillator but grow in time. i.e. $\phi \propto e^{kt}$. Physically this is because we are looking at an unstable system (when $\lambda = 0$) since the potential goes to negative infinity. We can't just throw

away λ because it's what stabilizes the potential. We cannot take it to be zero. The lesson is that we cannot just ignore this λ in the beginning.

The obvious thing to do is instead of ignoring λ which was basically equivalent to expansions around small perturbations around zero, what we need to do is expand about the actual minimum of the theory. This is the same thing you would do in classical mechanics. If you want to find the oscillations of a ball in such a potential well then you should expand about the minima.

Around the minima we have (from the Euler Lagrange equations)

$$\square\phi + \frac{\delta V}{\delta\phi} = 0 \quad (11.6)$$

$\phi_0 = +v$ or $\phi_0 = -v$. We have

$$\square\phi + \cancel{\frac{\delta V}{\delta\phi}} \Big|_{\phi_0} + \underbrace{\frac{\delta^2 V}{\delta\phi^2}}_{\equiv m^2} \Big|_{\phi_0} (\phi - \phi_0) + \underbrace{\frac{1}{2!} \frac{\delta^3 V}{\delta\phi^3}}_{\text{interactions}} \Big|_{\phi_0} (\phi - \phi_0)^2 + \dots = 0 \quad (11.7)$$

It's easy to show that all the interactions are proportional to λ . We define a new field $\xi(x) = \phi(x) - \phi_0$. So we have

$$\square\xi + m_\xi^2 \xi = 0 \quad (11.8)$$

up to $\mathcal{O}(\lambda)$. Our new field is ξ and it has solutions of

$$\xi \sim \alpha e^{-ik \cdot x} + \alpha^* e^{ik \cdot x} \quad (11.9)$$

with $k^2 = m_\xi^2$ where $m_\xi^2 = -2\mu^2$. The quantization is trivial. We do the same replacement we had before

$$\xi \sim \hat{a}_{\mathbf{k}} e^{-ik \cdot x} + \hat{a}_{\mathbf{k}}^\dagger e^{ik \cdot x} \quad (11.10)$$

We have the vacuum given by $\hat{a}_{\mathbf{k}} |0\rangle = 0$ for all \mathbf{k} and then $\langle 0|\hat{x}|0\rangle = 0$. However what is the condition based on our initial field? Recall that $\hat{\phi} = \phi_0 + \hat{\xi}$ and thus

$$\langle 0|\hat{\phi}|0\rangle = \phi_0 \quad (11.11)$$

This is called the vacuum expectation value (VEV) of the system. This is a huge difference in the physics! If for example the electric field (and hence A_μ) had a VEV there would be a background field everywhere which would mean if you had a charged particle it would be accelerated in some direction in space.

We can now create particles

$$\hat{a}_{\mathbf{k}}^\dagger |0\rangle = |\mathbf{k}\rangle \quad (11.12)$$

These are massive particles that obey $E^2 = \mathbf{k}^2 + m_\xi^2$.

You can insert the interactions in two different ways. One way is to rewrite the potential in terms of ξ :

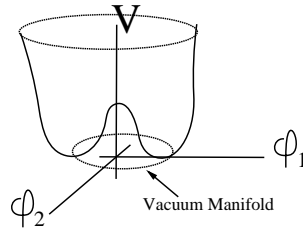
$$\begin{aligned} V(\phi) &= \frac{\mu^2}{2}\phi^2 + \frac{\lambda}{4}\phi^4 \\ &= \frac{1}{2}m_\xi^2\xi^2 + \underbrace{\lambda_3}_{\propto\lambda v}\xi^3 + \underbrace{\lambda_4}_\lambda\xi^4 \end{aligned} \quad (11.13)$$

Note that we have gained a cubic interaction out of a Lagrangian invariant under Z_2 .

