PARTICLE PHYSICS II
LECTURE NOTES

Lecture notes are largely based on a lectures series given by Yuval Grossman at Cornell University supplemented with by my own additions.

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Chapter 1

Preface

This lecture notes are based on a course given by Yuval Grossman at Cornell University. If you have any corrections please let me know at ajd268@cornell.edu.

Chapter 2

Introduction

High energy physics can be summarized with the question,

$$\mathcal{L} = ? \tag{2.1}$$

In theory you could start with the Lagrangian and get all the information that you need from a theory. To get the Lagrangian you need the following axioms,

1. Gauge symmetry

2. Irreducible representation of fermions and scalars

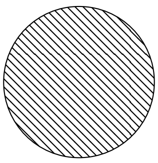
3. The spontaneous symmetry breaking pattern of the model (e.g. in the SM its in the input that the μ^2 parameter is negtative)

4. $\mathcal{L}$ is the most general Lagrangian that is renormalizable,

$$\mathcal{L} = \Lambda^4 \mathcal{O}_0 + \Lambda^3 \mathcal{O}_1 + \Lambda^2 \mathcal{O}_2 + \Lambda \mathcal{O}_3 + \mathcal{O}_4 + \frac{\mathcal{O}_5}{\Lambda} + \frac{\mathcal{O}_6}{\Lambda^2} \tag{2.2}$$

   where the linear term is always eliminated by a field redefinition and the constant term is gives rise the problem known as the cosmological constant problem.

Note that we don't typically impose global symmetries. The reason we only impose gauge symmetries is that we think that any global symmetry will be broken by gravity. The arguement is as follows. Consider a black hole,



A black hole obeys a no hair theorem. This theorem says that if you have something and you throw it into a black hole it will dissappear and have no memory of it. This arguement fails in the case of a gauge field. The difference between a gauge symmetry and a global symmetry is that a gauge symmetry has field lines. These field lines extend to infintiy. Hence a black hole can't "eat" the entire gauge charge. So if you have throw a charge of $1C$ into the black hole, the only memory that the black hole will have of this process is the charge. Therefore you can have,

$$p + BH \rightarrow e^{+} + \gamma \tag{2.3}$$

Electric charge must be conserved, but not global charges (such as baryon number), can't be.

Every model you have will have a set of free parameters. For example there are 18 free parameters in the SM (19 included θ_{QCD}, but since it appears to be zero we neglect this parameter). In principle these 18 parameters can be anything. The value of these paramters is not well defined since we can't measure bare quantities in the Lagrangian. However, you can measure the physical parameters. These are the parameters you can measure in a lab. Once all these 18 parameters are measured we can then start making predictions.

2.1 The Standard Model

The gauge symmetry of the SM is

$$SU(3)_c \times SU(2)_L \times U(1)_Y \tag{2.4}$$

The fermions are given by,

particle	rep
Q_L	$(3,2)_{1/6}$
U	$(3,1)_{2/3}$
D	$(3,1)_{-1/3}$
L	$(1,2)_{-1/2}$
E	$(1,1)_{-1}$
ϕ	$(1,2)_{-1/2}$
$\tilde{\phi} \equiv i\sigma_2\phi^*$	$(1,2)_{1/2}$

The spontaneous symmetry breaking (SSB) pattern is specified by the Higgs potential,

$$V(\phi) = \mu^2 \left|\phi\right|^2 + \lambda \left|\phi\right|^4 \tag{2.5}$$

μ^2 parameter being less than zero. The most general Lagrangian is given by as follows:

$$\mathcal{L}_{SM} = \mathcal{L}_{kin} + \mathcal{L}_{Higgs} + \mathcal{L}_{Yuk} \tag{2.6}$$

To get the kinetic term we make the modification,

$$\bar{\psi}\partial\!\!\!/\psi \rightarrow \bar{\psi}D\!\!\!\!/\psi \tag{2.7}$$

where the covariant derivative takes the form,

$$D^{\mu} = \partial^{\mu} + ig_s G_a^{\mu} L_a + igW_b^{\mu} T_b + ig' B^{\mu} Y \tag{2.8}$$

where,

$$\begin{aligned} &L_a \to \text{Gellman matrices} \\ &T_b \to \text{Pauli matrices} \\ &Y \to \text{number} \end{aligned}$$

where for the Higgs we have,

$$|D_{\mu}\phi|^2 \tag{2.9}$$

The gauge bosons G_a^{μ} are the gluons, W_b^{μ} and B^{μ} are mixtures of the electroweak bosons, $W^{\pm\mu}, Z^{\mu}, A^{\mu}$. Note that the kinetic term has 3 free parameters. These parameters are real as a consequence of requiring the Lagrangian to be real.

The Higgs potential is,

$$\mathcal{L}_{Higgs} = \mu^2 |\phi|^2 + \lambda |\phi|^4 \tag{2.10}$$

where we don't have a cubic term due to $SU(2)$ invariance. Furthermore, note that we take $\lambda > 0$ in order to require that a bounded potential. In principle one can instead use nonrenormalizable interactions to stabilize the potential, though this is not the case in the SM.

Another way to write the Higgs potential is to write it as,

$$\lambda \left|\phi^2 - v^2\right|^2 \tag{2.11}$$

These two are equivalent up to a constant. In the first method our two parameters are λ, μ^2 and in the second way the two parameters are λ, v.

Note that the μ^2 (or v) parameter are the only dimensionful parameter in the SM. Its important to understand that this is only one scale in the SM Lagrangian. This makes the theory not conformally invariant. This is actually an oversimplification we also have the scale Λ_{QCD}. However, Λ_{QCD} is scale with a very different and somewhat strange origin. If you took the SM without the Higgs then it would have no scales in it, but it would still have a scale such that the coupling runs to 1 (this is known as the Landau pole). The process of getting this scale is known as "dimensional transmutation".

The idea is as follows. If you look at the classical theory with no dimensionful couplings you would never be able to generate a scale. The only reason you generate a scale in such theories is because of the renormalization group (RG) equations. Because the RG equations always have an associated scale, from the β functions defined as,

$$\beta = M \frac{dg}{dM} \, . \tag{2.12}$$

The running generates a new dimensionful scale (where M is the renormalizable scale). In QCD the scale is generated in the infrared. So in the SM we really have two scales,

v and Λ_{QCD}. These two scales differ by about a factor of 100. Why these two scales of very different origin differ by only a factor of 100 is a mystery.

Lastly we have the Yukawa interactions,

$$\mathcal{L}_{Yuk} = Y_D \bar{Q}_L \phi D_R + Y_U \bar{Q}_L \tilde{\phi} U_R + Y_E \bar{L}_L \phi \tilde{E}_R \tag{2.13}$$

There turns out to be just 13 independent parameters in total for these 3×3 matrices.

The (exact to dimension four) accidental symmetries of the SM are,

$$U(1)_B \times U(1)_e \times U(1)_\mu \times U(1)_\tau \tag{2.14}$$

2.2 Spontaneous symmetry breaking

$$Q = T_3 + Y \tag{2.15}$$

which gives the masses for the gauge bosons,

$$m_W^2 = \frac{g^2 v^2}{4} \quad m_Z^2 = \frac{1}{2}(g^2 - g'^2)v^2 \quad m_A^2 = 0 \tag{2.16}$$

Then we define the Weinberg angle,

$$\tan\theta_w = \frac{g'}{g} \tag{2.17}$$

This is basically a basis rotation,

$$\begin{pmatrix} Z \\ A \end{pmatrix} = \begin{pmatrix} c_w & -s_w \\ s_w & c_w \end{pmatrix} \begin{pmatrix} W_3 \\ B \end{pmatrix} \tag{2.18}$$

After SSB we have,

$$\begin{aligned} m_d &= Y_d v \\ m_u &= Y_u v \\ m_e &= Y_e v \\ m_\nu &= 0 \end{aligned}$$

Before SSB we really can't tell an electron from a neutrino. This is the case in the same way that we can't tell a red quark from a blue quark. Only because of SSB we can't tell up and down or electron from neutrino.

Another important point is that in the SM all the fermions are massless prior to SSB. This is unique the SM. In other BSM theories this is not the case and we can write such tree level mass terms.

Now lets further examine the masses of the gauge bosons. We define a parameter called the "rho" parameter,

$$\rho = \frac{m_W^2}{m_Z^2 \cos\theta_w} \tag{2.19}$$

The SM predicts that this is equal to 1. The electroweak gauge sector is described by 4 parameters,

$$\{g, g', \lambda, v\}$$

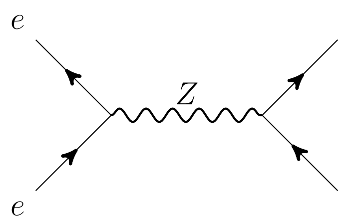
The ρ relation has nothing to do with the self coupling of the Higgs, λ, therefore it only depends on the 3 parameters,

$$\{g, g', v\} \tag{2.20}$$

Instead of using this parameter set we prefer to use a set that is really well measured,

$$\{\alpha, G_F, m_Z\} \tag{2.21}$$

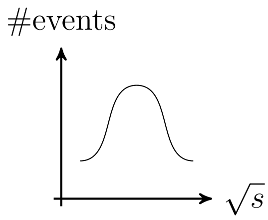
α is measured from atomic physics, G_F is measured using the muon lifetime ($\Gamma_\mu \sim \frac{m_\mu^5 G_F^2}{3 \cdot 2^6 \cdot \pi^3}$), and the mass of the Z boson is measured at LEP through,



which has the propagator,

$$\frac{1}{\left|q^2 - M_Z^2 - i\Gamma M_Z\right|^2} \tag{2.22}$$

We measure,



In fact the peak is shifted from the peak of the Lorentzian because there are other diagrams such as the photon channel which shift the peak.

The way we can think of the $\rho = 1$ measurement is that we take the three inputs,

$$\{\alpha, G_F, m_Z\} \tag{2.23}$$

Once these three parameters are measured they can be used to calculate ρ. They must give $\rho = 1$ to verify the SM.

The coupling of the photon is,

$$-\mathcal{L}_\alpha = (eq)\bar{\psi}\not{A}\psi \tag{2.24}$$

where e is the gauge coupling, the fundamental parameter in α and q is the representation under electromagnetism, $T_3 + Y$. We can note that following,

1. This is a vectorial interaction (it is parity conserving) such it is governed by a γ_μ.
2. The coupling is flavor diagonal (there are no couplings e.g. between the muon and the electron)
3. Typically we define the q of the electron to be -1. However, there is no absolute normalization. We can always multiply e by 2 and divide q by 2 and nothing will change. We cannot do this for the nonabelian case because there it has self coupling which forces a normalization of the generator and fixes q.

The W boson Lagrangian is,

$$-\mathcal{L}_W \propto \frac{g}{\sqrt{2}}\bar{\nu}_L \not{W} \ell_L^- \tag{2.25}$$

1. We only have left handed coupling (i.e. $V-A$ coupling with $\gamma_\mu - \gamma_\mu\gamma_5$). This breaks parity invariance.
2. This interactions isn't diagonal like the photon interaction.
3. Can we build a model where right handed fields couple to the W? Of course not, since by definition right handed fields are $SU(2)_L$ singlets.
4. This interaction is flavor conserving.
5. We have universality. This is a consequence of the fermions being in the same representations of the gauge group

The Z boson interaction is,

$$\mathcal{L}_Z = \frac{g'}{s_w}\left[T_3 - Q s_w^2\right]\left(\psi_L \not{Z} \psi_L + \psi_R \not{Z} \psi_R\right) \tag{2.26}$$

1. The $Q\sin\theta_w$ part is vectorial but the T_3 part is chiral.
2. It couples both to left and right handed fields and is flavor conserving.
3. Note that unlike for the photon, its possible to build a BSM model that is not flavor conserving for the Z boson. This is one of the dangers in model building.
4. Further, note that the Z couples equally to the electrons as it does to muons. This is called universality of gauge couplings.

We have,

$$\frac{\Gamma(Z \to e^+e^-)}{\Gamma(Z \to \mu^+\mu^-)} \approx 1 \tag{2.27}$$

Note that we don't have chirality suppression of m_e/m_μ. This is because the Z boson is a spin 1 particle. Such a suppression is going to exist for pion decay. The first order correction to this is due to the mass of the muon, m_μ/M_Z, which gives a larger phase space to the electrons. A similar ratio is,

$$\frac{\Gamma(\pi^+ \to \mu^+\nu)}{\Gamma(\pi^+ \to e^+\nu)} \approx \frac{m_\mu^2}{m_e^2} \tag{2.28}$$

However, in this case because of helicity suppression this ratio is about 10^4, while naively you would expect the electron to have a larger branching ratio. The idea is the neutrino is only left chiral, while the charged lepton, because the decaying particle is a scalar must then have right handed helicity. But since the W only couples to left handed particles you need a left handed charged lepton to mix into a right handed charged lepton and this mixing is proportional to the mass. Therefore the suppression is,

$$\sim \left(\frac{m_\mu}{m_e}\right)^2 \sim 4 \times 10^4 \tag{2.29}$$

which after accounting for the additional phase space of the electron is closer to 10^4. This chirality suppression can be eliminated if a photon is also added into the final state.

The W boson quark coupling is given by,

$$-\mathcal{L}_W = \frac{g}{\sqrt{2}}\bar{u}_L^i\gamma^\mu V_{ij} d_L^j W_\mu^+ + h.c. \tag{2.30}$$

where V_{ij} is the CKM matrix. This matrix encodes flavor and CP violation. The CKM has 10 parameters. This is very peculiar since we started with 36 (18 for each Yukawa matrix). However, it turns out that many of these are unphysical. We now do this counting expilcitly. To understand how to do this counting we take a detour into symmetries.

The global symmetries of the SM are,

$$U(1)_B \times U(1)_e \times U(1)_\mu \times U(1)_\tau \tag{2.31}$$

These symmetries are accidental, meaning that they are not imposed but a consequence of the gauge group and field content. Note that such symmetries are only preserved at the renormalizable level and hence are also a consequence of choosing to truncate the Lagrangian at a given order.

Recall that the fermion Lagrangian is

$$\mathcal{L} = \mathcal{L}_{kin} + \mathcal{L}_{Yuk} \tag{2.32}$$

The symmetries of the kinetic term is much larger then the symmetries of the Yukawa interaction. As an example lets consider the kinetic term of the up type quarks,

$$\mathcal{L}_{kin}^u = i\bar{U}^i \not{D} U^i \tag{2.33}$$

This Lagrangian is invariant under an $U(3)$ rotation between the up flavors (if U were real then the symmetry would be $O(3)$). This rotation for example means if we want to use u and c as our light up type quarks we are equally well to use,

$$\left\{\frac{u \pm c}{\sqrt{2}}\right\} \tag{2.34}$$

This symmetry is not satisfied by the Yukawa interaction because the Yukawas, $Y_{ij}\bar{Q}^i\phi U^j$, are not universal. After a rotation we have,

$$R_{ki}Y_{ij}R_{j\ell}\bar{Q}^k\phi U^\ell \tag{2.35}$$

and in general

$$R_{ki}Y_{ij}R_{j\ell} \neq Y_{k\ell} \tag{2.36}$$

This is a simple Lie group,

$$U(3) = SU(3) \times U(1) \tag{2.37}$$

The global symmetry of the kinetic term of the quarks is given by,

$$U(3)_Q \times U(3)_D \times U(3)_U \tag{2.38}$$

Under these three global symmetries we have,

$$Q = (3,1,1) \tag{2.39}$$
$$D = (1,3,1) \tag{2.40}$$
$$U = (1,1,3) \tag{2.41}$$

An important tool in this business is the spurion. A spurion is a number (not a field!) but we assign this number the transformation properties of some group. As an example consider the Higgs. We can write the Higgs as,

$$\begin{pmatrix} 0 \\ v \end{pmatrix} \tag{2.42}$$

We can say this that this number transforms as a doublet under $SU(2)_L$. We could have written this without ever knowing that the Higgs actually transforms in this way.

Now lets think of the Yukawas as spurions. Consider for example the quark Yukawa,

$$Y_U\bar{Q}U\phi \tag{2.43}$$

Under a flavor rotation we have,

$$Y_U\bar{Q}U\phi \rightarrow R_Q^\dagger Y_U' R_U \bar{Q}U\phi \tag{2.44}$$

This term is invariant if

$$Y_U \rightarrow R_Q Y_U R_U^\dagger \tag{2.45}$$

i.e. if

$$Y_U = (3,1,\bar{3}) \tag{2.46}$$

Similarly we have,

$$Y_D = (3,\bar{3},1) \tag{2.47}$$

Even though this symmetry doesn't exist in SM, its stil useful to know how terms break the symmetry. For example in the electroweak sector we still talk about $SU(2) \times U(1)$

even though its broken. In the same way even though this flavor symmetry is broken we still use these spurion transformation properties.

We can finally move onto counting parameters. As an easy example lets consider the Zeeman effect. There we put an atom in a magnetic field,

$$\mathbf{B} = B_x \hat{x} + B_y \hat{y} + B_z \hat{z} \tag{2.48}$$

In quantum mechanics classes we then move on and say the field is in the $\hat{z}$ direction. Why are we able to make this simplification? The answer is although its true we have 3 parameters, we only have 1 physical parameter which is the magnitude of the magnetic field. We have,

$$\# \text{ of param} = 1 \text{ physical} + 2 \text{ unphysical}$$

The reason we were able to make this simplificaition is because of symmetry. The unperturbed Hamiltonian is $SO(3)$. After applying a magnetic field the symmetry becomes $SO(2)$ (for us the unperturbed symmetry is that of the kinetic term and the perturbed one is the symmetry of the Yukawa interaction). We reduced the symmetry of the problem by applying the magnetic field. $SO(3)$ has 3 generators, while $SO(2)$ has only one. Therefore we broke 2 generators. We can use the broken generators to rotate away unphysical parameters. In this case rotate around x and y axes to eliminate B_x and B_y. We always have,

$$\# \text{ of phys param} = \# \text{ total} - \# \text{ broken gen}$$

Lets consider the number of parameters in the up type quark sector. The initial symmetry was $U(3)_Q \times U(3)_U \times U(3)_D$ which has $9+9+9=27$ parameters. The final symmetry is $U(1)$ which has 1 parameter. Therefore there are 26 broken generators. The total number of parameters is

$$\underbrace{3 \times 3}_{\text{3 by 3 mat.}} \times \overbrace{2}^{\text{Re + Im}} \times \underbrace{2}_{\text{up+down}} = 36\,. \tag{2.49}$$

Therefore there are $36 - 26 = 10$ physical parameters, 6 quark masses, 3 mixing angles, and 1 phase. You can even find how many real and how many imaginary parameters. The yukawa matrices had 18 real and 18 imaginary parameters. The $U(3)$ matrices have 5 real parameter and 4 imaginary (recall there are 5 real Gell-mann matices) resulting in 15 broken real generators and 11 broken imaginary generators ($U(1)_B$ is an unbroken imaginary generator). This gives, $18 - 15 = 3$ real parameters in the quark sector and $18 - 11 = 7$ imaginary parameters.

Lets do the same trick for the lepton. The initial symmetry is $U(3)_L \times U(3)_E$ which has 18 parameters and the final symmetry is $U(1)_e \times U(1)_\mu \times U(1)_\tau$ which has 3 parameters. Therefore the number of physical parameters is,

$$18 - (18 - 3) = 3 \tag{2.50}$$

These are just the lepton masses since in the SM we don't have a PMNS matrix (and no neutrino masses).

2.3 Beyond the Standard Model

Problems with the Standard Model can be divided into two types "data" and "beauty". Data problems are the traditional type of problem in science, however, theoretical high energy physics also has beauty or hierarchy problems.

1. Gravity - "theoretical problem"
2. Hierarchy problems
3. Data - neutrino masses, dark matter, dark energy, baryogenesis, inflation

The hierarchy problems in the SM are as follows:

1. $m_\phi \ll M_{Pl}$ - "10^{-16} problem"
2. Flavor puzzle - " 10^{-6}"
3. Strong CP problem - "10^{-10}"
4. Cosmological constant problem - "10^{-120}"
5. Gauge unification

Fine tunings can take two distinct form, problems which are "technically natural" (e.g., the flavor problem) or not-"technically natural" (e.g., the hierarchy problem). A small number is technical natural if a symmetry is restored in the limit that the number goes to zero. For example consider the mass of the electron. In the limit that $m_e \to 0$, QED has a new symmetry, a chiral symmetry, $e \to e^{i\gamma_5} e$. This is not true for the SM higgs mass. This feature is important because any corrections to the parameter must then be proportional to the parameter. For example,

$$m_e^{\text{1-loop}} = m_e^{\text{tree}} (\log(...)...) \tag{2.51}$$

$$(m_h^{\text{1-loop}})^2 = \Lambda^2 (\log(...) + ...) \tag{2.52}$$

The higgs mass corrections are proportional to the physics at the UV while the electron mass corrections are proportional to the electron. If there is a solution to a technically natural problem it can live at any scale, problems that are not technically natural must be solved at the scale of the parameter.

We can write the SM Lagrangian as,

$$\mathcal{L}_{SM} = \underbrace{\mathcal{L}_R}_{\text{renormalizable}} + \underbrace{\mathcal{L}^{(5)} + \mathcal{L}^{(6)} + ...}_{\text{nonrenormalizable}} \tag{2.53}$$

There is a single unique dimension five operator in the SM,

$$\mathcal{L}^{(5)} = \frac{(LH)^2}{\Lambda} \tag{2.54}$$

There is of order 100 new operators that can show up at dimension six,

$$\mathcal{L}^{(6)} = \frac{\mathcal{O}}{\Lambda^2} \tag{2.55}$$

One such operator is,

$$u_R u_R d_R e_R \tag{2.56}$$

This will give rapid proton decay and spoils the first success that we listed of SM. To avoid this term we have two options,

1. Impose a symmetry (e.g. R parity)
2. Take $\Lambda \gg v$.
3. Approximate symmetry

Chapter 3

Electroweak Precision

For this section we use the references,

1. Peskin - chapter 22
2. PDG
3. hep-ph/0412166
4. Skiba - TASI 2009

In electroweak we have,

$$SU(2)_L \times U(1)_Y \to U(1)_{EM} \tag{3.1}$$

and its charatorized by

$$\{g, g', v\} \tag{3.2}$$

These 3 parameters completely describe the electroweak sector of the SM at **tree level**. Basically what we want to do is overconstrain these three parameters and test our theory. At tree level the gauge sector is governed by only three parameters. After loop corrections we have to worry about more parameters due to new processes happening at one loop. The most important parameters that are going to be important are the ones that are large,

$$\{m_h, m_t, g_s\} \tag{3.3}$$

As an example of electroweak prceision (EWP) lets think about how to measure the weinberg angle. There are three ways to define the Weinberg angle. The first two are,

$$\sin^2\theta_W = 1 - \frac{m_W^2}{m_Z^2} \tag{3.4}$$

$$\sin^2 2\theta_0 = \frac{4\pi\alpha(m_Z^2)}{\sqrt{2}G_F(m_\mu^2)m_Z^2} \tag{3.5}$$

The capital W on θ_W reminds us that this definition arises from the mass of the W boson and not just the weinberg angle and the 0 subscript is just conventional.

The third definition arises from the coupling of the Z to the fermions which is given by,

$$\mathcal{L}_{\psi_i\psi_i Z} = -\frac{g}{2\cos\theta_w}\bar{\psi}_i\gamma_\mu\left(g_V^i - g_A^i\gamma_5\right)\psi_i Z^\mu \tag{3.6}$$

then [1],

$$g_V = T_3 - 2q\sin^2\theta_* \tag{3.8}$$

$$g_A = T_3 \tag{3.9}$$

Taking the difference of these equations and dividing by $2q_i$ we get,

$$\sin^2\theta_*^i \equiv \frac{g_A^i - g_V^i}{2q_i} \tag{3.10}$$

In principle we have 9 of these definitions of the Weinberg angle (one for each fermion which is charged) and so θ_* should have a flavor index on it. But to leading order the correction is universal. The best way to measure the θ_* is to use polarization information since then it differentiates g_V from g_A.