11.2 SSB of Continuous Global Symmetry

Classical Analysis

Consider $V(\Phi) = \mu^2 |\Phi|^2 + \lambda |\Phi|^4$ where $\Phi = \frac{\phi_1 + i\phi_2}{\sqrt{2}}$. The symmetry of this Lagrangian is $\Phi \rightarrow e^{i\alpha}\Phi$ where α is a constant. If $\mu^2 > 0$ then the VEV is at $\Phi = 0$. If $\mu^2 < 0$ then we have



and $\frac{\delta V}{\delta \Phi} = \frac{\delta V}{\delta \Phi^*} = 0$ which gives $|\Phi|^2 = \frac{-\mu^2}{\lambda}$. Note that before we had two minima but now we have an infinite number of vacua. All the different vacua are known as the vacuum manifold. We can get all the vacuums by applying the transformations on the VEV:

$$\Phi_\alpha = e^{i\alpha} \frac{1}{\sqrt{2}} \sqrt{\frac{-\mu^2}{\lambda}} = e^{i\alpha} \Phi_0 \quad (11.14)$$

This is a very general property of spontaneous breaking. We have spontaneous breaking of $U(1)$.

Quantization

Our equation of motion are

$$\square\phi_1 + \frac{\delta V}{\delta\phi_1} = 0 \quad (11.15)$$

$$\square\phi_2 + \frac{\delta V}{\delta\phi_2} = 0 \quad (11.16)$$

Due to the symmetry the physical results will not depend on the vacuum we choose. We choose to expand about $\phi_1 = v$ and $\phi_2 = 0$. The equations of motion give

$$\cancel{\square\phi_1 + \frac{\delta V}{\delta\phi_1}\Big|_{(\phi_1,\phi_2)=(v,0)}} + \frac{\delta^2 V}{\delta\phi_1^2}\Big|_{(v,0)} \left(\phi_1 - \frac{v}{\sqrt{2}}\right) + \frac{\delta^2 V}{\delta\phi_1\delta\phi_2}\Big|_{(v,0)} + \cancel{\left(\text{interaction terms}\right)} = 0 \quad (11.17)$$

$$\square\xi + \frac{\delta^2 V}{\delta\phi_1^2}\Big|_{(v,0)} \xi + \frac{\delta^2 V}{\delta\phi_1\delta\phi_2}\Big|_{(v,0)} \phi_2 = 0 \quad (11.18)$$

in terms of $\xi \equiv \phi_1 - \frac{v}{\sqrt{2}}$. We also have a second equation:

$$\square\phi_2 + \frac{\delta^2 V}{\delta\phi_2\delta\phi_1}\Big|_{(v,0)} \xi + \frac{\delta^2 V}{\delta\phi_2^2}\Big|_{(v,0)} \phi_2 = 0 \quad (11.19)$$

We define a mass matrix: $m_{ij}^2 \equiv \frac{\delta^2 V}{\delta\phi_i\delta\phi_j}\Big|_{(v,0)}$. We can put our equations together:

$$\square \begin{pmatrix} \xi \\ \phi_2 \end{pmatrix} + M \begin{pmatrix} \xi \\ \phi_2 \end{pmatrix} = 0 \quad (11.20)$$

For our example we have $V = \frac{\mu^2}{2} (\phi_1^2 + \phi_2^2) + \frac{\lambda}{4} (\phi_1^2 + \phi_2^2)^2$. We can take the derivatives:

$$\frac{\delta^2 V}{\delta\phi_1^2} = \mu^2 + 3\lambda v^2 = -2\mu^2 \quad (11.21)$$

$$\frac{\delta^2 V}{\delta\phi_1\delta\phi_2} = 2\lambda\phi_1\phi_2 \xrightarrow{(\phi_1,\phi_2)=(v,0)} 0 \quad (11.22)$$

Lastly

$$\frac{\delta^2 V}{\delta\phi_2^2} = \mu^2 + \frac{\lambda}{2} + \lambda\phi_1^2 + 3\lambda\phi_2^2 \xrightarrow{(\phi_1,\phi_2)=(v,0)} 0 \quad (11.23)$$

Hence not only is the mass matrix diagonal but one of the eigenvalues is zero. In the language of quantum field theory the $\frac{\delta^2 V}{\delta\phi_i}$ become masses. We have two particles with masses $m_\xi^2 = -2\mu^2$ and $m_{\phi_2} = 0$. The massless particles are known as Goldstone bosons. This mass also turns out to be included in the quantum corrections.