To find the corrections to the different definitions we consider the corrections to the propagator (these will effect the mass of the gauge bosons and hence the weinberg angles).

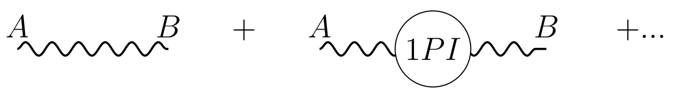
1. $$Z \to \ell^+\ell^- \tag{3.11}$$

   This only works for the b or τ since they decay and then you can measure their polarization from the angular distributions. But you it doesn't work for the other flavors.

2. One can polarize $e^+e^- \to Z$ as was done at SLAC.

When we talk about oblique corrections we mean corrections that are the same for all the flavors.

The corrections to the propagator are,



where A and B are the initial and final particles. These measurements are sensitive to BSM physics because new particles can propagate in the loops. The infinite sum is given by,

$$-\frac{i}{q^2}\left[1 + i\Pi(q^2)\cdot\frac{-i}{q^2} + ...\right] \tag{3.12}$$

[1]The vector and axial coupling are related to the left and right couplings through,

$$g_L = g_V - g_A \quad , \quad g_R = g_V + g_A \tag{3.7}$$

Here we are being a little hasty. Outside of QED it is not clear what exactly is meant by $\Pi(q^2)$ since in general the propagator correction is,

$$\Pi^{\mu\nu}_{AB}(q^2) = \Pi_{AB}(q^2)g^{\mu\nu} - \Delta(q^2)q^\mu q^\nu \tag{3.13}$$

We define the scalar, $\Pi_{AB}(q^2)$ as the quantity preceding the $g^{\mu\nu}$ in the propagator correction. In the results that follow the Δ dependent terms will always be proportional to

$$q_\mu j^\mu = 0 \tag{3.14}$$

due to gauge invariance. Therefore, we can drop this contribution for the purposes of our discussion.

We can work in either the physical basis,

$$\{W^\pm, \gamma, Z\} \tag{3.15}$$

or in the unphysical basis,

$$\{W^+, W^3, B\} \tag{3.16}$$

In general we have four propagator that we need to calculate,

$$\Pi_{W^-W^-}, \Pi_{W^+W^+}, \Pi_{\gamma\gamma}, \Pi_{ZZ} \tag{3.17}$$

We now make the assumption that we work in a region with small q^2 compared to the heavy particles,

$$\Pi(q^2) = \Pi(0) + q^2\Pi'(0) + ... \tag{3.18}$$

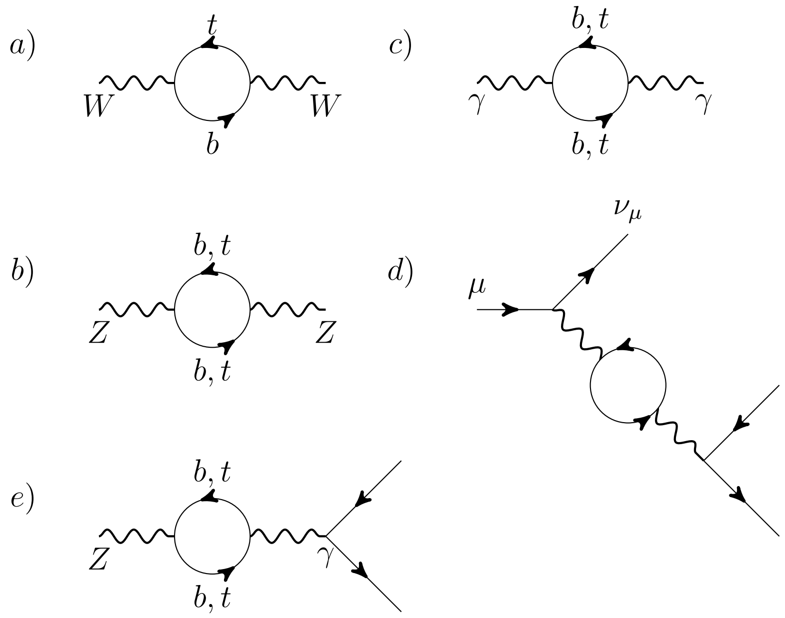
We have the identities,

$$m_W^2 = m_W^{(tree)\,2} + \Pi_{WW}(m_W^2) \tag{3.19}$$
$$\Pi_{\gamma\gamma}(0) = 0 \tag{3.20}$$
$$\Pi_{\gamma Z}(0) = \Pi_{\gamma Z}(m_Z^2) = 0 \tag{3.21}$$

where the last two identities follow since particles can only mix while both particles are off-shell. [Q 1: I don't think the last identity is true, $\Pi_{\gamma Z}(m_Z^2) = 0$ since we use it later...]

This isn't a very good assumption for the massive gauge bosons mass corrections (they are about a factor of 2 ligher then the top), but a terrific one for the photon propagator correction

Explicitly we will consider the corrections,



Note that (a) and (d) are really same calculation however we write them separately since they are evaluated at different q^2. (a) is evaluated at the Higgs mass while (d) is evaluated at the muon mass.

Lets consider these corrections. The massive gauge boson masses are,

$$m_W^2 = \frac{g^2 v^2}{4} + \Pi_{WW}(m_Z) \tag{3.22}$$

$$m_Z^2 = \frac{(g^2 + g'^2) v^2}{4} + \Pi_{ZZ}(m_Z) \tag{3.23}$$

Then we can calculate θ_W (the "W boson weinberg angle"),

$$\sin^2 \theta_W = 1 - \frac{m_{W,0}^2 + \Pi_{WW}}{m_{Z,0}^2 + \Pi_{ZZ}} \tag{3.24}$$

$$= \underbrace{\sin^2 \theta}_{\text{tree level result}} - \frac{1}{m_Z^2} \left[\Pi_{WW} - \cos^2 \theta \Pi_{ZZ} \right] \tag{3.25}$$

where $\cos^2 \theta$ is just the tree level result since the corrections to this are a higher order effect, and we used a subscript 0 to denote the tree level masses. Π_{AB} is calulated in appendix 3.A. Lets try to study the divergences of diagram (a). The diagram is roughly given by,

$$\sim \int d^4k \frac{(\not{k} + m)(\not{k} + m)}{(k^2 + m^2)^2} \sim \int \frac{d^4k}{k^2} \sim \Lambda^2 \tag{3.26}$$

However, this divergence is artificial and isn't existent due to the Ward identity. It may not be obvious that the massive gauge boson still is protected by a Ward identity but it turns out to be the case. The intuition behind this is that the UV divergence doesn't see the masses of the particles (at very large momenta you can set the masses to zero). However, there is a log divergence[2]. However, the divergence between two definitions of θ always cancels.

The calculation for θ_W was relatively easy since the modifications of m_W and m_Z are obvious. This isn't always the case. To find the correction to θ_* we need the correction to α, G_F, and m_W. To find the corrections for α recall that due to gauge invariance we have[3],

$$e = \sqrt{1 + \Pi(q^2)/q^2} e_0 \tag{3.27}$$

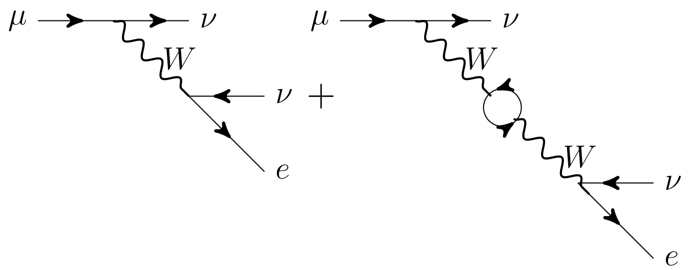
Taylor expanding the propagator we have,

$$\Pi(q^2) \approx \left[\cancelto{0}{\Pi(0)} + q^2 \Pi'(0) + ... \right] \tag{3.28}$$

where the first term vanishes due to gauge invariance and we drop terms of order q^4/m_t^4. Rewriting in terms of the fine structure constant we have,

$$4\pi\alpha = \frac{g^2 g'^2}{g^2 + g'^2} \left(1 + \Pi'_{\gamma\gamma}(0)\right) \tag{3.29}$$

Next we want to find the correction to G_F. To find the expression for the correction consider a muon decay process at one loop,



The sum of the internal part of the diagram is given by (taking $q^2 \ll M_W^2$ and denoting the vector Lorentz indices by σ and ρ),

$$\left(\frac{-i g^{\sigma\rho}}{M_W^2} \right) g^2 \left[1 - \frac{\Pi_{WW}(q^2)}{M_W^2} \right] \tag{3.30}$$

The Fermi constant is given by the coupling at $q^2 \rightarrow m_\mu^2 \approx 0$. Thus we can write,

$$\Pi_{WW}(q^2) = \Pi_{WW}(0) + q^2 \Pi'_{WW}(q^2) \approx \Pi_{WW}(0) \tag{3.31}$$

Furthermore, the tree level Fermi constant is just,

$$\frac{G_F^{tree}}{\sqrt{2}} = \frac{1}{8} \frac{g^2}{M_W^2} = \frac{1}{2v^2} \tag{3.32}$$

[2]Note that naively you may think that we have a linear, Λ divergence however this is absent due to charge conservation.

[3]See for example, *Quantum Field Theory* by Peskin and Schroeder (pg 246).

Therefore the sum of the diagrams is,

$$-i\frac{8G_F^{tree}}{\sqrt{2}}g^{\sigma\rho}\left[1-\frac{\Pi_{WW}(0)}{m_W^2}\right] \tag{3.33}$$

which implies that

$$\frac{G_F}{\sqrt{2}}=\frac{1}{2v^2}\left[1-\frac{\Pi_{WW}(0)}{M_W^2}\right] \tag{3.34}$$

We are finally in position to calculate $\sin^2 2\theta_0$ at one loop. Using equation 3.5,

$$\sin^2 2\theta_0 = \frac{g'^2}{g^2+g'^2}\left(1+\Pi'_{\gamma\gamma}(m_Z^2)\right)\left[\frac{1}{v^2}\left(1-\frac{\Pi_{WW}(0)}{M_W^2}\right)\left(\frac{(g^2+g'^2)v^2}{4}+\Pi_{ZZ}(m_Z^2)\right)\right]^{-1} \tag{3.35}$$

$$= \frac{4g'^2g^2}{(g^2+g'^2)^2}\left[1+\Pi'_{\gamma\gamma}(0)+\frac{\Pi_{WW}(0)}{M_W^2}-\frac{\Pi_{ZZ}(0)}{m_Z^2}\right] \tag{3.36}$$

where the factor in front is just another way of writing $\sin^2 2\theta_w$ at tree level. While this expression is perfectly fine it is more convenient to have an expression for $\sin^2\theta_0$. To find the new expression we make the convenient definitions,

$$A\equiv 4\sin^2 2\theta_w \quad \epsilon\equiv\Pi'_{\gamma\gamma}(0)+\frac{\Pi_{WW}(0)}{M_W^2}-\frac{\Pi_{ZZ}(0)}{m_Z^2} \tag{3.37}$$

Then we have,

$$\sin^2\theta_0\left(1-\sin^2\theta_0\right)=A(1+\epsilon) \tag{3.38}$$

which after using the quadratic equation give,

$$\sin^2\theta_0=\frac{1}{2}\left(1\pm\sqrt{1-4A(1+\epsilon)}\right) \tag{3.39}$$

for $\theta_w\approx 20^o$ the physical root is the negative one. Identifying $1-\sqrt{1-4A}$ with $\sin^2\theta_w$ we have,

$$\sin^2\theta_0=\sin^2\theta_w+\frac{\cos^2\theta\sin^2\theta}{\cos 2\theta}\left[\Pi'_{\gamma\gamma}(0)+\frac{\Pi_{WW}(0)}{M_W^2}-\frac{\Pi_{ZZ}(0)}{m_Z^2}\right] \tag{3.40}$$

In an analogous way to the calculation leading to equation 3.25 one can show that,

$$\sin^2\theta_*-\sin^2\theta_{tree}=-\frac{\sin\theta\cos\theta\Pi_{\gamma Z}(m_Z^2)}{m_Z^2} \tag{3.41}$$

An explicit calculation gives

$$\sin^2\theta_0-\sin^2\theta_*=\frac{3\alpha}{16\pi\cos^2 2\theta}\frac{m_t^2}{m_Z^2} \tag{3.42}$$

$$\sin^2\theta_W-\sin^2\theta_*=\frac{-3\alpha}{16\pi\sin^2\theta}\frac{m_t^2}{m_Z^2} \tag{3.43}$$

where we have only kept terms quadratic in m_t. Note that differences of quantities that you renormalize is finite. This is always the case since the differences of two renormalized quantities is physical. This is analogous to absolute energies being unphysical but energy differences are physical. This is a prime example of how you can get sensitivity to a heavy particle (the top in this case) at one loop. We see that this result appears to violate the decoupling theorem which says that as the mass of a particle is taken to infinity it should dissapear from the theory. However, this is not true since taking the mass here to infinity would require taking the Yukawa coupling to infinity which is impossible since it would make the theory nonperturbative.

Through these types of calculations we were able to predict the mass of the top to be around 150GeV which was correct within 1σ. One can calculate the corresponding Higgs correction to this quantity such a correction gives,

$$s_0^2 - s_\theta^2 = \frac{3\alpha}{16\pi\cos^2 2\theta}\frac{m_t^2}{m_Z^2} + \#\log(\frac{m_H}{m_Z}) \tag{3.44}$$

The Higgs correction surprisingly is going to logarithmic. This is a unique one loop effect and is due to the fact that the Higgs is not just a regular scalar but the particle which breaks spontaneous symmetry breaking. This logarithm reduces the effectiveness in predicting the mass of the Higgs.

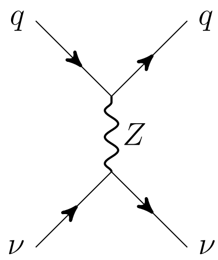
The general philosophy is to look at many different measurements as the one above. After you are done you can make a fit between all these results and estimate each parameter. This is done in the Gfitter website.

We can divide the actual measurements into two types of data, low energy and high energy data. For low energy data we have neutrino scattering. Basically you take a neutrino and you scatter it off a wall and you see the ejected electron moving. The sensitivity to these sorts of interactions is due to coupling to the Z. Other low energy measurements arise from parity violating atomic physics as well of course measurements of $G_F, \alpha, g-2, ...$

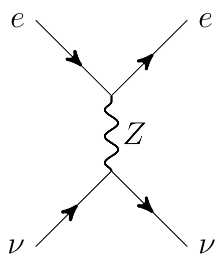
High energy data arose from LEP and SLAC. They calculated, $m_Z, \Gamma_Z, \Gamma_{Z\to had}, \Gamma_{Z\to\nu\nu}$, and A_{fb}. More recent high energy data arises from hadron colliders which aren't very good for precision measurements but designed for discovery instead.

3.1 Neutral Currents

Prior to discovery of the Z boson people thought about whether such a boson really exists. At this point people only new of the four fermi interaction which only couples left chiral fields. In fact you can build a model with $SU(2) \xrightarrow{SSB} U(1)_{EM}$ which will just have $W^\pm, \gamma$ but no Z. This model was ruled out by ν scattering. Consider ν scattering off a proton,



The signiture would be a proton in a sheet getting a kick due to a neutrino beam. One can device a similar experiment for scattering off an electron,



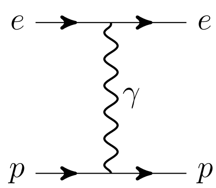
These involve a low energy Z exchange. The only time a low energy Z will be important will be when we use neutrinos since they don't interact with the photon.The cross-section for interaction with an electron is,

$$\sigma = \frac{G_F m_e E_\nu}{2\pi}\left[(g_V \pm g_\alpha)^2 + \frac{1}{3}(g_V \mp g_A)^2\right] \tag{3.45}$$

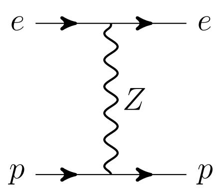
To calculate g_V and g_A you can take the ratio of different neutrino flavor interactions.

3.2 Atomic parity violation

Atomic physics is typically governed soley by QED, which of course doesn't have parity violation. However, there is one place where parity violating effects can exist and thats in the splitting of energy levels. The force that binds a proton to an electron can be drawn as,



but now there is a second interaction of Z exchange,



The reason we can typically ignore the second effect because it has a much smaller range. The Bohr radius is about $10^{-10}m$ while the range of the Z interaction is,

$$m_Z \sim 100\text{GeV} \sim 10^{-20}m \tag{3.46}$$

The way to see this effect is if the radius of the atom is very small. This will be the case for heavy elements. The heavier the element the worse we understand the background so there is a tradeoff. This experiment is useful because it has sensitivity to different contributions then LEP and SLC. For example if we have a new heavy neutral current that has weak coupling we could find it here, but we would have a much harder time finding it in a high energy experiment.

3.3 Using EWP - S, T and U parameters

In any new model the diagrams above should be calculated. However, calculating such diagrams is difficult. A much easier thing to do is to integrate out your high energy fields and then calculate their coefficients corresponding to higher dimensional operators of the SM. These coefficients can then easily be compared with the bounds on the SM.

One way to discuss this is in terms of whats known as the S, T, and U parameters. We want to make some assumption about the new physics. In principle if we have the SM then we have 3 parameters at tree level as well as m_t and m_h. All the other parameters aren't significant.

We make the following assumptions,

1. We assume $M_{NP} \gg M_Z$ (new physics is at least a factor of 2 or 3 greater than the weak scale)

2. All effects are in the propagator corrections, Π_{AB}. This is roughly true due to the large mass of the top quark. The vertex corrections are always going to be smaller since they don't involve the mass of the top quark in the numerator. These are called oblique corrections.

3. At the weak scale we only have $SU(2)_L \times U(1)_Y$ symmetry.

To a good approximation these 3 conditions are satified for almost all BSM models. In many models even though they do not satisfy these 3 conditions, they do so to a good approximation. Once these three assumptions are satisfies it turns out there are only 3 parameters you need to calculate.

We have 8 total variables since we have $\Pi_{ZZ}, \Pi_{WW}, \Pi_{Z\gamma}, \Pi_{\gamma\gamma}$ each have a $\Phi(0)$ and Φ'. But we have 2 constraints arising from the Ward identity (from the fact that $\Pi_{\gamma\gamma}(0)$ and $\Pi_{\gamma Z}(0)$ are zero). Then we have 3 tree level parameters in the SM, $\{g, g', v\}$. The number of new parameters is,

$$8 - 2 - 3 = 3 \text{ new parameters}$$

We define,

$$T \equiv \frac{1}{\alpha}\left[\frac{\Pi_{WW}}{m_W^2} - \frac{\Pi_{ZZ}}{m_Z^2}\right] \tag{3.47}$$

$$S \equiv \frac{4\sin^2 2\theta}{\alpha}\left[\Pi'_{ZZ} - \frac{2\cos^2 2\theta}{\sin^2 2\theta}\Pi'_{Z\gamma} - \Pi'_{\gamma\gamma}\right] \tag{3.48}$$

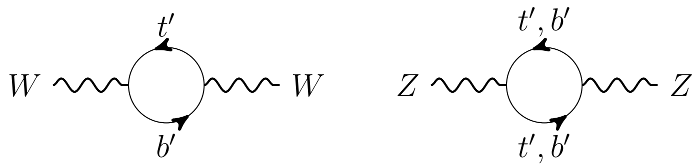
$$U \equiv \frac{4\sin^2\theta}{\alpha}\left[\Pi'_{WW} - \cos^2\theta\Pi'_{ZZ} - \sin 2\theta\Pi'_{Z\gamma} - \sin^2\theta\Pi'_{\gamma\gamma}\right] \tag{3.49}$$

[Q 2: Which θ do we use?]

A few remarks:

1. This is not really the proper definition. In the real definition you add a constant such that the SM value of T, S, U is zero.
2. The T parameter is closely related to the $\rho - \rho_{SM}$ parameter.
3. Practically we never hear about U, we only hear about T and S. The reason is that when you write these parameters in terms of higher dimensional operators, T and S are dimension 6 while U is dimension 8.

As an example lets consider a fourth generation. Lets first consider the effect on the T parameter. We have additional contributions to Π_{WW} and Π_{ZZ}:



Dimensional analysis gives[4],

$$\delta\Pi_{WW} \sim \frac{1}{16\pi^2}(2m_{t'}m_{b'}) \tag{3.50}$$

$$\delta\Pi_{ZZ} \sim \frac{1}{16\pi^2}(m_{t'}^2 + m_{b'}^2) \tag{3.51}$$

and so,

$$T \sim \frac{1}{16\pi^2 m_W^2}\left(m_{t'}m_{b'} - m_{t'}^2 - m_{b'}^2\right) \sim -\frac{1}{16\pi^2 m_W^2}\left(m_{t'} - m_{b'}\right)^2 \tag{3.52}$$

If the t' and b' were degenerate then the T parameter would be zero. The current bound on the the T parameter says that if we have a fourth generation then the splitting between the new quarks would have to be about 30 GeV. The S parameter,

$$S \sim \# \cdot \bar{m}_{t',b'} \tag{3.53}$$

[4] The factor of two in the charged interaction is due to the factors of $\sqrt{2}$ enhancement between charged and neutral current interactions (though this is obviously just a rough estimate).

where # is roughly the number of particles. This parameter essentially counts how many new generations you have. The S parameter has actually already ruled out such a possibility except for peculiar regions of parameter space. Note that the bounds on additional scalars are about 6 times weaker. This is why SUSY is still within these bounds.

3.4 Custodial symmetry

The custodial symmetry is an accidental symmetry which is only of the Higgs sector. For a extensive discussion on custodial symmetry, see `0410370` Here we outline the important points.

Consider the Higgs potential in the SM,

$$V = -\mu^2 \phi^\dagger \phi + \lambda (\phi^\dagger \phi)^2 \tag{3.54}$$

The Higgs doublet takes the explicit form [5],

$$\phi = \begin{pmatrix} \phi_1 + i\phi_2 \\ \phi_3 + i\phi_4 \end{pmatrix} = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} \quad \tilde{\phi} \equiv \epsilon \phi^* = \begin{pmatrix} \phi^{0*} \\ -\phi^- \end{pmatrix} \tag{3.55}$$

But the only variable in the potential is given by,

$$\phi^\dagger \phi = \phi_1^2 + \phi_2^2 + \phi_3^2 + \phi_4^2 \tag{3.56}$$

Since the potential depends only on the length of a four dimensional vector, there is actually a larger symmetry, $SO(4) \sim SU(2) \otimes SU(2)$. The length after SSB is given by,

$$\phi^\dagger \phi \to (\phi_1 + v)^2 + \phi_2^2 + \phi_3^2 + \phi_4^2 \tag{3.57}$$

Since there is still a rotational symmetry about ϕ_2, ϕ_3, ϕ_4 there is a residual $SO(3)$ symmetry. Therefore we have,

$$SO(4) \to SO(3) \sim SU(2)_D \tag{3.58}$$

where the D stands for "diagonal". This $SU(2)_D$ is what we call custodial symmetry.

Its easier to discuss this symmetry if we combine the Higgs, ϕ, and its conjugtate, $\tilde{\phi}$, into a matrix representation. We define[6],

$$\Phi = \frac{1}{\sqrt{2}} \begin{pmatrix} \tilde{\phi} & \phi \end{pmatrix} \tag{3.59}$$

[5] Recall that $\tilde{\phi}$ isn't an independent degree of freedom but just a convenient way to form $SU(2)_L$ doublets out of a Higgs with opposite hypercharge.

[6] Alternatively, one can think of projecting the Higgs vector, $\begin{pmatrix} \phi_1 & \phi_2 & \phi_3 \end{pmatrix}^T$ onto the Pauli matrices and then switching the charged basis.

Note that,

$$\Phi^\dagger\Phi = \frac{1}{2}\begin{pmatrix} \tilde{\phi}^\dagger\tilde{\phi} & \tilde{\phi}^\dagger\phi \\ \phi^\dagger\tilde{\phi} & \phi^\dagger\phi \end{pmatrix} \Rightarrow \mathrm{Tr}\left[\Phi^\dagger\Phi\right] = \phi^\dagger\phi \tag{3.60}$$

where we have used the fact that $\tilde{\phi}^\dagger\tilde{\phi} = \phi^\dagger\phi$. Deriving the kinetic term is more involved and the derivation is shown in appendix 3.B. The Higgs Lagrangian is given by,

$$\mathcal{L}_{Higgs} = \mathrm{Tr}D^\mu\Phi^\dagger D_\mu\Phi - \mu^2\mathrm{Tr}\left[\Phi^\dagger\Phi\right] + \lambda\mathrm{Tr}\left[\Phi^\dagger\Phi\right] \tag{3.61}$$

where the covariant derivative takes the form,

$$D_\mu\Phi = \partial_\mu\Phi - \frac{i}{2}gW^a_\mu\sigma^a\Phi + \frac{i}{2}g'B_\mu\Phi\sigma_3 \tag{3.62}$$

Note the extra σ_3 on the hypercharge term which is a consequence of ϕ and $\tilde{\phi}$ having different hyperchanges. Furthermore, note that σ_3 multiplies Φ from the right.

Since ϕ and $\tilde{\phi}$ both transform as doublets under $SU(2)_L$ we can write,

$$\Phi = \frac{1}{\sqrt{2}}\begin{pmatrix} \tilde{\phi} & \phi \end{pmatrix} \to L\frac{1}{\sqrt{2}}\begin{pmatrix} \tilde{\phi} & \phi \end{pmatrix} \tag{3.63}$$

while we the usual global transformation for the gauge field, $W^\mu \to LW^\mu L^\dagger$. The scalar potential, which is made out of $\mathrm{Tr}\Phi^\dagger\Phi$ is trivially invariant under $SU(2)_L$. For the kinetic term we have,

$$\mathrm{Tr}\partial_\mu\Phi^\dagger\partial^\mu\Phi \to \mathrm{Tr}\partial_\mu\Phi L^\dagger L\partial^\mu\Phi \tag{3.64}$$

$$\mathrm{Tr}\partial_\mu\Phi^\dagger W^\mu_a\sigma^a\Phi \to \mathrm{Tr}\partial_\mu\Phi^\dagger L^\dagger LW^\mu_a L^\dagger L\Phi \tag{3.65}$$

$$\mathrm{Tr}\partial_\mu\Phi^\dagger B^\mu\Phi\sigma_3 \to \mathrm{Tr}\partial_\mu\Phi^\dagger L^\dagger LB^\mu\Phi\sigma_3 \tag{3.66}$$

etc.

Note however, that due to the trace we almost have another symmetry, $\Phi \to \Phi R^\dagger$, where R is an $SU(2)$ transformation. The symmetry breaking term is the g' contribution. For example,

$$\mathrm{Tr}\partial_\mu\Phi^\dagger B^\mu\Phi\sigma_3 \to \mathrm{Tr}\partial_\mu\Phi^\dagger B^\mu\Phi R^\dagger\sigma_3 R \tag{3.67}$$

where we have the extra factor, $R^\dagger\sigma_3 R \neq \sigma_3$.

This symmetry is also broken by the Yukawa interactions which in our notation take the form,

$$\mathrm{Tr}\left[\begin{pmatrix} t_L^\dagger & b_L^\dagger \\ t_L^\dagger & b_L^\dagger \end{pmatrix}\begin{pmatrix} \phi^{0*} & \phi^+ \\ \phi^- & \phi^0 \end{pmatrix}\begin{pmatrix} y_t & 0 \\ 0 & y_b \end{pmatrix}\begin{pmatrix} t_R & 0 \\ 0 & b_R \end{pmatrix}\right] \tag{3.68}$$

The downtype quarks aren't made of $SU(2)$ multiplets so we can't just let them transform analagously to the t_L, b_L fields to keep the Yukawas invariant under the custodial symmetry.

We can identify one of the $SU(2)$ as the gauge symmetry. The symmetry of the theory is given by,

$$SU(2)_L \otimes SU(2)_R \tag{3.69}$$

$SU(2)_L$ is easy to identify. $SU(2)_R$ contains $U(1)_Y$ and two other global symmetry generators (i.e. they are not gauged). Note that in a left right symmetric theory this $SU(2)_R$ is also gauged. The Higgs transforms as,

$$\Phi \to L\Phi R^\dagger \tag{3.70}$$

The VEV of the Higgs takes the form,

$$\langle\Phi\rangle = \frac{1}{2}\begin{pmatrix} v & 0 \\ 0 & v \end{pmatrix} \tag{3.71}$$

This is not invariant under $SU(2)_L$ or $SU(2)_R$ (its easy to see that $L\langle\Phi\rangle \neq \langle\Phi\rangle$, and similarly for $SU(2)_R$). However, it is invariant under transformations such that $L = R$. We say that the resultant theory is invariant under diagonal or vector transformations,

$$\langle\Phi\rangle \to D\langle\Phi\rangle D^\dagger = \langle\Phi\rangle \tag{3.72}$$

The W_μ^a gauge bosons transform as a triplet under $SU(2)_L$ and a singlet under $SU(2)_R$ because forming a singlet out of $\mathrm{Tr} D_\mu \Phi^\dagger D^\mu \Phi$ relies on the W^μ remaining underchanged under $SU(2)_R$ and using the trace to cycle the transformation matrices. Therefore, they must transform as a triplet under $SU(2)_D$.

In the limit that $g' = 0$ we have 3 massive degenerate gauge bosons and one massless photon. Moving away from this limit, one boson remains massless while there is a splitting between the remaining 3. We have,

$$m_W^2 = \frac{1}{4}g^2 v^2 \quad m_Z^2 = \frac{1}{4}(g^2 + g'^2)v^2 \tag{3.73}$$

We also have the mixing angle,

$$\cos^2\theta_w = \frac{g^2}{g^2 + g'^2} \tag{3.74}$$

which controls how far we are from the $g' = 0$ limit ($g' = 0 \to \cos^2\theta_w = 1$). However, we still have the equality $m_{W_3} = m_{W_{1,2}}$. This results in the famous SM tree level prediction,

$$\rho \equiv \frac{m_W^2}{m_Z^2 \cos^2\theta_w} = 1 \tag{3.75}$$

where $m_{W_3} \equiv m_Z \cos\theta_w$ and $m_W = m_{W_1} = m_{W_2}$ as these masses are unaffected by the rotation in the neutral sector. The $\rho = 1$ relation is a consequence of the custodial symmetry since this is what gaurentees that $m_{W_{1,2}} = m_{W_3}$. If we had a higher representation of the Higgs then the masses of the W bosons would be different. In general each component of the W couples to a different combination of ϕ_i. Each of them gets a fraction of the VEV.

Since the custodial symmetry is broken by both the Yukawa interactions as well as g', radiative corrections to the $\rho = 1$ relation must be either proportional to the splitting of the quarks or g'.

3.5 Higher dimensional operators

This analysis follows `hep-ph/0412166`.

If you keep all the flavor indices in the SM then there are 2499 dimension 6 operators. If you forget about flavor indices there are 80 operators. However, it turns out that you can use the equations of motion to eliminate some of the operators. This reduces the number of operators to 59. Many of these still don't affect EWP but are only relevant in flavor physics. This is because the bounds from $K\bar{K}$ mixing are much stronger then from EWP. This brings us to only 28 dimension 6 operators.If you check what the data is sensitive to there are only 21 operators.

In the reference above they took the 21 operators and they found the matrix of bounds. So in your new physics model you can calculate all the 21 operators and compare with these bounds. Out of all these operators the most important ones are

$$S \leftrightarrow \mathcal{O}_{WB} \equiv \frac{H^\dagger W^{\mu\nu} B_{\mu\nu} H}{\Lambda^2} \tag{3.76}$$

$$T \leftrightarrow \mathcal{O}_{H} \equiv \frac{\left|H^\dagger D_\mu H\right|^2}{\Lambda^2} \tag{3.77}$$

$$Y \leftrightarrow \mathcal{O}_{BB} \equiv \frac{(D_\mu B^{\mu\nu})^2}{\Lambda^2} \tag{3.78}$$

$$W \leftrightarrow \mathcal{O}_{WW} \equiv \frac{(D_\mu W^{\mu\nu})^2}{\Lambda^2} \tag{3.79}$$

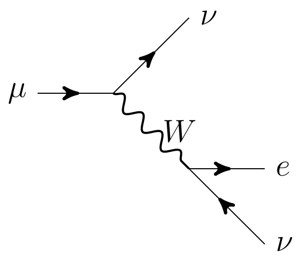
where the letters on the left indicate a name for these operators. [Q 3: There is some subtlety associated with how one defines $W_{\mu\nu}$. Look this up and comment.]

After electroweak symmetry breaking we can charactorize what symmetries the operators break:

	Custodial	$SU(2)_L$
S	$+$	$-$
T	$-$	$-$
Y	$+$	$+$
W	$+$	$+$

where a $+$ ($-$) indicates that it conserves (doesn't conserve) the symmetry. Note that operators that break a symmetry are much easier to probe. This is a very general property since in channels that break a symmetry we have lower backgrounds. This is for example the reason why we can probe the hyperfine splitting very precisely. With this in mind we see that T is the most important parameter while S is the second most important parameter for EWP.

We have two different expansions that we can consider. The first one is in our energy scale divded by the heavy physics energy scale. The other thing we can do is expand in q^2 instead. For example in the muon decay,



we have the propagator for the W boson,

$$\frac{1}{m_W - q^2} = \frac{1}{m_W^2}\left(1 + \frac{q^2}{m_W^2} + ...\right) \tag{3.80}$$

We typically only keep the lowest order term in the expansion. Note that the q^2 term can arise from effective operators with derivatives,

$$\sim G_F \mu \not{D} e \nu \not{D} \bar{\nu} \tag{3.81}$$

This discussion becomes a little more complicated when we introduce a Higgs. For example with the Higgs we have the dimension 8 term,

$$\frac{\mu e \nu \nu H^2}{\Lambda^4} \tag{3.82}$$

Without EWSB this term is irrelevant. However after EWSB we need to include this term. We can do similar expansions when looking for new physics beyond the SM. Here we have,

$$\Pi_{ab}(q^2) = \Pi(0) + q^2 \Pi'(0) + \frac{1}{2!} q^4 \Pi''(0) + ... \tag{3.83}$$

where $\Pi' \equiv d\Pi/dq^2$. Formally, this expansion is in q^2/v^2. This expansion is completely justified for muon decay. For measurements in the mass of the W we have,

$$q^2 \approx m_W^2 \tag{3.84}$$

and we have,

$$\frac{q^2}{v^2} = \frac{g^2}{4} \tag{3.85}$$

This means that the expansion parameter is parameterically small which justifies perturbation theory. Now lets truncate the series at q^2.

At tree level we have 3 parameters in the electroweak sector. At one loop we start with 6 parameters (one for each Π, Π' and for each combination of gauge bosons). To full desicrbe the one loop results in the SM we just need 3 parameters. We call these parameters, S, T, and U.

We can relate the Π's to the couplings,

$$g^{-2} = \Pi'_{W^+W^-} \tag{3.86}$$
$$g'^{-2} = \Pi'_{BB} \tag{3.87}$$
$$v^2 = -2\Pi_{W^+W^-} \tag{3.88}$$
$$g^{-2}m_W^2\hat{T} = \Pi_{33} - \Pi_{W^+W^-} \tag{3.89}$$
$$g^{-2}\hat{S} = \Pi'_{3B} \tag{3.90}$$
$$g^{-2}\hat{U} = \Pi'_{33} - \Pi'_{W^+W^-} \tag{3.91}$$

We define the "hatted" operators as rescaled versions of the operators we defined earlier,

$$S = \frac{4\sin^2\theta_w\hat{S}}{\alpha} \quad T = \frac{\hat{T}}{\alpha} \quad U = -\frac{4\sin^2\theta_w\hat{U}}{\alpha} \tag{3.92}$$

Unlike S and T, U doesn't arise from a dimension 6 operator. Y and W which we mentioned last time don't break any symmetries.

If we keep one more order in q^2 we have Π'' and we need four more parameters which we call $V(\text{dim}10)X(\text{dim}8)Y(\text{dim}6)W(\text{dim}6)$. In Π''' there are no more dimension 6 operators.

There is another seminal paper in this field, Arxiv: 0007265. In this paper they just calculate each operator one at a time. There they the bounds on these operators are given by,

$$\mathcal{O}_S \Rightarrow \Lambda \gtrsim 10\text{TeV} \tag{3.93}$$
$$\mathcal{O}_T \Rightarrow \Lambda \gtrsim 5\text{TeV} \tag{3.94}$$

This for example implies that in Randal Sundrum models (this is not 10 TeV because of a coupling),

$$m_{KK} \gtrsim 3\text{TeV} \tag{3.95}$$

(the reason we have 3TeV and not 10TeV is because we have some couplings here). As another example supersymmetry is affected by these bounds. However, because supersymmetric particles must appear in loops, these bounds are reduced by factors of $16\pi^2$.

A last example of the utility of EWP is looking at a fourth generation. In that case we introduce a t', b', e', ν'. The CKM becomes,

$$\begin{pmatrix} & & & 0 \\ & V_{CKM} & & 0 \\ & & & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{3.96}$$

The masses of the new quarks are $m_{t'}$ and $m_{b'}$. The T parameter is sensitive to breaking of the custodial symmetry so we would depend on the difference in their masses,

$$T \approx \frac{N_c}{12\pi\sin^2\theta_w\cos^2\theta_w}\frac{m_{t'}^2 - m_{b'}^2}{m_Z^2} \tag{3.97}$$

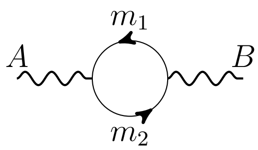
Due to the boudns of the T parameter we have to assume the new generation is roughly generate. However the S is just a number,

$$S \approx \frac{N_c}{6\pi}\left(1 - Y\log\frac{m_{t'}^2}{m_{b'}^2}\right) \approx \frac{N_c}{6\pi} \tag{3.98}$$

We say that S essentially counts the number of new flavors because for each new generation we would get a contribution of about $N_c/16\pi$ to the S parameter. The factor of N_c is equal to 3 for the quarks and 1 for the leptons. Lastly, note that this doesn't obey the decoupling limit. This is because we are assuming the fourth generation couples to the Higgs and the coupling is bounded at 4π.

3.A Gauge boson propagator at 1 loop

In this section we study the correction to the gauge boson propagator at 1 loop that connects boson A with boson B,



we allow for arbitrary couplings at each of these vertices, parametrized by g_L^i and g_R^i,

The diagram is given by,

$$-(i)^2 i^2 \int đ^d k \mathrm{Tr}\left[\gamma_\mu (P_L g_L^B + P_R g_R^B)\left(\frac{\not{k} - m_2}{k^2 - m_2^2}\right)\gamma_\nu (P_L g_L^A + P_R g_L^A)\left(\frac{\not{k} - \not{p} - m_1}{(k-p)^2 - m_1^2}\right)\right] \tag{3.99}$$

The denomenator is,

$$\frac{1}{D} = \frac{1}{k^2 - m_2^2}\frac{1}{(k-p)^2 - m_1^2} = \int dx \frac{1}{\left[(k-px)^2 - \Delta\right]^2} \tag{3.100}$$

where $\Delta \equiv -p^2 x(1-x) + (m_1^2 - m_2^2)x + m_2^2$. For simplicity we calculate each combination of left right projection operator simulataneously. We define $a, b = \pm 1$ and then we have,

$$\mathrm{Tr}\,[...] = \frac{1}{4}\Big\{\mathrm{Tr}\left[\gamma_\mu(\not{k} - m_2)\gamma_\nu(\not{k} - \not{p} - m_1)\right] + a\mathrm{Tr}\left[\gamma_\mu\gamma^5(\not{k} - m_2)\gamma_\nu(\not{k} - \not{p} - m_1)\right]$$
$$b\mathrm{Tr}\left[\gamma_\mu(\not{k} - m_2)\gamma_\nu\gamma^5(\not{k} - \not{p} - m_1)\right] + ab\mathrm{Tr}\left[\gamma_\mu\gamma^5(\not{k} - m_2)\gamma_\nu\gamma^5(\not{k} - \not{p} - m_1)\right]\Big\} \tag{3.101}$$

$$= \frac{1}{4}\Big\{(1 + ab)\mathrm{Tr}\left[\gamma_\mu\not{k}\gamma_\nu(\not{k} - \not{p})\right] + (1 - ab)m_1m_2\mathrm{Tr}\left[\gamma_\mu\gamma_\nu\right]$$
$$- (a + b)\mathrm{Tr}\left[\gamma_\mu\not{k}\gamma_\nu(\not{k} - \not{p})\gamma^5\right]\Big\} \tag{3.102}$$

$$= (1 + ab)\left(k_\mu(k - p)_\nu - g_{\mu\nu}k\cdot(k - p) + (k - p)_\mu k_\nu\right) + (1 - ab)m_1m_2g_{\mu\nu}$$
$$+ (a + b)i\epsilon_{\mu\sigma\nu\rho}k^\sigma(k - p)^\rho \tag{3.103}$$

Shifting the integration variable and dropping terms linear in k_μ we have,

$$\mathrm{Tr}\,[...] = (1 + ab)\left(2k_\mu k_\nu - 2p_\mu p_\nu(1 - x)x - g_{\mu\nu}(k^2 - p^2x(1 - x))\right) + (1 - ab)m_1m_2g_{\mu\nu} \tag{3.104}$$

where the $\epsilon_{\mu\sigma\nu\rho}$ term doesn't contribute since it multiplies a symmetric term. The relevant integrals are,

$$\int \not{d}^dk\frac{k_\mu k_\nu}{(k^2 - \Delta)^2} = \frac{-i/2}{(4\pi)^{d/2}}g_{\mu\nu}\Delta\frac{\Gamma(2 - d/2)}{\Delta^{2-d/2}} \tag{3.105}$$

$$\int \not{d}^dk\frac{k^2}{(k^2 - \Delta)^2} = \frac{-i}{(4\pi)^{d/2}}\Delta\frac{\Gamma(2 - d/2)}{\Delta^{2-d/2}}\frac{d/2}{1 - d/2} \tag{3.106}$$

$$\int \not{d}^dk\frac{1}{(k^2 - \Delta)^2} = \frac{i}{(4\pi)^{d/2}}\frac{\Gamma(2 - d/2)}{\Delta^{2-d/2}} \tag{3.107}$$

which gives,

$$-\int \not{d}^dk\sum_{a,b}g_a^Ag_b^B\,(...) = \sum_{a,b}g_a^Ag_b^B\frac{-i}{(4\pi)^{d/2}}\int dx\frac{\Gamma(2 - d/2)}{\Delta^{2-d/2}}\Big[g_{\mu\nu}\big(\,(1 + ab)\left(-\Delta + p^2x(1 - x)\right)$$
$$+ (1 - ab)m_1m_2\big) + p_\mu p_\nu(1 + ab)(-2x(1 - x))\Big] \tag{3.108}$$

If $A = B$ we have the relations,

$$\sum_{a,b}g_ag_b = (g_L + g_R)^2 \quad \sum_{a,b}abg_ag_b = (g_L - g_R)^2 \tag{3.109}$$

which gives the simplification,

$$= \frac{-2i}{(4\pi)^{d/2}}\int dx\frac{\Gamma(2 - d/2)}{\Delta^{2-d/2}}\Big[g_{\mu\nu}\big(\left(g_L^2 + g_R^2\right)\left(-\Delta + p^2x(1 - x)\right)$$
$$+ 2g_Lg_Rm_1m_2\big) + p_\mu p_\nu\left(g_L^2 + g_R^2\right)(-2x(1 - x))\Big] \tag{3.110}$$

If we assume a gauge invairant interaction then $m_1 = m_2$. In this case we have,

$$= \frac{-4i}{(4\pi)^{d/2}} \int dx \frac{\Gamma(2-d/2)}{\Delta^{2-d/2}} \left[(g_L^2 + g_R^2) x(1-x) \left(g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right) + g_{\mu\nu}(g_L^2 - g_R^2) m^2 \right] \tag{3.111}$$

Therefore a chiral gauge theory will always violate the Ward identity. For a nonabelian gauge group this is not an issue since the Ward identity doesn't exist for such groups. However, this shows that you can't have a chiral $U(1)$ gauge theory (this is a consequence of the well known axial anomaly).

For the photon we reproduce the expected prediction,

$$= \frac{-8e^2 i}{(4\pi)^{d/2}} \int dx \frac{\Gamma(2-d/2)}{\Delta^{2-d/2}} x(1-x) \left(g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right) \tag{3.112}$$

$$\Rightarrow \Pi_{\gamma\gamma}(p^2) = \frac{-8e^2}{(4\pi)^{d/2}} \int dx \frac{\Gamma(2-d/2)}{\Delta^{2-d/2}} x(1-x) \tag{3.113}$$

For other gauge bosons we still have the relation,

$$\Pi_{AB}(p^2) = \sum_{a,b} g_a^A g_b^B \frac{-1}{(4\pi)^{d/2}} \int dx \frac{\Gamma(2-d/2)}{\Delta^{2-d/2}} \left[(1+ab)\left(-\Delta + p^2 x(1-x)\right) + (1-ab) m_1 m_2 \right] \tag{3.114}$$

$$\Pi_{AA}(p^2) = \frac{-2}{(4\pi)^{d/2}} \int dx \frac{\Gamma(2-d/2)}{\Delta^{2-d/2}} \left[(g_L^2 + g_R^2)(-\Delta + p^2 x(1-x)) + 2 g_L g_R m_1 m_2 \right] \tag{3.115}$$

Using this formula we calculate the different gauge boson propagator corrections, e.g.,

$$\Pi_{ZZ} \approx \frac{-2}{(4\pi)^{d/2}} \int dx \frac{\Gamma(2-d/2)}{\Delta^{2-d/2}} \left[2(g_L^2 + g_R^2) p^2 x(1-x) - (g_L + g_R)^2 m_t^2 \right] \tag{3.116}$$

$$\Pi_{WW} \approx \frac{-2g^2}{(4\pi)^{d/2}} \int dx \frac{\Gamma(2-d/2)}{\Delta^{2-d/2}} \left[2p^2 x(1-x) - x m_t^2 \right] \tag{3.117}$$

$$\Pi_{\gamma Z} = \frac{1}{2} \frac{e}{s_\theta c_\theta} \left(\frac{1}{4} - \frac{2}{3} s_\theta^2 \right) \frac{-1}{(4\pi)^{d/2}} \int dx \frac{\Gamma(2-d/2)}{\Delta^{2-d/2}} \left[2p^2 x(1-x) \right] \tag{3.118}$$

3.B Matrix form of the Higgs

We can put the Higgs into matrix form using,

$$\Phi = \frac{1}{\sqrt{2}} \begin{pmatrix} \tilde{\phi} & \phi \end{pmatrix} \tag{3.119}$$

Its now easy to see that,

$$\mathrm{Tr}\Phi^\dagger \Phi = \frac{1}{2}(\tilde{\phi}^\dagger \tilde{\phi} + \phi^\dagger \phi) = \phi^\dagger \phi \tag{3.120}$$

which can be used to form the scalar potential.