The existence of the particle is easy to understand classically. Consider a mechanical system at the point near $(v, 0)$ (see diagram of potential above). It's clear that there is one direction along ϕ_1 that there is a restoring force. However we can also move along ϕ_2 along the vacuum manifold and not change the energy and hence there is no restoring force. In the language of oscillator that means you have an oscillator with zero frequency. The fact that the potential is flat was a consequence of the theory. This just follows from the symmetry of the theory. Since the symmetry holds to all orders quantum corrections do not change the mass of the Goldstone boson.

11.3 Linear Sigma Model

Now that we know how this works in the $U(1)$ case we will generalize these ideas in what's known as the linear sigma model. We start with N real fields, ϕ_i :

$$\mathcal{L} = \sum_{i=1}^N \frac{1}{2} (\partial_\mu \phi_i)^2 + V(\phi_i) \quad (11.24)$$

We consider a potential that is invariant under rotation of the fields about themselves: $\phi_i = R_{i,j} \phi_j$. Where R is orthogonal: $R^T = R^{-1}$. The only way this can occur is if the potential only depends on the magnitude of the vector

$$\boldsymbol{\phi} = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_N \end{pmatrix}$$

We take

$$V(\boldsymbol{\phi}) = \mu^2 \boldsymbol{\phi}^2 + \lambda \phi_4 \quad (11.25)$$

$$= \mu^2 \sum_i \phi_i^2 + \lambda \left(\sum_i \phi_i^2 \right)^2 \quad (11.26)$$

As before we take $\mu^2 < 0$. The vacuum manifold is given by $\boldsymbol{\phi}^2 = \frac{-\mu^2}{\lambda}$. The manifold is one dimension lower than the number of degrees of freedom in the system S_{N-1} . We pick a vacuum

$$\boldsymbol{\phi} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ v \end{pmatrix}$$

We define $\sigma(x) = \phi_N(x) - v$ and we get

$$V(\phi) = \frac{1}{2} (-2\mu^2) \sigma^2 + \left(\begin{array}{c} \text{cubic and} \\ \text{quartic} \\ \text{terms} \end{array} \right) \quad (11.27)$$

The key point is that we don't have any mass terms except for σ so we have $N - 1$ Goldstone bosons. We could have predicted how many Goldstone bosons will appear by just thinking about the symmetry of the theory. We can express the symmetry of the Lagrangian as $R = e^{i\alpha_i T_i}$ where T_i are the generators of $O(N)$. The dimensionality of the $O(N)$ group is $\frac{N(N-1)}{2}$. This comes from the condition that T_i 's need to be antisymmetric. Consider

$$R\phi_0 = (1 + i\alpha_i T_i + \dots)\phi_0 \quad (11.28)$$

We have two possibilities,

$$\begin{aligned} T_i \phi_0 = 0 & \quad \text{Unbroken gen.} \\ T_i \phi_0 \neq 0 & \quad \text{Broken gen.} \end{aligned}$$

The T_i matrices take the form

$$T_i = \begin{pmatrix} 0 & & & & \\ & \ddots & & & \\ & & \ddots & & \\ & & & \ddots & 1 \\ & & & -1 & 0 \end{pmatrix} \quad (11.29)$$

The number of broken generators is equal to the number of Goldstone bosons. This is called Goldstone theorem. We did two examples but you can do any group you want. In nature we don't think anything like this is happening exactly. But in QCD there are a lot of combinations of quarks. The pions for example are much lighter than the other particles. You can discuss the physics of pions as Goldstone bosons. You can add explicit breaking to give them some mass to correct that they are not massless. In many beyond the SM theories people speculate that different particles are approximate Goldstone bosons.

11.4 More on the Mexican Hat Potential

We now go back to the original Mexican Hat potential, $V(\Phi) = \mu^2 |\Phi|^2 + \lambda |\Phi|^4$, with $\mu^2 = -\lambda v^2 < 0$ and $\Phi \equiv \frac{\phi_1 + i\phi_2}{\sqrt{2}}$. If this were a mechanics problem then it would be clear that it is more convenient to use fields that obey the symmetry of the problem. In this case this would be a degree of freedom that represents the angle. There is nothing that stops you from doing this in Quantum Field Theory.