Building the kinetic term is more difficult. In this section we prove that the covariant derivative for the matrix Higgs must take the form,

$$D_\mu \Phi = \partial_\mu \Phi - \frac{i}{2} g W_\mu^a \sigma^a \Phi + \frac{i}{2} B_\mu \Phi \sigma^3 \tag{3.121}$$

which gives,

$$\mathcal{L}_{kin} = \mathrm{Tr} D_\mu \Phi^\dagger D^\mu \Phi \tag{3.122}$$

Expanding we have,

$$\mathcal{L}_{kin} = \mathrm{Tr}\left[\left(\partial_\mu \Phi - \frac{i}{2} g W_\mu^a \sigma^a \Phi + \frac{i}{2} B_\mu \Phi \sigma_3 \right)^\dagger \left(\partial_\mu \Phi - \frac{i}{2} g W_\mu^a \sigma^a \Phi + \frac{i}{2} B_\mu \Phi \sigma_3 \right) \right] \tag{3.123}$$

We now expand this kinetic term in terms of the component Higgs field. We have,

$$\Phi \Phi^\dagger = \frac{1}{2} \left(\phi^{0*} \phi^0 + \phi^- \phi^+ \right) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{3.124}$$

$$\Phi \partial_\mu \Phi^\dagger = \frac{1}{2} \begin{pmatrix} \phi^{0*} \partial_\mu \phi^0 + \phi^+ \partial_\mu \phi^- & \phi^+ \partial_\mu \phi^{0*} - \phi^{0*} \partial_\mu \phi^+ \\ -\phi^- \partial_\mu \phi^0 + \phi^0 \partial_\mu \phi^- & \phi^- \partial_\mu \phi^+ + \phi^0 \partial_\mu \phi^{0*} \end{pmatrix} \tag{3.125}$$

Using these two identities is straightforward to show that,

$$i \mathrm{Tr}\left[\partial_\mu \Phi^\dagger \sigma_3 \Phi \right] + h.c. = i(\phi^{0*} \partial_\mu \phi^0 + \phi^+ \partial_\mu \phi^-) + h.c. \tag{3.126}$$

$$i \mathrm{Tr}\left[\partial_\mu \Phi^\dagger \sigma_1 \Phi \right] + h.c. = i(\phi^0 \partial_\mu \phi^- - \phi^- \partial_\mu \phi^0) + h.c. \tag{3.127}$$

$$i \mathrm{Tr}\left[\partial_\mu \Phi^\dagger \sigma_2 \Phi \right] + h.c. = (\phi^0 \partial_\mu \phi^- - \phi^- \partial_\mu \phi^0) + h.c. \tag{3.128}$$

$$\mathrm{Tr}\left[\Phi^\dagger \sigma_a \sigma_b \Phi \right] = \delta_{ab} \left(\phi^{0*} \phi^0 + \phi^- \phi^+ \right) \tag{3.129}$$

Putting this all together gives,

$$\begin{aligned} \mathrm{Tr}\left[D_\mu \Phi^\dagger D^\mu \Phi \right] &= (\partial_\mu \phi^0)^* \partial^\mu \phi^0 + \partial_\mu \phi^- \partial^\mu \phi^+ \\ &+ (g^2 W_\mu^a W_a^\mu - 2 g g' W_\mu^3 B^\mu + g'^2 B_\mu B^\mu) \left(\phi^{0*} \phi^0 + \phi^- \phi^+ \right) \\ &- i g (W_1^\mu - i W_2^\mu)(\phi^0 \partial_\mu \phi^- - \phi^- \partial_\mu \phi^0) + h.c. \end{aligned} \tag{3.130}$$

This is just the ordinary kinetic term for the Higgs field.

Chapter 4

Flavor Physics

4.1 Flavor changing currents

The different bounds we have on new physics are,

1. EWP $\to \Lambda \gtrsim 10$TeV
2. Flavor physics $\Lambda \gtrsim 10^5$TeV
3. Baryon and lepton number violation, $\Lambda \gtrsim 10^{16}$TeV.

[Q 4: Fix the ordering of these paragraphs.]

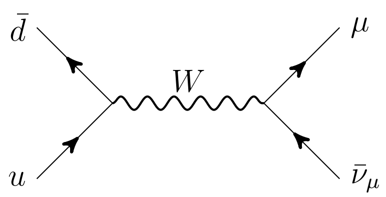
Any meson can be described in terms of flavor quantum numbers. For example pion can be described by,

$$\pi^+ \to \quad d = -1 \quad u = 1 \tag{4.1}$$

The decay of the pion is given by,

$$\pi^+ \to \mu^+ \nu \tag{4.2}$$

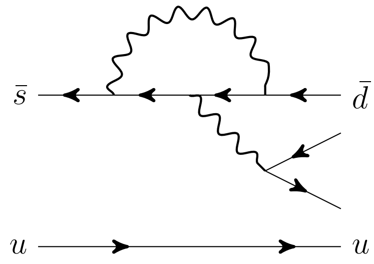
This occurs through a charged current,



The up and down are the same flavor so we call this a flavor changing charged current. On the other hand we can have (the electrons are needed sicne we can't have a $1 \to 1$ decay),

$$K^+ \to \pi^+ e^+ e^- \tag{4.3}$$

This corresponds to the decay, $\bar{s}u \to \bar{d}u$,



The s needs to change to a d which have the same charge but is a different generation so we call this a flavor changing neutral current (FCNC). The SM explains this by only have FCNC at one loop.

To understand the reason why the SM has FCNC we need to differntiate between universal and diagonal coupling matrices. A universal coupling matrix is very meaningful because it is independent of basis. This is not the case for diagonal couplings which are basis dependent. For example,

$$\underbrace{\begin{pmatrix} g & 0 & 0 \\ 0 & g & 0 \\ 0 & 0 & g \end{pmatrix}}_{\text{universal}} \quad \underbrace{\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}}_{\text{diagonal}} \tag{4.4}$$

Other examples of FCNCs are $K - \bar{K}$ mixing and $b \to s\gamma$ (which is seen through $B \to K^*\gamma$[1]). Note that we never look for flavor changing neutral current in $\rho \to K\gamma$ for example (which would correspond to $d \to s\gamma$). This is because the ρ decays to pions much more often than it does to $K\gamma$ since the pion decay is a strong decay while the $K\gamma$ decay is a weak decay.

We will think of FCNC as a tool for model building and putting constraints on new physics. The reason this provides strong constraints is beause in the SM these are absent at tree level. First we need to understand why there are no FCNC in the SM. The four types of neutral bosons are γ, Z, g, ϕ.

For the unbroken gauge bosons, γ, g we don't have FCNC because of gauge invariance. The vertex is always $A_\mu\bar{\psi}\psi$. To see why this is the case consider the kinetic term,

$$\mathcal{L}_{kin} = \bar{\psi}_i \not{D} \delta_{ij}\psi_j \tag{4.5}$$

As long as these are canonically normalized then they trivially couple equally to all flavors. Thus we don't have FCNC. Note this is true for the photon even though it is an output of spontaneous symmetry breaking! This arguement fails at the nonrenormalizable level since there we can have different Wilson coefficients and we are not reliant on gauge coupling.

Next lets study the Higgs. The reason the Higgs doesn't have FCNC is a consequence of having a single Higgs doublet model. In the single Higgs doublet there is an alignment

[1]We need a K^* because the B meson is a spin singlet and you can't have a photon which is a vector and a K which is a singlet form a singlet.

between the mass matrix and the Higgs couplings. As an example consider the down Yukawas,

$$Y\bar{Q}_L\phi D_R \to Y\bar{d}_L(\phi+v)d_R \tag{4.6}$$

The mass of the down-type quarks is,

$$m = Yv \tag{4.7}$$

where Y is a matrix. When we diagonalize the mass matrix, the Yukawas are also automatically diagonalized. Therefore there are no flavor changing contributions.

Now suppose we have two Higgs,

$$(Y_1H_1 + Y_2H_2)\bar{Q}_L D_R \tag{4.8}$$

The masses are given by,

$$m = Y_1v_1 + Y_2v_2 \tag{4.9}$$

while the matrix couplings to the down type quarks are Y_1 and Y_2 for h_1 and h_2 respectively. If the matrix U diagonalizes m then we have,

$$\mathcal{L}_{yuk} = \bar{Q}_L U^\dagger(Y_1H_1 + Y_2H_2)UD_R \tag{4.10}$$

but $U^\dagger Y_1 U \neq Y_1^{diag}$ since here the U's diagonalize the sum of Y_1v_1 and Y_2v_2 instead of a single one. So diagonalizing the mass matrix doesn't lead to diagonalization of the Yukawas and hence we have FCNC in the Yukawa interactions. This is often not an issue because of the necessarily small couplings of the Higgs to the fermions.

Finally lets consider the Z boson. We write the Z coupling in the interaction basis and then move to the mass masis. For example we have,

$$u_{L/R} \to V^u_{L/R}u_{L/R} \tag{4.11}$$
$$d_{L/R} \to V^d_{L/R}d_{L/R} \tag{4.12}$$

We have four rotations to do. The W interaction we have,

$$\mathcal{L}_W = i\bar{u}_L \not{W} d_L \to i\bar{u}_L V^\mu_L V^{d\dagger}_L \not{W} d_L \tag{4.13}$$

and we define,

$$V_{CKM} \equiv V^u_L V^{d\dagger}_L \tag{4.14}$$

Now lets consider what happens in the neutral current sector. In this case we have,

$$\mathcal{L}_Z = i\bar{u}\not{Z}Q_Z u \to i\bar{u}V^u Q_Z V^{u\dagger}\not{Z}u \tag{4.15}$$

The "CKM-like" matrix is just the identity. To see why this happened lets recall the Z charge,

$$Q_Z = T_3 - q\sin^2\theta_w \tag{4.16}$$

In general we can have some matrix,

$$Q_Z^i \bar{\psi}^i \psi^i \tag{4.17}$$

(we don't need to necessarily impose flavor universality because the symmetry is spontaneously broken regardless). We have,

$$Q_Z^{ij} = \left(\begin{array}{ccc} Q_Z^1 & 0 & 0 \\ 0 & Q_Z^2 & \vdots \\ \vdots & ... & \ddots \end{array} \right) \Rightarrow \mathcal{L}_Z = \bar{\psi}_i Q_{ij} \psi_j \tag{4.18}$$

where i, j are flavor indices. If Q_Z is universal then the Lagrangian is invariant under the rotations that diagonalize the fermions. However, if this is not universal (but only diagonal) then,

$$V Q_Z V^\dagger \neq \text{diagonal matrix} \tag{4.19}$$

The condition for particles to mix is that they be under the same representation under all symmetries of the lorentz group. In the SM it just so happens that every field that has the same quantum numbers (QN) under $SU(2) \times U(1)_Y$ has the same QN under electromagnetism. However, this is very unique to the SM. [Q 5: Why exactly does this happen?] This results in universal couplings also for the Z boson.

Its easy to cook up a model which doesn't have this property. Lets consider the 1 generation SM and consider the vector-like quark (that the left and right have the same representation),

$$\psi_L(3,1)_{-1/3} \quad \psi_R(3,1)_{-1/3} \quad Q_L(3,2)_{1/6} \quad d_R(3,1)_{-2/3} \tag{4.20}$$

We have mass terms,

$$m_1 \bar{\psi}_L \psi_R + m_2 \bar{\psi}_L d_R + Y_1 v \bar{Q}_L \psi_R + Y_2 v \bar{Q}_L d_R \tag{4.21}$$

The mass matrix is,

$$\left(\begin{array}{cc} m_1 & m_2 \\ Y_1 v & Y_2 v \end{array} \right) \tag{4.22}$$

In principle we have four couplings of the Z and we want to see whether they are diagonal. We still have,

$$Q_Z = T_3 - q \sin^2 \theta_w \tag{4.23}$$

[Q 6: How do we know that this equation still holds?]

Therefore we have,

Field	Q_Z
ψ_L	$\frac{1}{3} \sin^2 \theta_w$
ψ_R	$\frac{1}{3} \sin^2 \theta_w$
u_L	$\frac{1}{2} - \frac{1}{6} \sin^2 \theta_w$
d_L	$-\frac{1}{2} - \frac{1}{6} \sin^2 \theta_w$
d_R	$\frac{2}{3} \sin^2 \theta_w$

Therefore we have the down type quark matrices,

$$Q_Z^L = \begin{pmatrix} \frac{1}{2}\sin^2\theta_w & 0 \\ 0 & -\frac{1}{2} - \frac{\sin^2\theta_w}{2} \end{pmatrix} \tag{4.24}$$

$$Q_Z^R = \begin{pmatrix} \frac{1}{3}\sin^2\theta_w & 0 \\ 0 & \frac{1}{3}\sin^2\theta_w \end{pmatrix} \tag{4.25}$$

Since the Q_Z^L charge is not universal we would expect FCNC in the left sector.

The way the SM deals with the requirement of no FCNC is to make all the fermions copies of one another.

Now suppose we have a neutral heavy gauge boson that is not the Z boson (Z'). This Z' can be problematic because it can mediate FCNC. In order for this new boson not to create FCNC, we must require that all the SM fermions be under the same representation of this new boson.

Lets now consider the coupling of the W in this model, the CKM matrix. For the W coupling we have,

$$\tilde{u}_L V_L^{u\dagger} V_L^d d_L \tag{4.26}$$

The V_L^u is still a 3×3 matrix, but V_L^d is not a 4×4 matrix projected out to the states that interact with the W. The CKM is no longer a unitary matrix. In this kind of models where we get FCNC we also get nonunitarity in the CKM.

4.1.1 Unitarity triangles

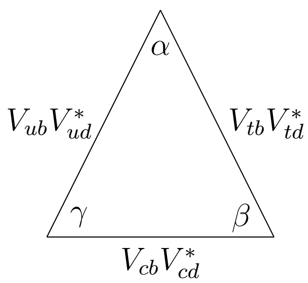
The unitarity of the CKM says that,

$$\sum_k V_{ik} V_{jk}^* = 0 \tag{4.27}$$

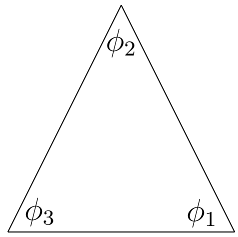
for any $i \neq j$.[Q 7: Does anyone study the $i = j$ case which is equal to 1?] This equation says that the sum of three complex numbers should equal to zero. If one thinks of complex numbers as vectors then these these numbers form a triangle. There is one triangle that is easy to measure,

$$\sum_k V_{dk} V_{bk}^* = 0 \tag{4.28}$$

It is known as **the** unitary triangle. It looks like (used at SLAC),



Unfortunately, there is also an additional naming scheme used at KEK:



If we can measure the lenghth of the sides of this triangle very precisely we can in fact measure the angles as well.

4.1.2 Wolfenstein parameterisation

We know that the CKM has a lot of structure to it. In paritcular the off diagonal elements are highly surpressed. Furthermore, we know there are 4 degrees of freedom in the CKM. Wolfestein suggested the parametrisation,

$$V_{CKM} \sim \begin{pmatrix} 1-\lambda^2/2 & \lambda & A\lambda^3(\rho - i\eta) \\ -\lambda & 1-\lambda^2/2 & A\lambda^2 \\ A\lambda^3(1-\rho-i\eta) & -A\lambda^2 & 1 \end{pmatrix} + \mathcal{O}(\lambda^4) \tag{4.29}$$

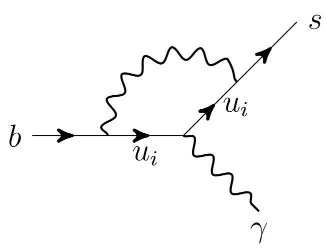
where, $\lambda \approx 0.22, A \approx 0.81, \rho \approx 0.14, \eta \approx 0.35$. To this order only the V_{bu} and V_{td} have a phase and are not symmetric to this order. λ is also referred to as $\sin\theta_{cabbibo}$.

4.2 FCNC at 1 loop

When we go and look for new physics we have to make sure we don't have FCNC at tree level, however this also needs to be checked at one loop. Consider the SM FCNC,

$$b \to s\gamma \tag{4.30}$$

which takes the form,



Lets estimate the amplitude,

$$i\mathcal{M} = \sum_i V_{bi}V_{is}^* \cdot f(m_i/m_W) \tag{4.31}$$

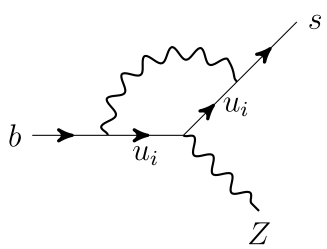
where we ignore the dependence on the masses of the external particles, which are in principle important but because we will in the end take the internal particle to be the top these will be a higher order effect.

Note that if $f(...)$ was equal to 1 the amplitude would be zero! due to unitarity of the CKM. We can write f as a Taylor expansion[2],

$$f\left(\frac{m_i^2}{m_W^2}\right) = f_0 + f_2\frac{m_i^2}{m_W^2} + f_4\frac{m_i^4}{m_W^4} + ... \tag{4.32}$$

Note that f_0 is unphysical. This is because the CKM gaurentees this term is zero. This is very important because f_0 is where the logarithmic divergence would live. Since there are no FCNC, there is no counterterm for this process. In fact QFT tells us that the lowest order contribution must be finite, which in this case is the one loop contribution.

Note that if we consider the model above and a process with a Z:



Since the CKM is not unitary, this diagram can diverge. However, here we also have FCNC at tree level and hence we have a counterterm.

Now lets return to the case of a unitary CKM matrix. The amplitude is already small since it occurs at one loop. However, if we assume m_i is small we can write,

$$f\left(\frac{m_i^2}{m_W^2}\right) \approx f_0 + f_2\left(\frac{m_i^2}{m_W^2}\right) \tag{4.33}$$

If the CKM is unitary we can forget about f_0 and we have quadratic dependence on the heavy field, m_t^2. This is essentially the GIM mechanism [Q 8: expand on this.] While this is not really relevant here since $m_t > m_W$ however it is used for $K \to \mu^+\mu^-$ for example.

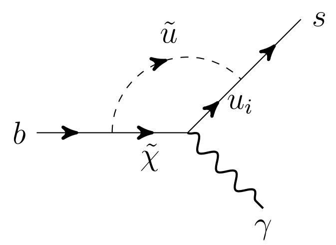
Notice that this is counterintiutive because the heavier the massive field, the more important it becomes. The decoupling theorem, that says that the heavier the field the less important it should be, is failing. The reason is happening because of the Higgs. What's really growing is not the mass of the top but the Yukawa couplings to the Higgs. So you can't really take the mass to infinity because then the theory becomes then the coupling becomes too large and the theory is nonperturbative.

Now suppose that quarks were all degenerate. If all the quarks were degerate then the mass can be taken out of the sum and the amplitude would again vanish by unitarity of the CKM. Therefore, we don't really have just the mass of the top, but also the small

[2]This expansion is not really justified for the top but lets ignore this caveat for now.

masses of the other quarks. In the end of the day amplitudes really depend on differences of masses.

There are generic ways to avoid new physics contributions from affecting FCNC too strongly at one loop. Consider for example,



There are three main ways to reduce these contributions:

1. Make the particles running in the loop to be very heavy.
2. Choose the mass of the particles running in the loop to be almost degenerate.
3. To ensure that the mixing angles are very small. This is an intricate topic that we will not discuss in detail.

4.3 Flavor and BSM

Here we follow 1002.0900. Recall that in EWP we were sensitive to about 10TeV. For example in Kaon physics we have,

$$\frac{(\bar{s}_L\gamma^\mu d_L)^2}{\Lambda^2} \tag{4.34}$$

This puts a bound of, $\Lambda \gtrsim 10^3$TeV. If we consider,

$$\frac{(\bar{s}_R d_L)(\bar{s}_L d_R)}{\Lambda^2} \tag{4.35}$$

then $\Lambda \gtrsim 10^4$TeV. This is well above anything we can probe directly.

To calculate CKM measurements we measure for example,

$$K \to \pi\ell\nu \sim |V_{us}| \tag{4.36}$$
$$B \to D\ell\nu \sim |V_{cb}| \tag{4.37}$$

4.4 Mixing and oscillations

Mixing refers to when you have interaction eigenstates which aren't equal to the mass eigenstates. For example in Kaons we have the flavor basis,

$$\left\{K, \bar{K}\right\} \tag{4.38}$$

and we also have the mass basis,

$$\{K_S, K_L\} \tag{4.39}$$

Oscillation, on the other hand, is that if we produce something which is not a mass eigenstate (for example if we produce a flavor eigenstate), then it will start to oscillate between the flavor eigenstates since the time evolution is not a trivial e^{-iEt}.

Note that K and $\bar{K}$ do have a mass in the Lagrangian and they are equal due to CPT symmetry. This is unique to the case of oscillation between a particle and its antiparticle. In general, this doesn't need to be the case. [Q 9: How can K and $\bar{K}$ be necesarrily degenerate while not mass eigenstates?] When two states are degenerate the mass matrix is,

$$\begin{pmatrix} a & \epsilon \\ \epsilon & a \end{pmatrix} \tag{4.40}$$

and mixing angle is,

$$\tan 2\theta \rightarrow \frac{\epsilon}{0} \Rightarrow \theta \sim 45^o \tag{4.41}$$

In neutrino oscillations, where the diagonal elements are distinct the mixing angle abides by the naive expective $\theta \sim \epsilon/a$.

K is just a complex scalar field. K_S and K_L are two nondegenerate real fields. The oscillation frequency in this case is just Δm (in the rest frame). This frequency is not Lorentz invariant and more generally is given by ΔE.

To produce such oscillations we need the amplitude to go from a K to be a $\bar{K}$. This is a FCNC and so can only occur at one loop in the SM. This amplitude represents how much mixing is between K and $\bar{K}$ (it takes the form of ϵ). We can write it schematically as,

$$\langle K|\mathcal{O}|\bar{K}\rangle \tag{4.42}$$

One such operator is,

$$\langle K|\bar{s}d\bar{s}d|\bar{K}\rangle \tag{4.43}$$

In general any meson system can mix. However, in practice there are only four mesons that can mix, K, B, B_s, D^0. We can't have pion oscillations since it is its own anti particle. We can mix the vector mesons,

$$\langle K^*|\mathcal{O}|\bar{K}^*\rangle \tag{4.44}$$

but the K^* decays much (much!) faster then the oscillation frequency. When you have a particle that decays strongly, it eliminates any chance to see oscillations which are necessarily a weak interaction effect.

In QFT we always assume that the state is stable and we put the interaction as a perturbation. There is another way to do particle physics and this is to assume that particle is not stable and add this effect to the particle propagator,

$$\frac{1}{p^2 - m^2} \rightarrow \frac{1}{p^2 - m^2 - i\Gamma m} \tag{4.45}$$

The original propagator is unitary, however this is not true for the decaying propagator. This requires a nonhermitian Hamiltonian,

$$H = M - \frac{i}{2}\Gamma \tag{4.46}$$

where M and Γ are hermitian (any nonhermitian Hamiltonian can be written as a sum of hermitian and antihermitian parts). We write,

$$H_{2\times 2} = \begin{array}{c c} & \begin{array}{cc} {\scriptstyle K} & {\scriptstyle \bar{K}} \end{array} \\ \begin{array}{c} {\scriptstyle K} \\ {\scriptstyle \bar{K}} \end{array} & \left[\begin{array}{cc} H_{11} & H_{12} \\ H_{21} & H_{22} \end{array} \right] \end{array} \tag{4.47}$$

Lets discuss each term one by one. H_{22} is equal to H_{11} by CPT and $M_{12} = M_{21}^*, \Gamma_{12} = \Gamma_{21}^*$. [Q 10: Check this statement].

$M_{11} = m_K, \Gamma_{11} = \Gamma_K$. This is the mass and width of the Kaon if there were no mixing. M_{12} is related to the matrix element that we wrote above. Any matrix element can be written in terms of its real and imaginary parts. The real part contributes to the mass and the imaginary corresponds to decay.

$$M_{12} \sim \mathrm{Re}_s \langle K|\mathcal{O}|\bar{K}\rangle \tag{4.48}$$
$$\Gamma_{12} \sim \mathrm{Im}_s \langle K|\mathcal{O}|\bar{K}\rangle \tag{4.49}$$

where here we take the real and imaginary parts of what we call *strong phase*. In general we like to distinguish between two kinds of phases, CP-even phase ("strong phase") and CP-odd phase ("weak phase"). A CP-even phase is one that does not change under CP. CP-odd phases arise from phases in the Lagrangian. A strong phase is typically due to time evolution. The phase due to time evoluation doesn't change under CP. The i in the Hamiltonian is a strong phase. If we have same electron with wavefunction, ψ, evolving in time,

$$\psi(t) = e^{-iHt}\psi(0) \tag{4.50}$$

If we have a positron it propagates with the same e^{-iHt} (there is no $i \to -i$ here).

Typically in QFT we just diagonalize the mass matrix, as oppose to the Hamiltonian. This is an approximation. The eigenvalues are,

$$\mu_\alpha = M_\alpha - \frac{i}{2}\Gamma_\alpha \tag{4.51}$$

where $\alpha = 1, 2$. Note that M_α and Γ_α are **not** the eigenvalues of the mass and decay matrices.

We write the mass eigenstates as,

$$K_L = p\,|K\rangle + q\,|\bar{K}\rangle \tag{4.52}$$
$$K_S = p\,|K\rangle - q\,|\bar{K}\rangle \tag{4.53}$$

such that $p^2 + q^2 = 1$.

Note that this rotation is not unitary. This results in,

$$\langle K_S | K_L \rangle = |p|^2 - |q|^2 \tag{4.54}$$

This means that you can produce a K_S but if you measure it afterwords you can measure a K_L.

Furthermore, one can show that if

$$\arg\left(M_{12}\Gamma_{12}^*\right) = 0 \tag{4.55}$$

then

$$|q| = |p| \tag{4.56}$$

[Q 11: Check]

If CP is a good symmetry then,

$$[H, CP] = 0 \tag{4.57}$$

If we have a nondegenerate state then the state must be an eigenstate of all the symmetries that commute with the Hamiltonian. Since K_L and K_S are nondegenerate then if CP commutes with Hamiltonian then we can choose,

$$CP(K_S) = \frac{1}{\sqrt{2}} CP(K + \bar{K}) = +K_S \tag{4.58}$$

$$CP(K_L) = \frac{1}{\sqrt{2}} CP(K - \bar{K}) = -K_L \tag{4.59}$$

We have seen that K_L can decay into 2π (which are CP even). Thus we know that the weak interaction has $\not{C}\!P$.We now define the eigenvalues,

$$\mu_\alpha = M_\alpha - \frac{i}{2}\Gamma_\alpha \tag{4.60}$$

which gives,

$$\Delta m = m_1 - m_2 \tag{4.61}$$

$$\Delta\Gamma = \Gamma_1 - \Gamma_2 \tag{4.62}$$

in the limit of CP we have,

$$\Delta m = 2\left|M_{12}\right| \tag{4.63}$$

$$\Delta\Gamma = 2\left|\Gamma_{12}\right| \tag{4.64}$$

We define,

$$x \equiv \frac{\delta m}{\Gamma} \quad y \equiv \frac{\Delta\Gamma}{2\Gamma} \tag{4.65}$$

and

$$\theta = \arg\left(M_{12}\Gamma_{12}^*\right) \tag{4.66}$$

We now assume CP conservation,

$$K_{L,S} = \frac{1}{\sqrt{2}}\left(K \pm \bar{K}\right) \tag{4.67}$$

We define $K(t)$ as the state at time t for a state that started out as a K.

$$K(t) = \cos\frac{\Delta mt}{2}\,|K\rangle + i\sin\frac{\Delta mt}{2}\bar{K} \tag{4.68}$$

The probability for measuring a K at time t is[3]

$$P(K \to K)\,[t] = \frac{1+\cos\Delta mt}{2} \tag{4.69}$$

$$P(K \to \bar{K}) = \frac{1-\cos\Delta mt}{2} \tag{4.70}$$

One can use this to measure Δm. Here we have ignored the decay amplitude. We were able to do this because of timescales. The relevant time scale in this problem is Δm.

Consider the case that $x \ll 1$. In this case the meson will decay very quickly and there is no time for things to oscillate. Now suppose that $\Delta x \gg 1$, in this case the oscillation is extremely fast and the probability averages to $1/2$. Oscillatons are important if $x \sim 1$. Experimentally it has been found that

$$\begin{aligned} x_K &\sim 1 \\ x_B &\sim 1 \\ x_D &\sim 10^{-2} \\ x_{B_s} &\sim 25 \end{aligned}$$

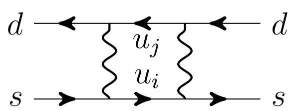
The fact that x_K and x_B are so close to 1 is accidental finetuning.

Lets now return to the braket we mentioned above,

$$M_{12} = \mathrm{Re}\,\langle k|\mathcal{O}|k\rangle \tag{4.71}$$

$\mathcal{O}$ needs to be a $\Delta s = 2$ operator - $s\bar{d}s\bar{d}$. Recall that a s field annihilates a free $|s\rangle$ state and creates a free $|\bar{s}\rangle$, and similarly for the d field. But suddenly we are asking our fields to act upon a bounded $|s\rangle$ and a bounded $|d\rangle$. We need to specify how our fields act on the bound state. To calculate this braket we need lattice QCD.

The diagram is,



$$\sim g^4\frac{s\bar{d}s\bar{d}}{M_W^2} \times \sum_{ij} V_{id}V_{is}^* V_{jd}V_{is}^* \times F\left(x_i, x_j\right)$$

[3] In practice we can differentiate between a K and $\bar{K}$ by its decay products since one decays with an electron and another with an electron.

where F is a finite function and $x_i \equiv \frac{m_i^2}{m_W^2}$. Here we see the GIM mechanism taking place because if we had no mass dependence in F then the diagram would be zero. Another interesting point is that this amplitude is g^4. We say this is second order in the weak interaction. [Q 12: Didn't we just say above that we can't calculate it because of QCD? How do you relate this calculation to the above?] In an explicit calculation one can show that,

$$F \sim x_i \sim \frac{m_c^2}{m_W^2} \tag{4.72}$$

where we only keep the charm due to CKM suppression. This calculation is only valid if none of the interactions in the loop are of $\mathcal{O}(\Lambda_{QCD})$. For the B quark this is a good assumption, however this is not the case in Kaon or charm physics. Therefore the calculation only holds up to order 1 corrections.

4.5 CP Violation

CP violation is a strong constraint on new physics. [Q 13: Show why complex numbers and CPV are related.] Fundamentally, C̸P̸ and flavor are two distinct phenomena, but in the SM they seem to come together. We build the SM such that we only have 1 C̸P̸ phase. The only other phase in the SM is θ_{QCD}.

In general if we have some process which takes us from $A \rightarrow B$ then,

$$\begin{aligned}
&P(A \rightarrow B)(x) \neq P(A \rightarrow B)(-x) \Rightarrow \not{P} && (4.73)\\
&P(A^+ \rightarrow B^+) \neq P(A^- \rightarrow B^-) \Rightarrow \not{C} && (4.74)\\
&P(A \rightarrow B) \neq P(B \rightarrow A) \quad \Rightarrow \not{T} && (4.75)\\
&P(A \rightarrow B) \neq P(\bar{A} \rightarrow \bar{B}) \quad \Rightarrow \not{C}\not{P} && (4.76)\\
&P(A \rightarrow B) \neq P(\bar{B} \rightarrow \bar{A}) \quad \Rightarrow \not{C}PT && (4.77)
\end{aligned}$$

where we use the common notation that the bar over the states, A and B, is both conjugation and reversal of $x \rightarrow -x$.

In 1964 people found $K_L \rightarrow \pi\pi$. This was a huge surprise because people knew that K_S decayed to $\pi\pi$ and if CP is a good quantum number then both couldn't decay to $\pi\pi$. K_L decays to $\pi\pi$ about 10^{-3} of the time. In 1967 people found,

$$\Gamma\left(K_L \rightarrow \pi^- e^+ \nu\right) \neq \Gamma\left(K_L \rightarrow \pi^+ e^- \nu\right) \tag{4.78}$$

Until this point there was no unambigious definition of the electron. After this experiment you can define the electron to be the outgoing particle that corresponds to the longer lifetime K_L decay.

In 1973 the third generation was predicted since it was needed for C̸P̸. In 1990's they found that $K_L \rightarrow \pi^+\pi^- \neq K_L \rightarrow \pi^0\pi^0$ [Q 14: What is the significance of this result?]

To know that $K_L \rightarrow \pi\pi$ is CP violating note the following. A symmetry is an operator that commutes with the Hamiltonian, $[\mathcal{O}, H] = 0$. Furthermore, for all non degenerate eigenstates of the Hamiltonian, $H|n\rangle = E|n\rangle$ we have $\mathcal{O}|n\rangle = \lambda_n|n\rangle$ for some λ_n. Now the point is that K_L is non degenerate. Furthermore, pions are CP odd and. Since K_L decays to both of these, it must break CP. Since the pions doesn't have spin and they are pseudoscalars and so they have CP,

$$\pi^0\pi^+\pi^- \rightarrow (-1)^L \tag{4.79}$$

[Q 15: Fix the above paragraph]

$\not{C}\not{P}$ is a rare phenomena. One reason this is the case is as follows. $\not{C}\not{P}$ means that,

$$\Gamma(A \rightarrow B) \neq \Gamma(\bar{A} \rightarrow \bar{B}) \tag{4.80}$$

However CPT tells us that,

$$\Gamma(A \rightarrow X) = \Gamma(\bar{A} \rightarrow \bar{X}) \tag{4.81}$$

where X stands for "anything"[Q 16: Why?]. CP violations says that the partial width should violate CP, but CPT invariance says that the total width still should be the same. Therefore, there must be an additional channel to cancel any violation.

To have CP violation we then we necessarily have an imaginary component in the Lagrangian[4]. If you impose CP then you can choose a basis for your fields such that the Lagrangian is completely real. To see why this is true consisder the interaction,

$$\mathcal{L} = \lambda\phi_1\phi_2\phi_3 + \lambda^*\phi_1^*\phi_2^*\phi_3\phi \tag{4.82}$$

CP takes $\phi_i \rightarrow \phi_i^*$. We have,

$$CP(\mathcal{L}) = \lambda\phi_1^*\phi_2^*\phi_3^* + \lambda^*\phi_1\phi_2\phi_3 \tag{4.83}$$

So we have, $\mathcal{L} = CP(\mathcal{L})$, if $\lambda = \lambda^*$ and hence the condition is that the couplings are real.

If we have a theory with many parameters we expect that we have roughly equal numbers of real and imaginary numbers. In the SM we only have 1 phase. This is due to the minimality of the SM. In practice we rarely see CP violation because in the Wolfenstein approximation the phase only appears at the off diagonal elements in the CKM (V_{dt}, V_{ub}).

To see a CP violating observable we need interference. Consider the amplitude for some process, $\mathcal{M}$, which has both a strong phase, δ, and a weak phase, ϕ. We have,

$$\mathcal{M} = 1 + re^{i(\delta+\phi)} \tag{4.84}$$

$$\bar{\mathcal{M}} = 1 + re^{i(\delta-\phi)} \tag{4.85}$$

A straight forward way to measure CP violation is to use,

$$a_{CP} \equiv \frac{|\bar{\mathcal{M}}|^2 - |\mathcal{M}|^2}{|\bar{\mathcal{M}}|^2 + |\mathcal{M}|^2} = 2r\sin\phi\sin\delta \tag{4.86}$$

for $r \ll 1$. There are two major difficulties with going from a physical quantity such as a_{CP} and extracting the weak CP violating phase:

[4] Recall that a Lagrangian must be hermitian, not real!

1. We need two amplitudes with both a strong and weak phase. If $\delta = 0$ then the interference dissapears! Thus using QCD is inevitable.
2. QCD - δ is a pure QCD while r has to do with both electroweak and QCD. Typically we can't calculate QCD quantities.

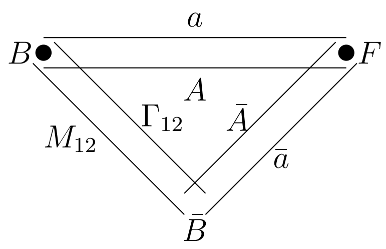
Two alternative methods which you can search for CP violation are:

1. Area of unitarity triangle (recall that if the area of the unitarity triangle is zero we don't have any CPV)
2. Electric dipole moment - every particle that cannot have an excited state (e.g. proton, electron, etc) that has an electric dipole moment is CP violating.

There are 3 types of $\not{CP}$

1. CPV in decay
2. CPV in mixing
3. CPV in interference between mixing and decay

Consider a B going to a final state, F,



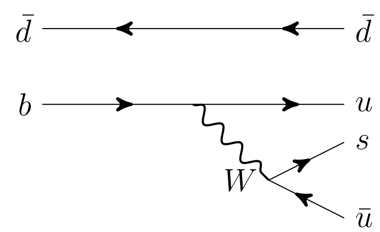
B can go to F through two different amplitudes, a and A while it can also first mix to a $\bar{B}$ which will then go into the same final state. The first kind of CP violation is interference between A and a. The second kind of interference is between M_{12} and Γ_{12}. The final kind of CPV arises from interference of a, A and with $M_{12}, \Gamma_{12}, \bar{a}, \bar{A}$.

CPV in decay can be extracted by taking,

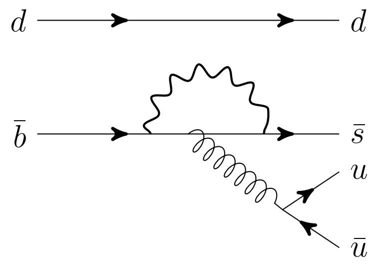
$$a_{CP} = \frac{\Gamma\left(\bar{B} \to \bar{F}\right) - \Gamma(B \to F)}{\Gamma\left(\bar{B} \to \bar{F}\right) + \Gamma(B \to F)} = \left|\bar{\mathcal{M}}/\mathcal{M}\right|^2 - 1 \tag{4.87}$$

as we measured above.

Lets consider the decay, $\bar{B} \to K^+\pi^-$, or equivalently, $\bar{d}d \to \bar{s}ud\bar{u}$. This can be accomplished with,



Similary at loop level we can have,



Lets track the phase of the diagram. For the tree level diagrams we have the CKM factors,

$$V_{bu}V_{us}^* \tag{4.88}$$

while the CKM factor for the loop diagram is,

$$V_{bt}V_{st}^* \tag{4.89}$$

However, the tree level diagram doesn't contribute a phase except for when we have V_{ub} which is highly surpressed. Therefore, to a very good approximation this diagram is real and so the two diagrams have a different weak phase.

A quick estimation gives,

$$\frac{\mathcal{M}(tree)}{\mathcal{M}(loop)} \sim \frac{V_{ub}V_{us}}{V_{tb}V_{ts}\frac{\alpha_s}{4\pi}} \sim \frac{4\pi\lambda^2}{\alpha_s} \sim \frac{1}{3} \tag{4.90}$$

[Q 17: Explain why both diagrams are surpressed by G_F.]

Here we are able to see CP violation. However, the issue is because of nonperturbativity we are not able to calculate the matrix element. This gives a measurement of γ.

4.5.1 Meson mixing with CP violation

Earlier we discussed meson mixing but we assumed that there was no $\not{C}\not{P}$. Now we redo the calculation without making that assumption. Here we have a nonHermitian hamiltonian. This is because the Hilbert space that we can considering is incomplete. Thus states can leak out of our Hilbert space. This is called an open system. However, any matrix can be written as a sum of Hermitian and antiHermitian matrices,

$$H = \begin{pmatrix} M_{11} & M_{12} \\ M_{12}^* & M_{11} \end{pmatrix} + i \begin{pmatrix} \Gamma_{11} & \Gamma_{12} \\ \Gamma_{12}^* & \Gamma_{11} \end{pmatrix} \tag{4.91}$$

where we still have $M_{11} = M_{22}$ and $\Gamma_{11} = \Gamma_{22}$ by CPT.

The eigenstates of the Hamiltonian are given by (here L stands for long and H stands for heavy),

$$|B_{L,H}\rangle = p\,|B\rangle \pm q\,|\bar{B}\rangle \tag{4.92}$$

where one can show that you can choose p, q are real and satisfy, $p^2 + q^2 = 1$. Here the states are not orthogonal:

$$\langle B_L | B_H \rangle = p^2 - q^2 \neq 0 \quad (\text{ in general}) \tag{4.93}$$

However, one can show that if you assume CP conservation then the states are still orthognal. Therefore if we have $|p/q| \neq 1$ it implies CPV.

By time evolving the state, $|B\rangle$, for a time t we get,

$$|B(t)\rangle = g_+(t) \, |B\rangle - \frac{q}{p} g_-(t) \, |\bar{B}\rangle \tag{4.94}$$

[Q 18: check.] where, $g_\pm(t) = \frac{1}{2}\left(e^{-im_H t - \frac{1}{2}\Gamma_H t} \pm e^{-im_L t - \frac{1}{2}\Gamma_L t}\right)$. We define a new parameter which is basis independent,

$$\lambda \equiv \frac{q}{p} \frac{\bar{A}_f}{A_f} \tag{4.95}$$

where,

$$A_f \equiv \mathcal{M}(B \to f), \qquad \bar{A}_f \equiv \mathcal{M}(\bar{B} \to f) \tag{4.96}$$

For simplicity we assume that,

$$y \equiv \frac{\Gamma_L - \Gamma_H}{\Gamma} = 0 \tag{4.97}$$

and that f is a CP eigenstate (e.g. K^+K^-). Then we get,[Q 19: check]

$$\Gamma(B \to f) \, [t] = e^{-\Gamma t} \left[(1 - |\lambda|^2) \cos xt - 2\text{Im} \, [\lambda] \sin xt \right] \tag{4.98}$$

$$\Gamma(\bar{B} \to f) \, [t] = e^{-\Gamma t} \left[(1 - \left|\lambda^{-1}\right|^2) \cos xt - 2\text{Im} \left[\lambda^{-1}\right] \sin xt \right] \tag{4.99}$$

The condition for CP conservation is then given by setting these two terms equal to one another. This implies that,

$$|\lambda|^2 = \left|\lambda^{-1}\right|^2 \qquad \text{Im}\lambda = \text{Im}\lambda^{-1} \tag{4.100}$$

The first equation requires the magnitude of λ to be 1 and the second requires the phase to vanish. Hence, we need $\lambda = \pm 1$.

To have $\lambda \neq \pm 1$ we can have,

1. CP violation in decay:

$$\left| \frac{\bar{A}}{A} \right| \neq 1 \tag{4.101}$$

2. CP violation in mixing:

$$\left| \frac{q}{p} \right| \neq 1 \tag{4.102}$$

3. The option is $|\lambda| \approx 1$ but $\mathrm{Im}(\lambda) \neq 0$.

To study case (2) we can use semileptonic decay since there we can differentiate between a particle and an antiparticle. Then we can know if the particle was a B or $\bar{B}$ at the time of decay. We define,[Q 20: check]

$$a_{SL}(t) \equiv \frac{\Gamma(B(t) \to \ell^+) - \Gamma(\bar{B}(t) \to \ell^-)}{\Gamma(B(t) \to \ell^+) + \Gamma(\bar{B}(t) \to \ell^-)} = \frac{1 - |q/p|^4}{1 + |q/p|^4} \tag{4.103}$$

where $\Gamma(B(t) \to \ell^+)$ really means a B meson oscillating into a B and then decaying into a ℓ^+. Note that the fact that this doesn't depend on time is a pure accident.

Alternatively we can find CP violation in mixing through looking at the decay of a mass eigenstate. This can only be done with Kaons since you can produce a collection of Kaons and after a short period of time all the K_S will decay. We study the ratio:[Q 21: check]

$$\frac{\Gamma(K_L \to \pi\ell^+\nu) - \Gamma(K_L - \pi^+\ell^-\nu)}{\Gamma(K_L \to \pi\ell^+\nu) + \Gamma(K_L - \pi^+\ell^-\nu)} = \frac{1 - |q/p|^2}{1 + |q/p|^2} \tag{4.104}$$

We can isolate K_L because if we produce a beam of Kaons (which starts with an equal mix of K_L and K_S) and wait long enough for the all the K_S to decay the beam will be a pure K_L.

Lets now study case (3). We assume $|\lambda| = 1$ and $\mathrm{Im}(\lambda) \neq 0$. One can show that,[Q 22: check]

$$a_{CP}[t] = \frac{\Gamma(\bar{B} \to f) - \Gamma(B \to f)}{\Gamma(\bar{B} \to f) + \Gamma(B \to f)} = (\mathrm{Im}\lambda) \sin xt \tag{4.105}$$

So the amplitude of the oscillation is the phase of λ. Unlike in the $B \to K$ decay we can measure this with very little hadronic uncertainty. [Q 23: Why does only this one have time dependence?] Roughly speaking the phase of q/p arises from the phase of the mixing matrix, so we can write it as,

$$\frac{q}{p} = arg(m_{12}\Gamma_{12}^*) \tag{4.106}$$

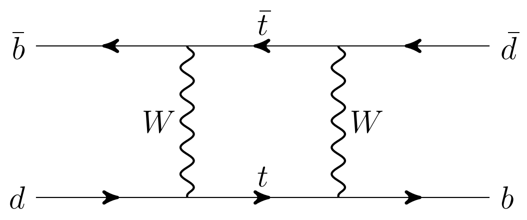
where we define $\phi_{m_{12}}$ as the phase of the m_{12} matrix element. The phase of $\bar{A}/A$ arises is,

$$\frac{A}{\bar{A}} = \frac{e^{i(\phi+\delta)}}{e^{i(-\phi+\delta)}} = e^{2i\phi} \tag{4.107}$$

Consider $B \to \psi K_S$ ($b \to c\bar{c}s$).Its easy to estimate the phase of this amplitude by its CKM factors,

$$V_{bc}V_{sc} \tag{4.108}$$

This doesn't have any phase since it doesn't include the first generation. The phase of q/p roughly arises from the box diagram,



The CKM factors are,

$$V_{bt}V_{td}V_{dt}V_{tb} \tag{4.109}$$

so the phase is twice the phase of $(V_{dt})^2$, which is just 2β. Therefore,

$$a_{CP}\left[B \to \psi K_S\right] = \mathrm{Im}\lambda = \sin 2\beta \sin xt \tag{4.110}$$

Notice that we have no idea what A_f is, but we know it has some phase. All we know is that $\bar{A}_f$ has the same magnitude as A_f, therefore it cancels. We have almost no hadronic uncertainty in this calculation. This arguement hinges on only having a single tree level diagram in A_f which means there are no extra interference terms. [Q 24: Exercise: Do the same but instead consider $B \to \pi\pi$]

4.5.2 Kaons

In the Kaon system we have a large heirarchy between the K_S and K_L lifetimes:

$$\frac{\Gamma_S}{\Gamma_L} \approx 580 \tag{4.111}$$

This implies that,

$$\Delta\Gamma \approx -\Gamma_S \quad \leftrightarrow \quad \Gamma \approx \frac{\Gamma_S}{2} \tag{4.112}$$

Furthermore we have,

$$\Delta m_K \approx 5.3 \times 10^9 Hz \tag{4.113}$$

and

$$\Gamma \approx 5.6 \times 10^9 Hz \tag{4.114}$$

which gives $x \approx 1.05$. These two numbers have no reason to be of the same origin. This is extremely finetuned by pure luck. We define,

$$\epsilon_K \equiv \frac{1+\lambda_{\pi\pi}}{1-\lambda_{\pi\pi}} \tag{4.115}$$

where $\lambda_{\pi\pi}$ is the λ value of $K \to \pi\pi$[Q 25: Is that true?] One can show that [Q 26: check],

$$|\epsilon_K|^2 = \frac{\Gamma(K_L \to \pi\pi)}{\Gamma(K_S \to \pi\pi)} \tag{4.116}$$

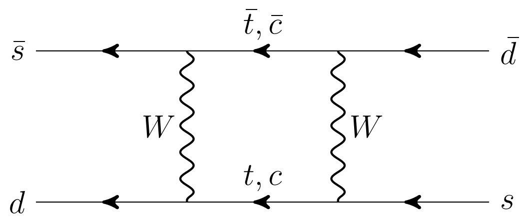
Furthermore we can measure the semileptonic assymetry,

$$a_{s\ell} = 2\mathrm{Re}(\epsilon) \tag{4.117}$$

These measurements give,

$$|\epsilon_K| = 2.3 \times 10^{-3} \tag{4.118}$$
$$\arg(\epsilon_K) = 43.7^o \approx \cot^{-1}(x) \tag{4.119}$$

Lets now estimate the phase of the box diagram,



where we keep both the top and charm contributions because of CKM suppression in the virtual top diagram. We work in the basis where V_{us} is a real, V_{cd} has a small phase, while V_{td} has a large phase, but is small. In practice you have to keep both these contributions when calulating the phase. We have,

$$\Delta m \sim \frac{g^4}{16\pi^2} \times \frac{m_c^2}{m_W^2} \tag{4.120}$$
$$\epsilon_K \sim \frac{g^4}{16\pi^2}\frac{m_c^2}{m_W^2} \times 10^{-3} \tag{4.121}$$

ϵ_K is a very good restriction on new physics since the SM result is so small. This requires that the scale on new physics be,

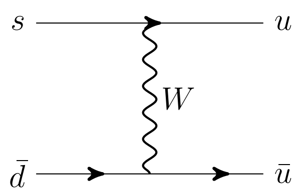
$$\epsilon_K \Rightarrow \Lambda \gtrsim 10^5 \tag{4.122}$$
$$\Delta m_K \Rightarrow \Lambda \gtrsim 10^4 \tag{4.123}$$
$$\Delta m_B \Rightarrow \Lambda \gtrsim 10^3 \tag{4.124}$$

These are the bounds on new physics assuming the most general type of model. In practice this means that whatever new model arises at the TeV scale, it can't have a generic flavor structure.