With this in mind we introduce a non-linear field redefinition

$$\Phi = e^{i\pi(x)/v} \frac{1}{\sqrt{2}} (v + \sigma(x)) \quad (11.30)$$

Our potential becomes

$$\begin{aligned} V(\Phi) &= \frac{\mu^2}{2} (v + \sigma)^2 + \frac{\lambda}{4} (v + \sigma)^4 \\ &= \frac{1}{2} m_\sigma^2 \sigma^2 + \lambda_3 \sigma^3 + \lambda_4 \sigma^4 \end{aligned} \quad (11.31)$$

where we drop the unimportant constant term and the linear term disappeared since

$\frac{\delta V}{\delta \phi} = 0$. The kinetic term is given by

$$\begin{aligned} |\partial_\mu \Phi|^2 &= \frac{1}{2} \left| i\partial_\mu \pi \left(1 + \frac{\sigma}{v}\right) + \partial_\mu \sigma \right|^2 \\ &= \underbrace{\frac{1}{2}(\partial_\mu \sigma)^2 + \frac{1}{2}(\partial_\mu \pi)^2}_{\text{can. normalization}} + \underbrace{\frac{1}{v}(\partial_\mu \pi)^2 \sigma + \frac{1}{2v^2}(\partial_\mu \pi)^2 \sigma^2}_{\text{interactions}} \end{aligned}$$

Notice that the interaction terms have couplings with dimensions of inverse mass! These interactions appear non-renormalizable. However that is very naive and power counting is not sufficient to test Renormalizability. This theory is still renormalizable (it is completely equivalent to the original ϕ^4 theory).

Further notice that π only enters the Lagrangian in terms of its derivative. Our $U(1)$ symmetry acts on π and σ in a very simple way:

$$\pi \rightarrow \pi + \alpha \quad \text{“shift”} \quad (11.32)$$

$$\sigma \rightarrow \sigma \quad (11.33)$$

To have an invariant Lagrangian we can't have any dependence on π , since π is not invariant. For example if we had a $a\pi^2$ term we could apply a shift and get $a(\pi + \alpha)^2$ which differs from the first Lagrangian. Hence we can only have (as we see above), $\mathcal{L}[\sigma, \partial_\mu \pi]$.

In the original theory renormalizability was clear and here the symmetry is clear.

Further we have

$$\langle \Omega | T(\tilde{\pi}(p) \dots) | \Omega \rangle \propto p^\mu \cdot (\dots) \quad (11.34)$$

11.5 SSB of Gauge Symmetries (“The Higg’s Mechanism”)

Abelian Case

Today we will consider the Abelian example of SSB. Later we will the more intricate SM $SU(2) \times U(1) \rightarrow U(1)$ mechanism. We have the same potential as before only now we consider a gauge symmetry:

$$\mathcal{L} = |D_\mu \phi|^2 - V(\Phi) - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \quad (11.35)$$

where $V(\Phi) = -\mu^2 |\Phi|^2 + \lambda |\Phi|^4$ (we make the minus sign on μ^2 explicit such that now $\mu^2 > 0$) and $D_\mu = \partial_\mu + ieA_\mu$. Our VEV is

$$v^2 = \frac{\mu^2}{\lambda} \quad (11.36)$$

and our vacuum is $\Phi_\alpha = e^{i\alpha} \frac{v}{\sqrt{2}}$ and $A_\mu = 0$.

We’ll start by solving this problem heuristically and follow up with a more rigorous derivation. We fix Φ to it’s vacuum value, $\Phi_0 = \frac{v}{\sqrt{2}}$. This gives

$$\mathcal{L}_{\Phi \text{ kin}} = |D_{\mu}\Phi|^2 = \left| ieA_{\mu}\frac{v}{\sqrt{2}} \right|^2 = \underbrace{\frac{e^2v^2}{2}A_{\mu}A^{\mu}}_{\text{mass term}} \quad (11.37)$$

We got a mass term for the gauge field, $m_A = ev$. When we first began talking about QED we rejected these terms because they were not gauge invariant ($A_{\mu} \rightarrow A_{\mu} - \frac{1}{e}\partial_{\mu}\alpha$). We started with a theory that’s completely gauge invariant by shifting the Φ field, $\Phi \rightarrow e^{i\alpha(x)}\Phi$ spontaneously broke gauge invariance. Picking the vacuum.