4.5.3 Kaon pion mixing - A lesson in QM

[Q 27: Write up a summary of the discussion of Kaon and pion mixing]

Above we discussed meson oscillating into their antiparticles. However, in principle this mixing can occur between any two particles of the same quantum numbers. For example we can have Kaon-pion mixing,



This is a much larger contribution then the $K-\bar{K}$ box diagram. However, the effects of this mixing are tiny. To see this suppose we have the mass matrix,

$$\begin{pmatrix} m^2 & \Delta^2 \\ \Delta^2 & M^2 \end{pmatrix} \tag{4.125}$$

The mixing angle is given by,

$$\tan\theta = \frac{2\Delta^2}{m^2 - M^2} \tag{4.126}$$

This mixing angle is small as long as m is very different from M and $\Delta \ll m, M$. This is the case for Kaon to pion oscillations. $m_\pi^2 \sim (100\text{MeV})^2$, M_K^2
$sim(500\text{MeV})^2$, and $\Delta \sim G_F \sim 10^{-2}\text{MeV}^2$. So we have a mixing angle of about 10^{-5}.

On the other hand for Kaons, $m = M$ by CPT. This produces a mixing angle of 45^o regardless of the value of Δ, but how much mixing actually occurs is still governed by Δ.

Chapter 5

Neutrino Physics

Unlike all the other topics discussed up to now, the SM doesn't encode the results of this chapter. Instead people invented the νSM (pronounced as "new Standard Model").

People have measured,

$$\Delta m_{12}^2 \sim 10^{-4}\text{eV}^2 \tag{5.1}$$

$$\Delta m_{23}^2 \sim 10^{-3}\text{eV} \tag{5.2}$$

which implies,

$$m_\nu \lesssim 10^{-1}\text{eV} \tag{5.3}$$

The massses of the neutrinos are roughly 10^{-7} smaller then the mass of the electron. The mixing angles are,

$$\theta_{12} \sim \theta_{23} \sim 1, \theta_{13} \sim 0.1 \tag{5.4}$$

which is very different then in the quark sector, where there is very little mixing between different generations.

5.1 Dirac vs Majorana Masses

A mass term couples a right handed field to a left handed field,

$$m\bar{\psi}_R\psi_L \tag{5.5}$$

If $\psi_L \neq \psi_R$ then we have a Dirac mass. Alternatively, we can define,

$$\psi^c \equiv C\bar{\psi}^T \tag{5.6}$$

which changes the chirality of a field. Then we can write,

$$m\overline{\psi_L^c}C\psi_L \tag{5.7}$$

However, for this term to exist the field must be neutral under all $U(1)$. Since $Q\left[\psi^c\right] = -Q\left[\psi\right]$. Now suppose that ψ is charged under some non-abelian group instead. Then we have,

$$\psi \to e^{iT_a\alpha_a}\psi \tag{5.8}$$

which gives,

$$\overline{\psi_L^c}\mathcal{C}\psi_L \to \overline{\psi_L^c}\mathcal{C}e^{iT_a^*\alpha_a}e^{iT_b\alpha_b}\psi_L \tag{5.9}$$

which is invariant if $-T_a^* = T_a$. This is the condition for having a real representation.

In the SM the only fields which can be Majorana are the neutrinos. In SUSY the gauginos as well as the neutral Higgsino can be Majorana.

Lets try to understand why neutrino masses are zero in the SM. Firstly we choose not to include right handed neutrinos which implies that there is no Dirac mass for the right handed neutrinos. Secondly, we assume only a doublet. If we had a triplet scalar, $\Delta(1,3)_1$, then we can have a Majorana mass for the neutrinos,

$$\overline{L_L^c}\Delta L_L \tag{5.10}$$

where L is the lepton doublet. We also choose the SM to be renormalizable. If we included dimension 5 operators then we could write,

$$(HL_i)(HL_j) \tag{5.11}$$

We also need to make sure that there is some symmetry to prevent the neutrinos from getting mass at loop level and the reason is because $U(1)_{B-L}$ is anomaly free (up to the effects of gravity).

There are essentially two ways to produce neutrino masses in the SM. We can add a fermion, ν_R and produce Dirac masses, or a scalar, Δ, which gives Majorana masses. The most naive guess might be Dirac masses,

$$M\overline{N_L^c}N_L \tag{5.12}$$

However, this is highly problematic since there is no good reason for $M = 0$.

If we have a scalar then the neutrino masses are,

$$m_\nu \sim \lambda \langle \Delta_{LL} \rangle \tag{5.13}$$

The problem is that Δ_{LL} needs to be very small ($\gtrsim$ GeV) not to interfere with a the ρ parameter of EWP and the coupling must also be small.

The modern way to think about adding these heavy degrees of freedom to the SM is we can write higher dimensional operators to the model. These can be written in terms of the dimension 5 neutrino mass operator above,

$$\frac{\lambda(HL)^2}{M} \Rightarrow m_\nu = \frac{mv^2}{M} \tag{5.14}$$

In this modern way of thinking neutrino masses must be there as long as the SM is an effective theory (unless the UV completian doesn't give the higher dimensional operator).

The scale of M is apriori unkown. The most naive guess would be $M = M_{Pl}$ however then neutrino masses turn out to be too low. There are an infinite number of UV completians that contain the dimension 5 operator as an output. The simplest option is the see-saw mechanism.

5.2 See-saw

In this model we add a right handed neutrino, $N_R(1,1)_0$, which implies that we have additional terms to the Lagrangian ($\tilde{H} \equiv \epsilon H^*$),

$$\Delta\mathcal{L} = i\bar{N}_R \partial\!\!\!/ N_R + Y \bar{L}_L \tilde{H} N_R + \frac{1}{2} M_N \overline{N_R^c} N_R \tag{5.15}$$

After SSB we assume that $M_N \gg v$. One may be concerned that this introduces a heiarachy problem. But we already have the heirarchy problem anyways so adding this in doesn't really make the problem any worse.

We make the one generation approximation. We get the mass matrix,

$$\begin{array}{cc} & \begin{array}{cc} \nu_L & N_R \end{array} \\ \begin{array}{c} \nu_L \\ N_R \end{array} & \begin{bmatrix} 0 & m_D \\ m_D^* & M_N \end{bmatrix} \end{array} \tag{5.16}$$

where $m_D \equiv Yv$. To a good approximation the eigenvalues are given by [1],

$$m_1 \approx -\frac{m_D^2}{M_N} \tag{5.17}$$

$$m_2 \approx M_N \tag{5.18}$$

A few remarks:

1. We have a negative mass. However, we can always redefine your fields to eliminate the phase. The only physical quantity is $m^\dagger m$.

2. The mixing angle is $\theta_{\nu N} \sim m_D/M_N \ll 1$. Therefore the mass eigenstate is to a very good approximation the same as the flavor eigenstate,

$$|1\rangle \approx |N_R\rangle \quad |2\rangle \approx |\nu_L\rangle \tag{5.19}$$

3. Matching with the dimension 5 operator we get,

$$M = M_N \quad \lambda = Y^2 \tag{5.20}$$

4. If we extend this to 3 flavors we find,

$$\frac{m_D^2}{M} \to m_D^T M^{-1} m_D \tag{5.21}$$

and,

$$U \to \left(\begin{array}{c|c} X_{NN} & X_{N\nu} \\ \hline X_{\nu N} & X_{\nu\nu} \end{array} \right) \tag{5.22}$$

$X_{N\nu}$ is the matrix that generalizes the mixing angle $\theta_{\nu N}$ so must be very small, $\mathcal{O}(m_D/M)$. Since U is the PMNS matrix, in the limit that $X_{N\nu} \to 0$ limit $X_{\nu\nu}$ is almost unitary.

[1] A quick way to diagonalize a two by two matrix is to note that the sum of the eigenvalues should be equal to the trace and the product of the eigenvalues is equal to the determinant.

5. This model has tree-level flavor changing neutral currents! You can have the processes:

$$N \to \nu Z, \nu H \tag{5.23}$$

These are typically small since they are suppressed by a mixing angle.

6. What if $m_D \gg M$? In this case you get "sterile neutrinos". These neutrinos are important at low energies but difficult to probe. They can be a form of DM.

This is often called seesaw type I. There are alternatives such as type II where we introduce a new Higgs as while as seesaw type III where we introduce a new triplet fermion. Each case gives small neutrino masses but has some different phenomenology.

5.3 Neutrino Mixing

Consider some general neutrino masses,

$$m_{ij}\bar{\nu}_L^i \nu_R^j \tag{5.24}$$

where we ν_R here is just the chirality symbol we don't really care here if it is actually a $SU(2)_L$ singlet. The charged current interaction is given by,

$$\mathcal{L}_W = -\frac{g}{\sqrt{2}}\bar{\ell}_{L,\ell}U_{\ell n}\gamma^\mu(\nu_L)_n W_\mu \tag{5.25}$$

where U is some matrix. $\ell = e, \mu, \tau$ and $n = 1, 2, 3$. To have neutrino mixing, U must be nondiagonal. The matrix breaks the lepton flavor symmetries:

$$U(1)_e \times U(1)_\mu \times U(1)_\tau \to U(1)_L \tag{5.26}$$

The mixing itself doesn't break lepton number (though lepton number violation often inevitably appears in the masses, e.g., in the right hand majorana neutrino masses). The mixing matrix has many names, "neutrino mixing matrix", "PMNS", "MNS", or "MNSP".

The mixing angles of the PMNS matrix are large,

$$\sin\theta_{23} \sim \sin\theta_{12} \sim \mathcal{O}(1) \quad \sin\theta_{13} \sim \mathcal{O}(0.1) \tag{5.27}$$

This is a big difference from the CKM matrix structure. One last difference is the order of the indices in the mixing matrix. Here we have $U_{\ell n}$ (charged lepton then neutrinos) ("down" then "up"). However, this is the opposite from the CKM which is defined as, $(V_{CKM})_{ud}$. This is just conventional, but important to keep in mind.

In the quark sector we have the up type and down type mass matrices. We choose to diagonalize m_u and m_d and we choose to keep the charged current nondiagonal. For the leptons however, we choose to diagonalize the charged leptons as well as the charged current, but leave the neutrino mass matrix nondiagonal. Therefore if for example we produce a muon neutrino then its going to oscillate. This is a consequence of a different basis choice then in the quark sector where we choose the basis such that the quarks are mass eigenstates. This is a smart basis choice since when we produce neutrinos from say the Sun, then it produces neutrinos of a single well defined flavor. If we chose to have a nondiagonal charged current then this would no longer be the case.

Assuming the neutrinos are Dirac neutrinos then just like the CKM the PMNS also has one phase. However, we can have more phases if the neutrinos are Majorana. The general Lagrangian in the lepton sector to dim-5 is,

$$\lambda_{ij}\phi L_i E_j + \frac{z_{ij}}{\Lambda}(HL_i)(HL_j) \tag{5.28}$$

The z_{ij} matrix is a complex symmetric matrix and λ_{ij} is a complex matrix so there are $18 + 12 = 30$ parameters. How many physical parameters? The symmetry breaking due to these terms is,

$$U(3) \times U(3) \to 0 \tag{5.29}$$

This gives 18 broken generator and hence we have 12 physical parameters. There are 9 real and 3 imaginary parameters, 6 masses, 3 mixing angles, 2 "Majorana phases" (denoted θ_{Maj}), and 1 "Dirac phase" (denoted δ_{Dirac}). In the Dirac mass case, we would have only one phase. This phase is the same as the "Dirac phase" that appears in the Majorana case, hence the name.

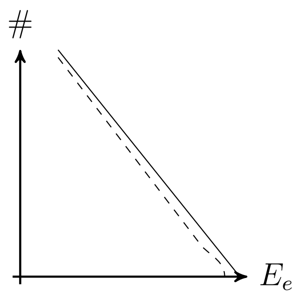
Measuring neutrino masses is hard. You can try to use pion decay,

$$\pi \to \mu\nu \tag{5.30}$$

which gives a bound, $m_\nu \lesssim 190$ keV. Much stronger bounds come from tritium decay through the process,

$$^3\mathrm{H} \to {}^3\mathrm{He} + e + \bar{\nu} \tag{5.31}$$

The kinematics of such a decay can be used to measure the neutrino mass. If the neutrinos were massless we have the solid line and if they gain a mass this gets modified to the dashed line in the sketch below:



Roughly speaking, the shape changes when the electron energy is m_ν away from the end point on the graph.

There are two effects. One is that the line shifts down and the other is the change of shape. Finding the change is shape is in practice much easier to measure. This is somewhat surprising because there are much more events in the rest of the plot. However, since knowing the normalization precisely is very difficult this isn't feasible. This has been done to energies, $m_{\nu_e} \lesssim 2.2\text{eV}$.

Another way to probe neutrino masses is done by neutrinoless double beta decay ($0\nu\beta\beta$). β decay is just,

$$n \to p + e + \nu \tag{5.32}$$

Here the neutrino must be present for angular momentum (and lepton number) conservation. Another process is $\nu\nu\beta\beta$,

$$2n \to 2p + 2e + 2\nu \tag{5.33}$$

In this case angular momentum can be satisfied with or without the neutrinos, they are only needed for lepton number conservation. $0\nu\beta\beta$ is,

$$2n \to 2p + 2e \tag{5.34}$$

Beta decay is very common. Two neutrino double beta decay is quite rare. It can only occur if two neutrinos come together and you also have huge phase space surpression. Neutrinoless double beta decay is even more surpressed since they only occur if neutrinos are Majorana fermions, but we do gain some on the phase space.

Remarks:

1. The rate for $0\nu\beta\beta$ obeys,

   $$\Gamma \propto m_\nu^2 \tag{5.35}$$

   since it must break lepton number by 2 units. If we have Dirac mass then we don't have lepton number violation and hence we can't have this process. $0\nu\beta\beta$ decay really measures lepton number violation. In the νSM the only source of lepton number is related to the neutrino masses and hence in this model this measures the neutrino mass. The mass is given by,

   $$m_{\beta\beta}^e = \sum m_i U_{ei}^2 \tag{5.36}$$

   Note there is no type here: there is no absolute value signs on U_{ei}.

2. There is theoreoretical uncertainty associated with nuclear matrix elements.

3. It turns out that the best way to study this effect is to use Germanium. In practice we actually build the detectors out of Germanium and let the Germanium decay.

4. The bounds from this experiment are

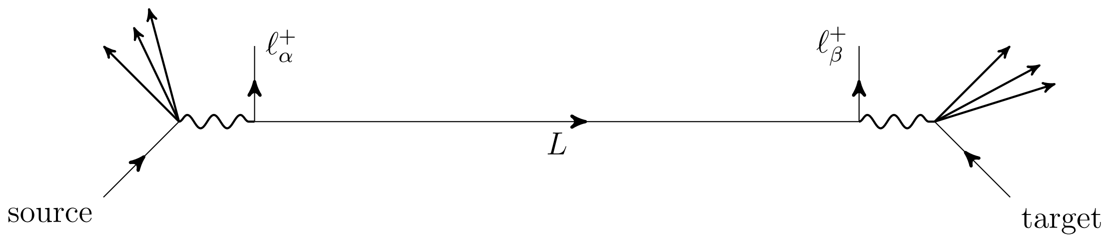
   $$m_{\beta\beta} \lesssim 0.4\text{eV} \tag{5.37}$$

5.4 Neutrino Oscillations

Neutrino oscillations are almost the same as meson mixing with a few major differences,

1. Neutrinos are ultrarelativistic
2. Neutrinos oscillate between 3 different states instead of 2
3. Mesons can decay but the neutrinos are stable (at least for all practical purposes)
4. Since here the particles oscillating into one another are not particles and antiparticles so we can't use CPT to restrict the Hamiltonian

While this doesn't actually describe nature, for simplicity we still make the two generation approximation. Consider a source and detector setup as shown below,



At $t = 0$ we have a neutrino with a known energy, E, and flavor, α. This can for example be done by decay pions which almost always decay into muons. At t_f we have find neutrinos with flavor β and energy E_f. Lastly we assume that the neutrinos are ultrarelativistic. In this case the neutrino velocity is equal to the speed of light.

When you produce neutrinos they are produced in a flavor eigenstate, say ν_ℓ for example. The neutrino is a linear combination of mass eigenstates:

$$|\nu_\alpha\rangle = \sum_\beta U_{\beta\alpha} |\nu_\beta\rangle \tag{5.38}$$

where $U_{i\ell}$ are entries from the PMNS matrix. When neutrinos propagate in space, the mass eigenstates are the ones that propagate. When the flavor is again measured after a while there is a probability to measure it as any of the flavor eigenstates. Thus lepton number for each generation is broken, though total lepton number is still conserved.

In the beginning we have,

$$\begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix} = U^*_{PMNS} \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix} \tag{5.39}$$

At some point farther away these states will pick up phases:

$$|\nu_i(\mathbf{p})\rangle \to e^{-iE_i t} |\nu_i(\mathbf{p})\rangle \tag{5.40}$$

where $E_i = \sqrt{\mathbf{p}^2 + m_i^2}$. Due to this interference between states we have a non-vanishing probability to find a new type of neutrino. This gives rise to the phenomena of neutrino oscillations. We now answer the question "if you produce a flavor α what is the probability that you will detect a flavor β a given distance away?" We label this probability $P(\ell_\alpha \to \ell_\beta)$.

We derive this result using simple quantum mechanics and treat the neutrinos as plane waves. At some initial time you produce a neutrino is a flavor state;

$$|\nu_\alpha(t_0)\rangle = \sum_i U^*_{\alpha,i} |\nu_i(\mathbf{p})\rangle \tag{5.41}$$

These states are eigenstates of the Hamiltonian and are on shell:

$$\hat{H} |\nu_i(\mathbf{p})\rangle = E_i |\nu_i(\mathbf{p})\rangle \tag{5.42}$$

$$E_i = \sqrt{m_i^2 + \mathbf{p}^2} \approx |\mathbf{p}| \left(1 + \frac{m_i^2}{\mathbf{p}^2}\right) \tag{5.43}$$

We evolve the initial state through the evolution operator:

$$|\nu_\alpha(t)\rangle = \sum_i U^*_{\alpha i} e^{-iE_i(t_f - t)} |\nu_i(\mathbf{p})\rangle \tag{5.44}$$

The probability is simply

$$P(\nu_\alpha \to \nu_\beta) = |\langle \nu_\beta | \nu_\alpha(t)\rangle|^2 \tag{5.45}$$

$$= \left| \sum_i U_{\beta,i} U^*_{\alpha,i} e^{-iE_i(t_f - t)} \right|^2 \tag{5.46}$$

The difference of the energies is,

$$E_j - E_i = \sqrt{|\mathbf{p}|^2 + m_j^2} - \sqrt{|\mathbf{p}|^2 + m_i} \tag{5.47}$$

$$\approx \frac{m_j^2 - m_i^2}{2|\mathbf{p}|} \tag{5.48}$$

Furthermore,

$$L = t_f - t \tag{5.49}$$

Inputting in these results gives the master formula:

$$P(\nu_\alpha \to \nu_\beta) = \sum_{ij} U^*_{\beta,j} U_{\alpha,j} U_{\beta,i} U^*_{\alpha,i} e^{i\Delta m^2_{ji} L/2|\mathbf{p}|} \tag{5.50}$$

The master formula gives the "probability of appearance" for $\alpha \neq \beta$ and the "probability of dissappearance" for $\alpha = \beta$. One can show that the master formula can be written in a slightly more famililar form using the Unitarity of the matrix: [Q 28: Derive this formula.]

$$P(\nu_\alpha \to \nu_\beta) = \delta_{\alpha\beta} - 4\sum_{i<j} \mathrm{Re}\left[U^*_{\alpha,i}U_{\beta,i}U_{\alpha,j}U^*_{\beta,j}\right]\sin^2\frac{\Delta m^2_{ij}L}{4E} - 2\sum_{i<j} \mathrm{Im}\left[U^*_{\alpha,i}U_{\beta,i}U_{\alpha,j}U^*_{\beta,j}\right]\sin\frac{\Delta m^2_{ij}L}{2E} \tag{5.51}$$

To compute the corresponding formula for neutrinos we just need to take the complex conjugate of the product of matrix. This gives,

$$P(\overline{\nu}_\alpha \to \overline{\nu}_\beta) = \delta_{\alpha\beta} - 4\sum_{i<j} \mathrm{Re}\left[U^*_{\alpha,i}U_{\beta,i}U_{\alpha,j}U^*_{\beta,j}\right]\sin^2\frac{\Delta m^2_{ij}L}{4E} + 2\sum_{i<j} \mathrm{Im}\left[U^*_{\alpha,i}U_{\beta,i}U_{\alpha,j}U^*_{\beta,j}\right]\sin\frac{\Delta m^2_{ij}L}{2E} \tag{5.52}$$

Thus to the extent that the imaginary term does not vanish we will have a difference in the oscillation probability for neutrinos and antineutrinos. In order to avoid this term the phases in the imaginary parts need to add to zero. It is a simple exercise to show that in the case that $\Delta m = 0$ for one set of the neutrinos this term vanishes.

For only two generations there is one angle in the mixing matrix and its given by,

$$U = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \tag{5.53}$$

Plugging in to the result above for disappearance[2]

$$P(\nu_\alpha \to \nu_\beta) = \sin^2 2\theta \sin^2\frac{\Delta m^2 L}{4E} = \sin^2 2\theta \sin^2\left(\frac{1.27\Delta m^2(\mathrm{eV})L(\mathrm{km})}{E(\mathrm{GeV})}\right) \tag{5.54}$$

This formula can be easily understood. You'll have more oscillation if the mass differences are greater or you have a longer beamline. Furthermore, we have a $1/E$ due to a time dialation factor. The greater the energy, the shorter the beamline appears to the neutrinos. The oscillation length is,

$$\lambda \equiv \frac{2\pi E}{\Delta m^2} \tag{5.55}$$

Lets consider some limits:

$$\begin{cases} L \ll \lambda & P(\nu_\alpha \to \nu_\beta) \to 0 \\ L \gg \lambda & P(\nu_\alpha \to \nu_\beta) \to \frac{1}{2}\sin^2 2\theta \\ L \sim \lambda & P(\nu_\alpha \to \nu_\beta) \to \sin^2 2\theta \sin^2\frac{\Delta m^2 L}{4E} \end{cases} \tag{5.56}$$

For GeV neutrinos with a beamline of 1 km we see 1 oscillation. To get more oscillations we want to minimize the energy and large beamlines. Then we can probe smaller Δm^2.

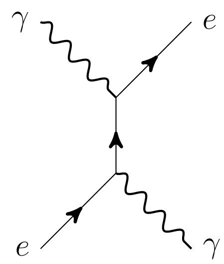
[2] It helps to note that for $i > j$ we can only have $i = 1, j = 2$.

5.5 Neutrino oscillations in matter

To zeroth order approximation, neutrinos don't care about matter and just pass through it. However, occasionally they can scatter off of matter. Is this a relevant effect? Consider neutrinos from the sun passing through the Earth. The energy is GeV and the distance is around 10^4km. The cross-section is roughly given by $G_F^2 E^2$. The penetration length is, [Q 29: compute this.]

When light goes through a material it scatters and it has an index of refraction. A similar effect occurs here.

When a photon passes through a material we have the diagram (and the crossed diagram),



This four point function is,

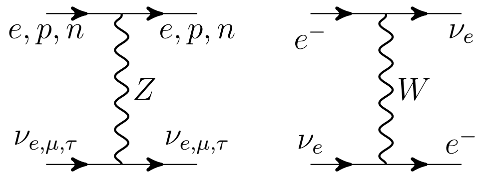
$$\langle 0 | T [A_\mu A_\nu e e] | 0 \rangle \tag{5.57}$$

We can take a semiclassical approximation by treating the electrons with a mean field,

$$\langle 0 | T [A_\mu A_\nu] | 0 \rangle \langle e e \rangle \tag{5.58}$$

To do this formally we need to understand which interactions we have.

For neutrinos the relevant diagrams are given by,



Its important to note that the left diagram is universal, while the right one can only occur for the electron. For this reason we only need to consider the second diagram.

To find the effect of the electrons on the neutrinos we use the mean field approximation. The Hamiltonian is given by,

$$H_{CC} = \sqrt{2} G_F \bar{e} \gamma_\mu (1 - \gamma_5) \overline{\nu}_e \gamma_\mu (1 - \gamma_5) \nu_e \tag{5.59}$$

We can trace out the effects of the background electron field using,

$$H_{eff} = \mathrm{Tr}_e [\rho H_{CC}] \tag{5.60}$$

where ρ is the density matrix and is given by,

$$\rho = \int d^3pf(E_e)\,|e(p)\rangle\,\langle e(p)| \tag{5.61}$$

The trace is taken only over the background field, i.e.

$$\mathrm{Tr}_e A \equiv \int d^3p\,\langle e(p)|A|e(p)\rangle \tag{5.62}$$

Inserting these relations into the above we have the Hamiltonian,

$$H_{eff} = \sqrt{2}G_F \int d^3pf(E_e)\,\langle e(p)|\bar{e}(1-\gamma_5)e|e(p)\rangle\,\bar{\nu}_e(1-\gamma_5)\nu_e \tag{5.63}$$

We now need to simplify the expression for the electron braket and put it in terms of quantities we are familiar with. To do that we expand the electrons in terms of plane waves,

$$\langle e(p,s)|\bar{e}\gamma_\mu(1-\gamma_5)e|e(p,s)\rangle = \frac{1}{V}\,\langle e(p)|\bar{u}_s(p)a_s^\dagger(p)\gamma_\mu(1-\gamma_5)a_s(p)u_s(p)|e(p,s)\rangle \tag{5.64}$$

We can identify the number operator as $\hat{N}(p) = a^\dagger(p)a(p)$, which gives,

$$= \frac{1}{V}\,\langle e(p,s)|\bar{u}_s(p)\gamma^\mu(1-\gamma_5)u_s(p)\hat{N}_e(p)|e(p,s)\rangle \tag{5.65}$$

$$\tag{5.66}$$

Averaging over electron spins and using

$$\sum_s \bar{u}_s(p)\gamma_\mu(1-\gamma_5)u_s(p) = \mathrm{Tr}\left[\gamma_\mu(1-\gamma_5)(\not{p}-m)\right] = p_\mu \tag{5.67}$$

$$\langle e(p,s)|e(p,s)\rangle = \frac{1}{E_e} \tag{5.68}$$

we have,

$$H_{eff} = \sqrt{2}G_F\bar{\nu}_e\gamma_\mu(1-\gamma_5)\nu_e \int d^3pf(E_e)N_e(p)\frac{p^\mu}{E_e} \tag{5.69}$$

If we assume an isotropic medium then there should be no preferential direction in space, i.e.,

$$\int d^3p\vec{p}f(E_p) = 0 \tag{5.70}$$

and we have the final expression,

$$H_{eff} = \sqrt{2}G_F N_e\bar{\nu}_e\gamma_0(1-\gamma_5)\nu_e \tag{5.71}$$

where we have used,

$$\int d^3pf(E_e) = 1 \tag{5.72}$$

[Q 30: Go over these steps.] [Q 31: Show quantitatively when this approximation is valid.]

The effective mass is then given by,

$$m \sim \sqrt{2} G_F N_e \tag{5.73}$$

Notice that this is a strange mass term since it involves spin structure. Unlike a typical mass this corresponds to a vector and not scalar potential.

As expected the interaction is stronger, if we have a large density of electrons. We can change this to a density using,

$$\sqrt{2} G_F N_e \sim \frac{N_e}{N_e + N_p} \rho \frac{eV}{10^{13} g/\mathrm{cm}^3} \tag{5.74}$$

For the Earth we have $\rho \sim 1 g/\mathrm{cm}^3$ while for the sun, $\rho \sim 10^2 g/\mathrm{cm}^3$. So for neutrinos travelling through the Sun (Earth) they gain an effective mass of order 10^{-11} eV (10^{-13}eV).

The neutrino mass is 10^{-2}eV or 10^{-3}eV. Since this is so much larger then the effective mass naively one would expect this to be a tiny effect that we wouldn't care about. The reason this is important is because this potential is actually a vector potential. When we have a scalar potential we add it to the Hamiltonian through,

$$p_\mu p^\mu = (m - V)^2 \tag{5.75}$$

However, for a vector potential due to Lorentz invariance we must add is through,

$$(p_\mu - V_\mu)(p^\mu - V^\mu) = m^2 \tag{5.76}$$

where for us $V_\mu \sim \sqrt{2} G_F N_e g_{\mu 0}$. The Hamiltonian that we calculated is the zero component of the vector potential so we have,

$$(E - V)^2 = p^2 + m^2 \tag{5.77}$$

$$E \approx p + V + \frac{m^2}{2p} \tag{5.78}$$

$$= p + \frac{2EV + m^2}{2p} \tag{5.79}$$

The effective mass is $m^2 + 2EV$. Therefore the relevant comparison is not m to V, but instead m^2 to $E \times V$. Therefore, at high neutrino momenta this effective mass becomes importnant.

A careful calculation gives [Q 32: Do this calculation...],

$$\Delta m^2_{eff} = \sqrt{(\Delta m^2 \cos 2\theta - A)^2 + (\Delta m^2 \sin 2\theta)^2} \tag{5.80}$$

where $A \equiv 2mV$ and

$$\tan 2\theta_{eff} = \frac{\Delta m^2 \sin 2\theta}{\Delta m^2 \cos 2\theta - A} \tag{5.81}$$

Note that in the limit that $A \to 0$ we get a the vacuum result as expected.

If $A \to \infty$ we have the limit,

$$\Delta m^2_{eff} \sim A \quad \tan 2\theta_{eff} \sim 0 \tag{5.82}$$

In this case the electron neutrino doesn't mix with the other neutrinos! Therefore, in this case the flavor eigenstate becomes the mass eigenstate. We have "flavor locking". When we go in a very dense area, the electron neutrino becomes a mass eigenstate.

In the limit that $A \to \Delta m^2 \cos 2\theta$,

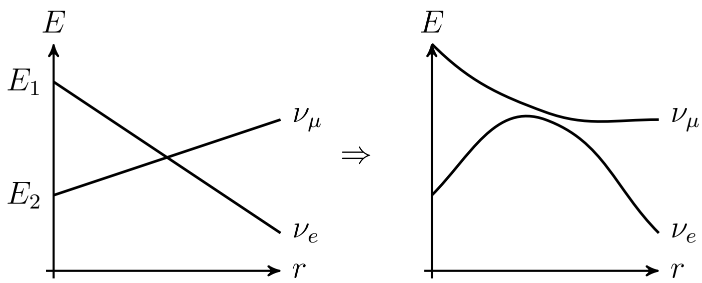
$$\Delta m^2_{eff} \to \Delta m^2 \sin 2\theta \quad \tan 2\theta_{eff} \to \infty \tag{5.83}$$

and so we have maximum mixing. Therefore even if Δm^2 is very small there will still be a special value of A such that we have maximum mixing.

Now suppose that a neutrino is passing through a material with a background that has a changing density. This gives rise to the "MSW effect". There are two regimes where we can solve this problem, if the change is slow (adiabatic) or the change is sudden. In the adiabatic approximation, which we consider here, the state remains in the same eigenstate but the energy changes. This is in fact the case for an neutrino passing through the sun, which is much denser in the core then it is on the surface. We have,

$$Q(r) = \frac{\lambda}{A}\frac{dA}{dr} \tag{5.84}$$

If $Q(r) \ll 1$ then we are in the adiabatic limit and when $Q(r) \gg 1$ then we are in the suddent limit. In the sudden approximation we take the energy to be unchanged but the wavefunction changes when the change occurs. Suppose we have two neutrinos, ν_e and ν_μ. The energy of the neutrinos changes with the coupling, G_F. In quantum mechanics, eigenvalues that interact with one another don't cross as you change the coupling. [Q 33: Expand and clarify this section...]



where the splitting between the energy levels is just $\Delta E \sim \Delta m$. One can show that the probability for a transition between an electron to a muon as A increase is known as the Parke formula,

$$P_{e\mu} = \frac{1}{2}\left[1 + (1 - 2P_{LZ})\cos 2\theta_{r=0} \cos 2\theta_{r\to\infty}\right] \tag{5.85}$$

where P_{LZ} is a probability constant. [Q 34: A should be negative for antineutrinos. Show that this is the case from the amplitude calculated above.]

5.6 Neutrino oscillation experiments

The big issue with neutrino oscillation experiments is that the cross section of neutrino interactions with the detectors is very small. To solve this problem we can do two things,

1. Increase the size of the target
2. Bring the source close to the target so all the neutrinos hit the target

On the other hand the oscillation frequency scales inversely with the length of the experiment. Therefore, we don't want to bring the target too close that we don't see oscillation but not too far so we don't lose too many neutrinos.

Next consider the neutrino energy. The cross section of Fermi interaction increases with the energy. Furthermore, the higher energy beam that you have, the easier it is to focus the beam. On the other hand, we also would like to have low energy neutrinos since lower energy neutrinos oscillate more. This is just because of special relativity. For very fast moving neutrinos, the length of the beam line is contracted by a γ factor. Therefore, similarly for the length of the beamline, we again have a competition.

For an initial beam we want to know the energy, the flux, as well as the flavor. We then detect each of these quantity for the final beam.

In general there are two types of neutrino oscillation experiments. The one type is called appearance exeriments. In this set of experiments you start with some flavor $N_\alpha(t=0)=0$ and find that at some time t_f after $N_\alpha(t_f)\neq 0$. For this experiment the flux cancels out when calculating physical quantities. Therefore, the important properties for these types of experiments is the energy of length of the beamline. For disappearance experiment we start with $N_\alpha(t=0)$ and check how many neutrinos of flavor α we have at some time t_f. Here the flux is important.

An interesting test of neutrinos comes from looking at the Sun. The photons in the sun have a very difficult time getting out of the Sun due to all the interactions with matter. The time it takes a photon to leave the center of the Sun and arrive on Earth is roughly a billion years. On the other hand, because neutrinos don't interact the time it takes neutrinos to go from the Sun to the Earth is about 8 minutes.

At the sun we know the flavor of the neutrinos very well since its dominated by electron neutrinos. Note that fusion tends to come with neutrinos while fission tends to come with antineutrinos. The energy is roughly given by the scale of nuclear physics, MeV. We certainly can't produce muons which have a mass of about 100MeV. The basic process is given by,

$$4p \to {}^4\text{He} + 2e^+ + 2\nu_e + 2\gamma \tag{5.86}$$

The 98.5% of the neutrinos have an energy, $E_\nu < 0.$. The problem is that experimentally is very hard to go to such small neutrino energies. The first generation of experiments wasn't able to probe these low energy neutrinos.

One detection mechanism is known as the charged current interaction,

$$\nu_e + n \to p + e^- \tag{5.87}$$

This can't produce muons since the energy difference between the proton and the neutron is about 1 MeV and the maximum energy of the neutrinos is about 10 MeV. So this is insufficient to produce muons.

The second detection is known as elastic scattering and its given by,

$$\nu_\ell + e^- \to \nu_\ell + e^- \tag{5.88}$$

All neutrinos can participate in this interaction, but the cross sections for the electron are different then for the muon and tau since while the muon and tau have to interaction via a Z boson, the electron neutrino can interact via the W boson.

The last detection mechanism is known as the neutral current interaction,

$$\nu_\ell + N \to \nu_\ell + N\,,\; N = n, p \tag{5.89}$$

In order to have this interaction we need to break into the nucleus. This is most easily done with Deuterium. With heavier water, we have neutral current detection through the breaking of the Deuterium,

$$\nu + {}^2\mathrm{H} \to n + p + \nu\,. \tag{5.90}$$

Since this interaction mechanism is mediated by the Z boson, it is the same for all neutrino flavors.

We now define

$$R = \frac{N_{obs}}{N_{MC}} \tag{5.91}$$

where N_{obs} is the number of observed neutrinos and N_{MC} is the number of Monte Carlo neutrinos assuming no oscillations. $R = 1$ corresponds to no oscillations.

The first experiment that looked for this is called the homestake experiment. The idea is to look for,

$$\nu_e + {}^{37}\mathrm{Cl} \to {}^{47}\mathrm{Ar} + e^- \tag{5.92}$$

The neutrinos are assumed to come from the Sun which have energy, $E_\nu > 0.814\mathrm{MeV}$. The experiment found, $R \approx 0.3$. Therefore, the experimentors saw that some neutrinos dissappeared.

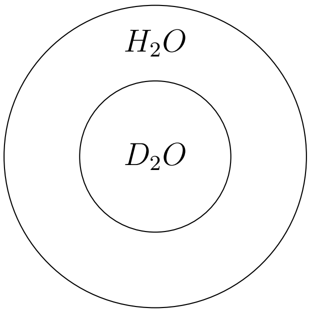
To test whether this effect was just due to a misunderstanding of the sun people produced the Gallium experiment,

$$\nu_\ell + {}^{71}\mathrm{Ga} \to {}^{71}\mathrm{Ge} + \ell \tag{5.93}$$

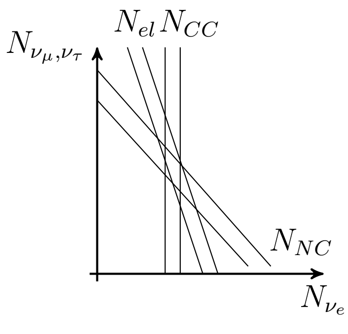
Here you are sensitive to neutrino energies, $E_\nu > 0.233\mathrm{MeV}$. Here they found that $R \approx 0.6$. Here the mass difference of the isotopes is about 700MeV. Therefore, they are able to produce muons as well as electrons (though this was not the conclusion at the time). Now people saw that we can measure, $R(E_\nu)$.

The next set of results came out of Superkamiokande. They have a huge tank of water. When a neutrino passes through the water it can emit Cherenekov radiation. They are sensitive to $E_\nu \gtrsim 5\mathrm{MeV}$. They also have directionality and time resolution in their experiment. They found that $R \approx 0.5$. They directionality really allowed them to see that the neutrinos truely came from the Sun.

More recently, the SNO experiment is Sudbury was undertaken. The basic idea is as follows:



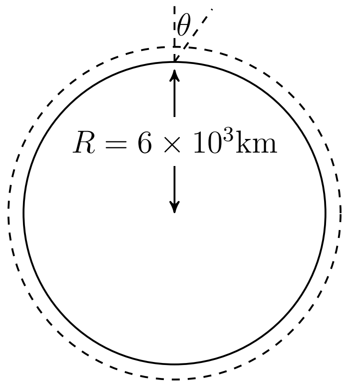
where D_2O is just heavy water. Here neutrinos can interaction using charged current, elastic, and neutral current scattering. the charged current interactions only measure electrons. There are some fixed number of muon and tau neutrinos due to oscillation. The neutral current interactions measure the different neutrino flavors equally. The elastic scatterings measure primarily the electron neutrinos with some sensitivity to muon and tau neutrinos. The measured cross section of each source was used to plot the number of muon and tau neutrinos vs the number of electron neutrinos. They found that the 3 sources agree perfectly. The three measurements are sketched below:



confirming that the neutrinos really came from one source.

5.6.1 Atmospheric neutrinos

The idea of atmoshpheric neutrinos is the following. Consider the Earth,



The atmosphere is roughly 20-30 km. In the upper atmoshpere we have cosmic rays hitting the Earth that come from a well known spectrum. When these protons hit the atmosphere hit the atmosphere (mostly protons) they produce primarily charged pions,

$$p + p \to \pi^+ + ... \to \mu^+ + \nu_\mu + ... \to (e^+\nu_e\bar{\nu}_\mu)\nu_\mu + ... \tag{5.94}$$

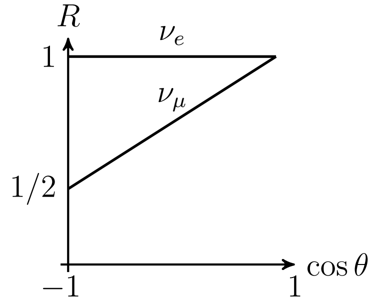
To leading order we expect (at these energies we can't really distinuish neutrinos and antineutrinos),

$$N_{\nu_\tau} : N_{\nu_\mu} : N_{\nu_e} \simeq 0 : 2 : 1 \tag{5.95}$$

The cosmic muons reach the Earth with a large boost (high energy pions rarely reach the Earth since they have a lifetime of about a factor of a 100 smaller due to phase space differences (2 body vs 3 body decay)) therefore there we don't have any electron neutrinos. The energy scale here is roughly about a GeV.

Depending on which directly the neutrinos come from they will propagate through matter a lot more. If the neutrino arises from the sky then they travel for roughly 30km, but if they travel through the Earth they propagate about 10^4 times more.

In reality experiments that were indifferent to the direction of the neutrinos found $R \approx 1$. This was known as the atmoshperic neutrino puzzle. Since then SuperK found the following variation of R as a function of the angle:



where θ is define as shown in the diagram above.

This showed that muon neutrinos are dissapearing for longer beamline distance. However, they are not changing into electron neutrinos. Therefore, the muon neutrinos are changing into tau neutrinos. This is a consequence of having $\theta_{12} \gg \theta_{13}$ and $\theta_{23} \gg \theta_{12}$.

5.6.2 Reactors neutrinos

The idea of reactor neutrinos is that we have a nuclear reactor with an unstable element such as uranium which decays and emits neutrinos. The reactors are typically built to produce power for industrial purposes, but we can use the inevitable emission of neutrinos from the reactors to study neutrinos oscillations. The decays are well known so we know the flavor of the produced neutrinos very well. Furthermore, we also understand the energy spectrum of the neutrinos very precisely. Lastly, we can turn off our reactors when we please. This allows a better calibration and understanding of background.

For reactors you put your detector next to your reactor. To really see the oscillation you need to go a minimum of a few km's from the reactor.

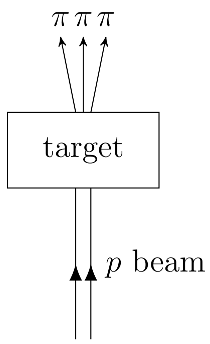
In Japan using an experiment called kamland for $L \sim 180$km they measured Δm^2_{12} very presisely. More recently, in Daya bay they measured Δm^2_{13}. From atmospheric neutrinos we measured Δm^2_{23}. Using these we can then calculate all the mixing angles and the PMNS matrix.

5.6.3 Accelerator neutrinos

Using accelerators we can get much higher energy neutrinos and so we can get much larger flux. However, to go see oscillations at such high energies we have to go further. These have oscillations lengths on the order of 10^3km and are called long baseline (LBL) experiments. The production at accelerators is done by,

$$p + p \to \pi^+ ... \to (\mu + \nu) + ... \to ((e + \nu) + \nu) + ... \tag{5.96}$$

The advantage of accelerator neutrinos is we know the spectrum and flux of neutrinos relatively well. Another advantage is that you can use beams which are produced for other reasons, as is the case at the LHC.



The first long baseline experiment was K2K which was only 250 km. This is relatively short which made it hard to see the oscillations. There are two other long baseline experiments, at CERN - Gran Sasso and at Fermilab - Soudan. These completely coincidently both have about the same baseline length (732km). Detection of accelerator neutrinos is primarily done using water Cherenkov detectors. This has the advantage that it has directional information. The future of neutrino physics in the US is the Deep Underground

Neutrino Experiment (DUNE). The most important result of LBL is the confirmation of atmospheric neutrino results. At the time of this writing there is also a 2σ detection of CPV in the neutrino sector.

The short baseline experiments have several outstanding anomalies. The most famous is from an experiment known as LSND. The neutrinos here are much less energetic as they arise from pions decaying at rest. The beam line is about 30 meters. They look for $\mu \to e$ oscillation and they detected a positive signal at 3.5σ. If you assume a 2 generation model, you can get a good fit for $\Delta m^2 \simeq 1$ eV and a relatively small θ. Shortly after, an experiment called KARMEN with a somewhat different setup ruled out most, but not all, of this parameter space.

To solve this puzzle the Mini-Boone experiment was built at Fermilab. They ruled out the remaining parameter space for the simple 2-flavor model but saw a positive signal somewhere else. Furthermore, they looking at neutrinos and not antineutrinos. This is still an on-going problem.

5.7 Beyond the νSM

There are three questions which we ask when thinking beyond the νSM.

1. Are there more light states, possibly sterile neutrinos?
2. What are UV completions of the νSM?
3. Are ν interactions as in the standard model, known as Non-standard Neutrino Interactions (NSI)

We won't saying anything about the final question.

5.7.1 Sterile neutrinos

Sterile neutrinos are additional states are that singlets under the gauge group of the SM, $N(1,1)_0$, which gives an additional light mass ($m \lesssim$ MeV) eigenstate. We know how many active neutrinos from measuring the invisible width of the Z. However, you can't measure this width directly since you can't measure the neutrinos. Instead you can look for a process with initial state radiation,

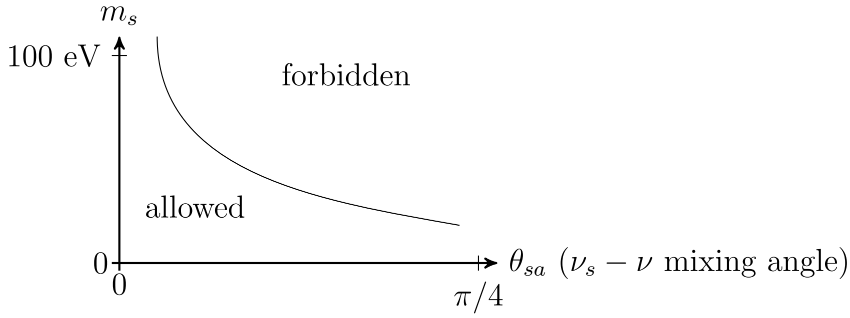
$$e^+e^- \to Z\gamma \to (\bar{\nu}\nu)\gamma \tag{5.97}$$

Alternatively you can measure the total width of the Z by scanning over energies of the incoming state. Then you measure the visible width using the known production cross-section. Subtracting the total width from the visible width we can obtain the invisible width.