We now attack this problem more carefully. We begin by shifting our field:

$$\Phi = e^{i\pi/v}\frac{1}{\sqrt{2}}(v + \sigma(x)) \quad (11.38)$$

This shift gives

$$\begin{aligned} |D_{\mu}\phi|^2 &= \frac{1}{2}\left|\partial_{\mu} + \pi\left(1 + \frac{\sigma}{v}\right) + \partial_{\mu}\sigma + ieA_{\mu}(v + \sigma)\right|^2 \\ &= \frac{1}{2}(\partial_{\mu}\sigma)^2 + \frac{1}{2}(\partial_{\mu}\pi + evA_{\mu})^2\left(1 + \frac{\sigma}{v}\right)^2 \end{aligned} \quad (11.39)$$

This is the same kinetic term as before except for the extra gauge field terms:

$$ev\partial_{\mu}\pi A^{\mu} + \frac{1}{2}e^2v^2A^{\mu}A_{\mu} \quad (11.40)$$

Note that 11.39 is still manifestly gauge invariant since $\sigma \rightarrow \sigma, \pi \rightarrow \pi + \alpha$ and $A_{\mu} \rightarrow A_{\mu} - \frac{1}{e}\partial_{\mu}\alpha$.

The second mass term we knew would be there based on our earlier heuristic arguments. However there is a new interesting term, $ev\partial_{\mu}\pi A^{\mu}$. This term is very strange. This leads to propagators starting as one field and turning into another. This means that we are probably not using a good basis. There is probably more then one way to solve this problem and understand what the physics is. We will use the standard solution. The idea is instead of trying to redefine fields, we have another approach. Recall that in QED $\mathcal{L}_{\text{gauge fix}} = -\frac{1}{2\xi}(\partial_{\mu}A^{\mu})^2$. This allowed us to compute a propagator. This is a remnant of the choice we made that said that $G[A] = \partial_{\mu}A^{\mu} = 0$. This was a good choice but it was not unique.

Having said that, with theories with spontaneously broken symmetries we use what’s known as R_{ξ} gauge. Our gauge fixing Lagrangian is

$$\begin{aligned} \mathcal{L}_{\text{gauge fix}} &= -\frac{1}{2\xi}(\partial_{\mu}A^{\mu} - iv\xi\pi)^2 \\ &= -\frac{1}{2\xi}(\partial_{\mu}A^{\mu})^2 + ev\pi(\partial_{\mu}A^{\mu}) - \frac{e^2v^2\xi}{2}\pi^2 \\ &= -\frac{1}{2\xi}(\partial_{\mu}A^{\mu})^2 - ev(\partial_{\mu}\pi)A^{\mu} - \frac{e^2v^2\xi}{2}\pi^2 \end{aligned} \quad (11.41)$$

where we have integrated by parts. The middle term cancels the term we didn't like $ev\partial_\mu\pi A^\mu$ in equation 11.40. However there is a new strange mass term for the π field. Based on physical arguments we there should not be any mass for π .

We have three propagators:

$$\begin{aligned} \mu \text{---} \text{wavy} \text{---} \nu &= \frac{-i}{k^2 - m_A^2} \left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2 - \xi m_A^2} (1 - \xi) \right) \\ \pi \text{---} &= \frac{1}{p^2 - m_A^2} \\ \sigma \text{---} &= \frac{1}{p^2 - m_\sigma^2}, \quad m_\sigma^2 = 2\mu^2 \end{aligned}$$

Note that we have two poles in the gauge boson propagator. One makes sense since it corresponds to that we have a massive gauge boson. However we have a second gauge dependent pole. This gauge dependent pole is cancelled by the π field. The propagators always come together:

$$\begin{array}{c} \diagup \text{---} \text{wavy} \text{---} \diagdown \\ \text{---} A \text{---} \\ \diagdown \text{---} \text{wavy} \text{---} \diagup \end{array} + \begin{array}{c} \diagup \text{---} \text{dashed} \text{---} \diagdown \\ \text{---} \pi \text{---} \\ \diagdown \text{---} \text{dashed} \text{---} \diagup \end{array} \rightarrow \text{gauge independent}$$

π cannot be real fields. We cannot have diagrams with external π fields.

To understand our system better we introduce Unitary gauge, $\xi \rightarrow \infty$. In this limit π drops out of the theory completely. The photon propagator becomes

$$\mu \text{---} \text{wavy} \text{---} \nu \rightarrow \frac{-i}{k^2 - m_A^2} \left(g_{\mu\nu} - \frac{k_\mu k_\nu}{m_A^2} \right) \quad (11.42)$$

To understand the particle we consider the case of the particle on shell:

$$\mu \text{---} \text{wavy} \text{---} \nu \rightarrow k^2 \rightarrow m_A^2 \frac{-i}{k^2 - m_A^2} \sum_{s=1}^3 \epsilon_\mu^s(p) \epsilon_\nu^{*s}(p) \quad (11.43)$$

We started with the theory of the photon that has two degrees of freedom and a complex scalar field with two degrees of freedom. The overall number of degrees of freedom was $2 + 2 = 4$. After the symmetry is broken the picture changes, the gauge boson gets a mass and has three degrees of freedom while the σ is a real scalar field and has 1 degree of freedom. So we still have $3 + 1 = 4$ degrees of freedom. The gauge boson "ate" the π field and became massive.

Non-Abelian Case

Consider a scalar field, Φ , which is charged under some non-Abelian gauge symmetry, G . In other words that

$$\Phi \rightarrow \Phi' = \Lambda \Phi \quad (11.44)$$

where $\Lambda^\dagger \Lambda = 1$ and $\Lambda = e^{i\alpha_a(x)T^a}$ ($a = 1 \dots N_g$) with $N_g = \dim(\text{adj}(G))$. Our Lagrangian becomes

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^a F^{a,\mu\nu} + |D_\mu \Phi|^2 - V(\Phi) \quad (11.45)$$

where $D_\mu = \partial_\mu - igA_\mu^a T^a$. The vacuum state (Φ_0) is where

$$\left. \frac{\delta V}{\delta \Phi_i} \right|_{\Phi_0} = 0 \quad (11.46)$$

If Φ_0 is a solution then $\Phi_\alpha = e^{i\alpha^a T^a} \Phi_0$ is also a solution. There are two possibilities. Either we have $T^b \Phi_0 \neq 0$ or $T^a \Phi_0 = 0$. The first case corresponds to the broken generators while the second corresponds to unbroken generators. We N_b such that $b = 1, \dots, N_b$ then the vacuum manifold is an N_b -dimensional space. Each broken generator gives a Goldstone boson.

Just as we did last time parametrize our field non-linearly as

$$\Phi = e^{i\Pi^b T^b/v} \Phi_0 [(\sigma - \text{fields})] \quad (11.47)$$

The gauge transformation gives $\pi^b(x) \rightarrow \pi^b + \alpha^b(x)$, and $\sigma \rightarrow \sigma$. Expanding about the vacuum, Φ_0 gives a term

$$|D_\mu \Phi|^2 \rightarrow g^2 \sum_{a,b=1}^{N_b} A_\mu^a A^{b,\mu} (T^a \Phi_0)^\dagger (T^b \Phi_0) \quad (11.48)$$

Note that the sum is over broken generators since the unbroken ones gives 0. We now examine this term more closely. Claim: $T^a \Phi_0$ (where a runs over unbroken generators) are eigenvectors of the mass matrix $M_{i,j}^2 \equiv \left. \frac{\delta^2 V}{\delta \Phi_i \delta \Phi_j} \right|_{\Phi_0}$ with zero eigenvalue. Proof: Suppose that $\Phi \rightarrow e^{i\alpha_a T^a} \Phi$ is a symmetry. Then if we consider infinitesimal α such that

$$\begin{aligned} \Phi &\rightarrow \Phi + \delta\Phi \\ \delta\Phi_i &= \alpha^a (T^a \Phi)_i \end{aligned}$$

Since V must be invariant under this transformation (by assumption) then

$$\delta V = \frac{\delta V}{\delta \Phi_i} \delta\Phi_i = 0$$

which implies that

$$\alpha^a (T^a \Phi)_i \frac{\delta V}{\delta \Phi_i} = 0$$

Taking the derivative of both sides gives:

$$\alpha^a \frac{\delta}{\delta \Phi_j} (T^a \Phi)_i \frac{\delta V}{\delta \Phi_i} + \alpha^a (T^a \Phi)_i \frac{\delta^2 V}{\delta \Phi_i \delta \Phi_j} = 0$$

Plugging in $\Phi = \Phi_0 \frac{\delta V}{\delta \Phi_i} \Big|_{\Phi_0}$ gives

$$(T^a \Phi_0)_i \frac{\delta^2 V}{\delta \Phi_i \delta \Phi_j} \Big|_{\Phi_0} \quad (11.49)$$

as required.

The Mass matrix, $M^{i,j}$ is Hermitian, so eigenvectors can be chosen to be orthonormal:

$$(T^a \Phi_0)^\dagger (T^b \Phi_0) \propto v^2 \delta^{ab} \quad (11.50)$$

which implies that $g^2 v^2 \sum_b A_\mu^b A^{a,\mu}$ is a mass term for the gauge bosons. The number of massive gauge bosons is equal to the number of broken generators. We don't have any Goldstone bosons left (all have been "eaten"). The masses are $m^2 \propto |T^a \Phi_0|^2$.

As an example we consider $SU(2)$. We define Φ is a fundamental "spin-1/2" representation, $\Phi = \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix}$. The potential is given by $V = -\mu^2 \Phi^\dagger \Phi + \lambda (\Phi^\dagger \Phi)^2$ where $\Phi^\dagger \Phi = |\Phi_1|^2 + |\Phi_2|^2$. The Lagrangian is invariant under $\Phi \rightarrow \Lambda \Phi$ where $\Lambda_\alpha = e^{-i\alpha \cdot \sigma/2}$. The vacuum is given by

$$\Phi_\alpha = \Lambda_\alpha \begin{pmatrix} 0 \\ v/\sqrt{2} \end{pmatrix} \quad (11.51)$$

where $v^2 = \frac{\mu^2}{\lambda}$.

We have

$$\sigma^a \Phi_\alpha \neq 0 \quad (11.52)$$

for $a = 1, 2, 3$ so we have three broken generators and hence 3 Goldstone bosons. Our three gauge bosons are

$$\left| \frac{\sigma_a}{2} \begin{pmatrix} 0 \\ v/\sqrt{2} \end{pmatrix} \right|^2 = \frac{v^2}{8} \quad (11.53)$$

for all $a = 1, 2, 3$ which implies that $m_A^2 = \frac{g^2 v^2}{4}$.

This is very similar to the situation in the Standard Model, except in that case we have an extra $U(1)$ symmetry. In the Standard Model we see weak interactions which give masses of $m_W \sim 80\text{GeV}$, $m_Z = 90\text{GeV}$ and only left handed fermions feel the weak interaction. The Lagrangian is given by

$$\mathcal{L} = i\psi_L^\dagger (\partial_\mu - igW_\mu^a t^a) \bar{\sigma}^\mu \psi_L + i\psi_R^\dagger \partial_\mu \sigma^\mu \psi_R \quad (11.54)$$

where $\psi_L = \begin{pmatrix} \psi_{L,1} \\ \psi_{L,2} \end{pmatrix}$.

The naive guess for the mass terms (the Dirac mass terms) is $m (\psi_{L,a}^\dagger \psi_R + \psi_R^\dagger \psi_{L,a})$. However this is not gauge invariant! We need the Higg's mechanism to fix this problem.

We have $H = \begin{pmatrix} H^* \\ H^0 \end{pmatrix}$. The vacuum is given by

$$\langle H^0 \rangle = \frac{v}{\sqrt{2}} \quad (11.55)$$

which implies that $m_W = \frac{gv}{\sqrt{2}}$. The Yukawa couplings are given by

$$\mathcal{L}_{Yuk} = \lambda \left(\psi_L^\dagger H \psi_R + \psi_R^\dagger H^\dagger \psi_L \right) = \underbrace{\frac{\lambda v}{\sqrt{2}}}_{m_F} \left(\psi_{L,1}^\dagger \psi_R + \psi_R^\dagger \psi_{L,1} + \dots \right) \quad (11.56)$$

so but adding the Yukawa couplings to the Higgs field and give it a VEV we immediately get fermion masses. Note that to derive this formula we require the transformation

$$H = e^{i\pi^a t^a / v} \begin{pmatrix} 0 \\ \frac{v}{\sqrt{2}} + \sigma \end{pmatrix} \text{ where } \sigma \text{ is the physical Higg's boson.}$$

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