These measurements give a number of active neutrinos of,

$$N_\nu = 3.00 \pm 0.05 \tag{5.98}$$

Sterile neutrinos can also effect neutrino oscillation experiments. Rough bounds look something like:



There are several bounds on sterile neutrinos

1. Kinematics - bounds from decays, e.g., β decay

2. Short Baseline neutrino experiments - use short baseline ($\mathcal{O}$(10m)) experiments and look for disappearance

3. Oscillation experiments - possible modifications to long baseline experiments where you have known flavor oscillations

4. Big Bang Nucleosynthesis (BBN) - if we have sterile neutrinos they can increase the density of the universe,
$$\rho_{\rm tot} = \rho_{\nu_s} + \rho_{\rm SM} \tag{5.99}$$
which controls the expansion of the universe through the Freidmann equation. BBN constrains the expansion rate around an MeV due to the competition between Helium dissociation and neutron decay. The observed abundances of Deuterium, Helium, etc. and depend sensitivity on the expansion rate and in turn the density of the universe.

5.8 Lepton flavor problem

In the lepton sector we have the parameters,

$$\begin{aligned} &\frac{m_\mu}{m_e} \simeq 200 && \frac{m_\tau}{m_\mu} \simeq 17 \\ &\Delta m_{12}^2 \simeq 7\times 10^{-5}\ \mathrm{eV}^2 && \Delta m_{23}^2 \simeq 3\times 10^{-3}\ \mathrm{eV}^2 \\ &\theta_{23} \simeq \frac{\pi}{4} && \sin^2\theta_{12} \simeq \frac{\pi}{3} \\ &\theta_{13} \ll 1 \end{aligned}$$

This sector contains some hierarchy between parameters but it doesn't smack you in the face as the hierarchy problem does. Its possible that these arise from a completely anarchic assignment of parameters. Alternatively, there could be some underlying structure. Lets consider one such example known as the Tribimaximal mixing scenario which gives,

$$\sin^2\theta_{12} = \tfrac{1}{3}(0.31) \qquad \sin^2\theta_{23} = \tfrac{1}{2}(0.42) \qquad \sin^2\theta_{13} = 0(0.03)$$

This has reasonable agreement with experiment so people suggest that this is a leading order approximation. The simplest way to get this pattern is through what are known as A_4 models.

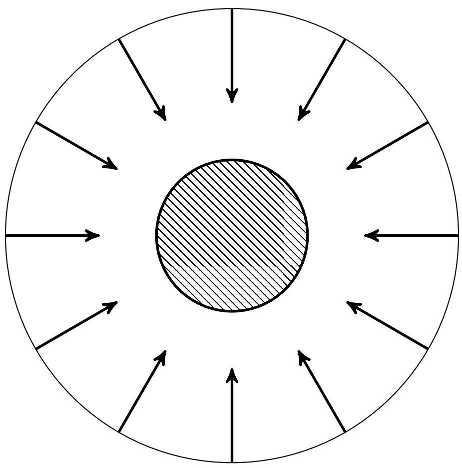
5.9 Additional constraints on neutrinos

5.9.1 Neutrinos in supernova

We can use astrophysics to try to probe neutrinos. If we find agreement with our astrophysical model with neutrinos then we are happy. If there is disagreement there is a question of whether the astrophysics or the particle physics is wrong. A powerful probe of neutrinos is supernova. There are different types of supernova. At the initial state the pressure in the star is so big that the protons and electrons combine,

$$p + e \to n + \nu_e \tag{5.100}$$

The idea is the density increases, the momentum of the electrons (i.e., the Fermi surface), also grows. As the energy of the electrons grows its kinematically favorable for the proton and electron to combine emitting the neutrino. This process takes a fraction of a second. After the initial emission of neutrinos, the star begins to develop a core:



Because it is so dense the neutrinos are not able to escape the core. The rate of collisions is faster than the inverse of the radius of the core (about 10km), resulting in neutrinos in equilibrium, with a temperature of $T \sim 30$ MeV.[3] This makes the supernova core opaque

[3]More precisely its quasi-equilibrium since in ~ 10s the core cools down.

to neutrinos. This is analogous to what happens in the sun for photons. The photons in the sun are stuck since the rate of collisions in so large.

The supersnova core (also known as the "neutrino sphere") is different from the electron and muon (or tau) neutrinos. The reason is that cross-section to collide with electrons is different for the different type of neutrinos.

The most famous supernova is probably supernova 1987A. This was a supernova in the Mangellanic cloud. It was not in our galaxy but it was close enough. Superkamiokande was able to observe about 20 events. They saw potentially a faint signal from the initial supernova collapse but saw a very clear signal from the supernova core. There are a few bounds we can derive from this event,

1. Stability: since the distance to the supernova was $\sim 170\,000$ light years we know the neutrino lifetime is roughly longer then 10^5 years.

2. Mass difference: to our experimental sensitivity we found the neutrino beams come all in the single bunch. If there mass difference was large, then we would get 3 separate bunches. This allowed us to put a constraint of $\mathcal{O}(10 \text{ eV})$ on the mass difference

3. Neutrino magnetic moment: If the neutrino is a Dirac neutrino it will have a magnetic moment which will interact with the magnetic field of the universe.

4. Lorentz invariance violation: Lorentz invariance tells us that the velocity of the neutrino will be independent of the direction of the proton that created it. We can check this using these supernova neutrinos. This is actually the best bounds we have on Lorentz invariance violation.

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5.9.2 Neutrino dark matter

Neutrinos are a form of dark matter so in principle they could account for the extra matter in the universe. If the neutrino were heavier, with a mass around 20 eV, then it could account for the extra energy density observed today in dark matter. However this is ruled out for additional reason, because we observe small scale structure which would be inconsistent with hot dark matter. We can use this to put an additional constraint on neutrino masses

5.10 Bounds on new physics

All this section we discussed the SM unique dimension 5 operator,

$$\frac{LHLH}{M} \tag{5.101}$$

There are many dimension 6 operators. Flavor conserving operators are bound by EWP, flavor violating operators are bound by meson measurements, and baryon/lepton number violation is bound through proton decay.

We spoke in depth about all of the operators except the ones correponding to proton decay bounds. One such operator is,

$$\frac{QQQL}{M^2} \tag{5.102}$$

This could give proton decay through,

$$p \rightarrow e^+\gamma, p \rightarrow e^+\pi^0 \tag{5.103}$$

In principle since these operators are dimension 6 they should give a much weaker bound on the scale M. However experimentally, proton decay is very easy to probe. For this reason, the strongest bound on M (from proton decay) gives, $M \gtrsim 10^{16}$GeV.

To have proton decay we actually need both baryon and lepton number violation. The reason is that the proton needs a (assymptotically free) fermion to decay to and in the SM the only fermions lighter then the proton are the leptons. This is actually a bit too strong a statement since we could have for example decays to the gravitino which would be invisible.

The proton decay operators are $\Delta B = \Delta L = 1$. There are also experiments which probe $\Delta B = 2, \Delta L = 0$. One such experiment looks for $n - \bar{n}$ oscillation. Another example looks for,

$$^{16}O \rightarrow {}^{16}C + \pi^+\pi^- \tag{5.104}$$

The bound on these are much weaker then on proton decay.

Chapter 6

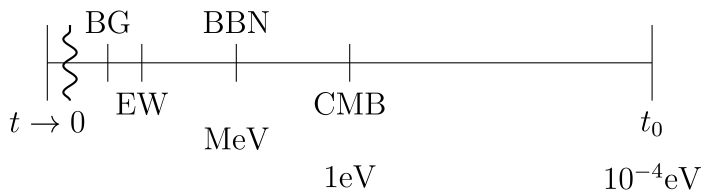
Cosmology

In this course we aren't going to go into detail to the fundamentals of cosmology but instead we focus on how we use cosmology to put bounds on new physics. Historically we used the SM to make predictions for cosmology. Now that cosmology has become more mature we can do the inverse and use cosmology to put bounds on new physics.

The basic idea is as follows. We know that the universe was extremely hot at "the beginning" and slowly the universe became bigger and colder. When the temperature of the universe is about the mass of a particle then this particle can be created spontaneously after this point this can no longer happen and we say that these particles freeze out.

The way we'll think about cosmology is to take our current universe and begin going backward in time. Right now the universe is at $2.7K$. When we go back up to the weak scale we think we know almost all that happens since we know which particles are and aren't relevant (particles that are frozen out become largely irrelevant for the structure of the universe.

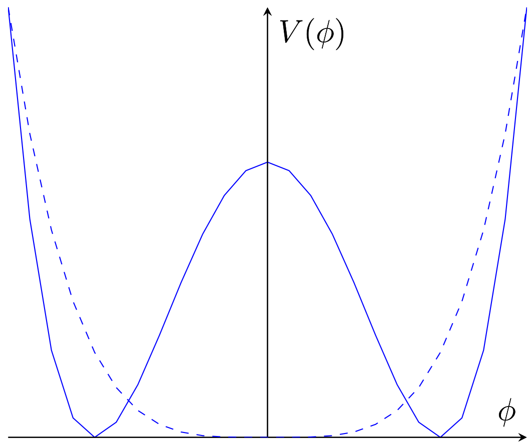
From the weak scale until the Planck scale, we at least know that we have quantum field theory. Past the Planck scale we really don't know how to even describe nature. The basic picture is (BG - baryogenesis, EW - electroweak phase transition, BBN - big bang nucleosynthesis, CMB - cosmic microwave background),



We never really take time to 0 since just know that at that point everything breaks loose and all our foundations of physics fail. When we look at $t \to 0$ there is a lot of cosmology that is irrelevant for particle physics. The first time when particle physics begins to be important is when we produce the CMB. Further back we have big bang nucleosynthesis, when we produce the primordial elements.

Before this we have the electroweak phase transition when the Higgs potential changed from a mexican hat to a potential with a minima at 0. This is far from a trivial statement.

Its very clear that when we go to a very high energy we don't care about the minima being away from zero, but a phase transition is very different - its a morphing of the potential,



where the dashed line is the new effective potential after the phase transition. This effective potential arises from integrating over the background, which is a very hot plasma. Whether this phase transition is first or second order is still unclear.

The last relevant part for us is baryogenesis, when protons were made. This explain why we have more baryons then antibaryons.

In this course we don't discuss inflation, which while is very interesting, cannot be used to put bounds on new physics.

One other way to describe time in astrophysics is through the redshift (z). When light is propagating through an expanding space its wavelength gets longer causing a redshift. This can be a useful parameter to describe how long ago the light was produced, because since we know that space is expanding, this is a unambigious way to specify time. We define $z = 0$ for light emitted now and the longer ago the light was produced the larger the redshift. Light produced at the time of the CMB has $z \sim 1100$. Since z grows so high in cosmology it isn't used very often in this field but very often used to describe stars and galaxies.

Now we describe the basics of cosmology. The basic assumption is that we have dynamic, isotropic, and homogenous. This means that the only evolution of the universe is through time. This gives the invariant interval,

$$ds^2 = dt^2 - a^2(t) d\Sigma^2 \tag{6.1}$$

where $d\Sigma^2$ is essentially the same as dr^2 but with a small modification which we will not worry about. Its far from trivial that $a(t)$ is only a function of t, however this is fundamental assumption.

We also assume that this completely describe the evolution of the universe, so if we know $a(t)$ then we know everything in terms of cosmology. From there we know the size of the universe and hence the temperature and all its properties.

This $a(t)$ is related to the redshift and the wavelength of light emitted at time t ($\lambda(t)$) by,

$$1 + z \equiv \frac{a(t_0)}{a(t)} = \frac{\lambda(t_0)}{\lambda(t)} \tag{6.2}$$

We define the Hubble constant,

$$H(t) = \frac{\dot{a}(t)}{a(t)} \tag{6.3}$$

This is not a constant in time, but in direction. H tells you how fast the universe is changing. H tells you the typical time scale for things that happen in the universe. When we talk about cosmology, each process has some time, e.g. a neutron decaying. When we talk about BBN we can compare the neutron decay rate (15min) to the Hubble constant. If $\Gamma_{neutron} \ll H$ then we can treat the neutron as stable.

The current value of H is,

$$H_0 \sim 10^{-33} \mathrm{eV} \tag{6.4}$$

This is a very small number compared to quantities we are used to in particle physics. Thus we almost never have to worry about the effects of this expansion. Taking the inverse of this quantity give the lifetime of the universe, ~ 13.7 billion years.

The Hubble law says that,

$$H_0 d \approx z \quad (z \ll 1) \tag{6.5}$$

where d is the distance to a given object.[Q 35: Derive this]

In order to solve $a(t)$ we need to solve Einstein's equations which are related to the matter in the universe. We define the critical density of the universe as,

$$\rho_c \equiv \frac{3H_0^2}{8\pi G_N} \tag{6.6}$$

This is the density that the universe would have collapsed (there is so much mass that it will implode).

Furthermore we define,

$$\Omega_i = \frac{\rho_i}{\rho_c} \tag{6.7}$$

For example $\Omega_{DM} \sim 0.23$. In general one wouldn't expect ρ_i to have anything to do with ρ_c. However, it turns that that $\sum_i \rho_i \approx \rho_c$. This is a very important piece of evidence for inflation (though its not trivial why this is the case).

We can separate the different densities through,

$$\Omega = \Omega_{matter} + \Omega_{DM} + \Omega_{CC} \tag{6.8}$$

How the universe expands depends on what particles are inside it. If we have a box with a gas of a certain density and we increase the length of each side of the box by 2, then the density will go down by a factor of 2^3. So we see that for regular matter,

$$\rho \propto L^{-3} \tag{6.9}$$

where L is the length of the sides of the box. For the universe the length of the box is essentially given by a. Therefore we have,

$$\rho \propto a^{-3} \tag{6.10}$$

For photons the same reduction in number density will happen. However, here the energy of the photons will change when we increase the size of the box due to the redshift. Therefore, we have,[1]

$$\rho \propto a^{-4} \tag{6.11}$$

The cosmological constant doesn't change with scale. So we have,

$$\rho_{CC} = a^0 \tag{6.12}$$

This is very peculiar. For radiation and for the cosmological constant, the energy of the universe changes when we increase the size of the universe. Therefore, in this case we don't have energy conservation. This is because energy conservation is a consequence of time translation invariance. However, we don't have this symmetry here since we have expansion so we don't have energy conservation!

We can get these relationships from the equation of state which relates the density to the pressure in the universe,

$$p = w\rho \tag{6.13}$$

where w is a constant. For matter $w = 0$. For radiation, $w = 1/3$, and for the cosmological constant, $w = -1$. [Q 36: Derive these from thermo.] This gives,

$$\rho \sim a^{-3(1+w)} \tag{6.14}$$

So knowing how the density scales with a is the same as knowing w.

From here you can find how $a(t)$ is dependent on time,

$$a = t^{2/(3(1+w))} \tag{6.15}$$

It isn't trivial that we can define a temperature for the universe. This is because typical definitions of temperature require things to be in equilibrium.

Lets think how the number density of electrons scales with temperatures. If the temperature is below the electron mass, then it is a constant. On the other hand, the number density of photons always scales with the temperature, $n_\gamma \propto T^3$ and so $\rho_\gamma \propto T^4$, since they never become non-relativistic.

Suppose we are in the radiation dominated era of cosmology. Then we have the energy density of radiation,

$$\rho_R = \frac{\pi^2}{30} g_* T^4 \tag{6.16}$$

[1] This is easy to see from thermodynamics,

$$\rho \sim \int dp \frac{p^3}{e^{p/T} + 1} \sim T^4$$

where g_* is the effective number of degrees of freedom, $g_* = g_B + 7g_F/8$, and g_B, g_F are the number of degrees of freedom of bosons and fermions. You can kind of understand why the fermions are less important then the bosons. You can't put two fermions in a single level, so you can't place them close together which makes you have a hard time to increase the density. g_* changes as a function of time, but it changes very slowly.

Say we start with a radiation dominated universe. Since $\rho_{rad} \propto a^{-4}$ and $\rho_{mat} \propto a^{-3}$ the universe will at one point become matter dominated instead.

In general we think of the evolution of the universe as,

$$\Omega_{CC} \rightarrow \Omega_{rad} \rightarrow \Omega_{mat} \rightarrow \Omega_{CC} \tag{6.17}$$

In general Ω_i is equal to 1 for a single type of energy at one point in time. Transitions periods are very short. Baryogensis for example, occurs in the radiation dominant era.

Right now we are in between matter domination and cosmological constant domination. It is peculiar that we live in this special time that we are in between two regions. This is an unsolved problem of cosmology. [Q 37: Draw the different density curves.]

Currently we say that the temperature of the universe is 2.7 K. This means that the photons from the big bang at 2.7 K (as opposed to e.g. the neutrinos).

Through more algebra one can show that,

$$H(T) = T^2\sqrt{g_*}\left(\frac{7\pi^3 G_N}{90}\right)^{1/2} \tag{6.18}$$

The very last definition that we have is the entropy density,

$$S = \frac{\rho + p}{T} = \frac{2\pi^2}{4\pi} g_* T^3 \tag{6.19}$$

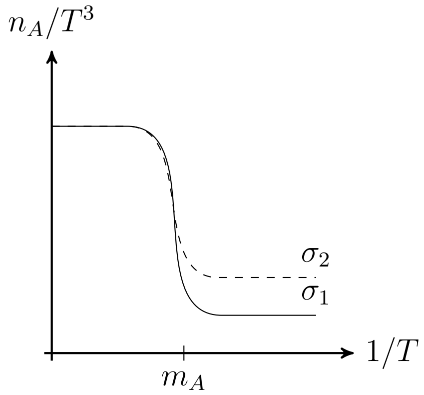
As of today the number density of photons is related to the entropy by,

$$S \simeq 7 n_\gamma \tag{6.20}$$

The number density also scales like T^3 so we often use these interchangeably.

Decoupling or "freezing out" is the last important topic we need to understand. Say you have a universe with a plasma and a particle A with temperature, $T \gg m_A$ where m_A is the mass of the particle. The particle will contantly interact with its surroundings (i.e., in equilibrium). This is independent of the details of the plasma, but just requires that the cross section of interaction will be large compared to how fast the universe is expanding. In this case the number density goes as, $n_A \sim T^3 e^{-m_A/T}$. If $m_A \ll T$ then the exponential is harmless and $e^{-m_A/T} \sim 1$. However, if $m_A \sim T$ then the exponential becomes important the number density sharply drops.

However, this is not the end of the story. At one point the particles interaction cross section will become so low that it effectively doesn't interact with the thermal bath at all. At what point this will occur isn't obvious but usually will occur at some point $T \ll m_A$. The larger the interaction cross section is (relative to the mass) the longer the particle will remain in thermal equilibrium and hence be exponentially suppressed. This is shown below:



where we depict the process for the same particle but with $\sigma_2 > \sigma_1$. There is small cavaet to this picture which is to do with the chemical potential.[Q 38: expand on this.] Here we have made the important assumption that after freezing out the number density remains constant at some value. As we now discuss this is an oversimplification.

6.1 Big bang nucleosynthesis

Big bang nucleosynthesis occured when the universe was roughly at 1 MeV. Above this point the proton and neutron were free, but below they form nuclei. The energy scale is an MeV because this is the binding energy of nuclear physics. By that time the number of antiprotons goes to zero since MeV $\ll$ GeV (they completely decoupled). At this point we basically have,

$$p, n, e^-, e^+, \gamma, \nu \tag{6.21}$$

but the neutron and proton are now at their frozen out temperatures, while the electron, photon, and neutrino are still in equilibrium.

Recall that the difference in the proton neutron masses are given by,

$$\Delta m \equiv m_n - m_p = 1.293\text{MeV} \tag{6.22}$$

When $T \gg \Delta m$ we can't differentiate a neutron and a proton (prior to freezing out the plasma interacts with all the particles are varying timescales but all at much larger then the timescale of the expansion of the universe). We must have,

$$\frac{n_n}{n_p} = 1 \tag{6.23}$$

We start by assuming that the neutron is stable. Even though its stable there is still a thermal bath that allows it to convert into a proton. Statistical mechanics tells us that,

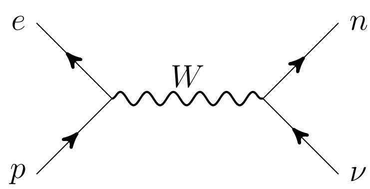
$$\frac{n_n}{n_p} = e^{-\Delta m/T} \tag{6.24}$$

This isn't completely true since a neutron and proton can also form a Helium or Deterium nucleus. The idea is that the neutron travels around it can produce a Helium and then its safe. Since it takes about 2MeV to break a deuterium they are not very safe. The probability of breaking a Helium is much less then the probability of breaking the deterium. [Q 39: How is this relevant?]

Freeze out happens when this rate becomes smaller then H. To find the freeze out time for the neutron we equate the Hubble constant with the width for neutron to proton conversion.

$$\Gamma(n \to p) \sim G_F^2 T^5 \tag{6.25}$$

This needs to be derived from finite temperature quantum field theory, but it turns out that you can get the right expression by dimensional analysis. The dominant process that takes a neutron to a proton and the reverse at finite temperature is,



The potentially important dimensful quantities in this rate are given by $m_n, m_p, \sqrt{s}$, and T. First we note that $\sqrt{s}$, the energy of the process in equilibrium is equivalent to T. So we already eliminate one variable. Furthremore, if we assume that $T \ll m_n, m_p$ then the rate for a $2 \to 2$ process should no longer depend on the masses of the outgoing particles since the phase space for the process isn't going to change with the masses. However, what is important is the mass difference, $\Delta m \equiv m_n - m_p$. Finally if we assume that $T \gg \Delta m$ (the regime where this is valid) then we can also drop Δm in our list of dimensful quantities. This leaves $\Gamma \sim G_F^2 T^5$ by dimenal analysis.

Furthermore, we have $H \sim \sqrt{g_* G_N} T^2$. We will roughly have a freezeout when $\Gamma \sim H$, or equivalently,

$$G_F^2 T^5 = \sqrt{g_* G_N} T^2 \tag{6.26}$$

$$\Rightarrow T = \left(\frac{\sqrt{g_* G_N}}{G_F^2} \right)^{1/3} \tag{6.27}$$

This is because roughly the energy scale of the universe is given by the Hubble constant. Much later the universe becomes too dilute and we can't keep ourselves in equilibrium. [Q 40: Sharpen this discussion]

We can calculate the temperature of when this transition happens [2],

$$T_{freeze}^{n \to p} = 0.7 \text{MeV} \tag{6.28}$$

[Q 41: Using the variables above I get the wrong answer by almost a factor of 100, but I was sloppy with many factors. Repeat this more carefully.] By complete accident this

[2] I calculate $G_N \sim 10^{-16}\text{GeV}^{-2}$ and we have $G_F \sim 10^{-5}\text{GeV}$.

is close the neutron proton mass difference, 1.293MeV. The number of neutrons that we have in the universe at this time are given by,

$$e^{-\Delta m/T_{freeze}} \sim \mathcal{O}(1) \tag{6.29}$$

Its interesting that this number isn't exactly 1 nor is it exactly 0. Taking into account neutron decay you get a more precise estimate.

We can calculate this much more generally. To do this we make the following assumptions:

1. We have a radiation dominated universe
2. The number of neutrons is roughly equal to the number of protons which are equal to the half the number of baryons (this assumes $T \ll \Delta m$)
3. The correct theory is the SM of particle physics
4. The couples and masses are the same

We define the weighted number density (relative mass) of some particle, i, as

$$y_i = \frac{n_i A_i}{n_B} \tag{6.30}$$

We of course have,

$$\sum_i y_i = 1 \tag{6.31}$$

The boltzmann equation is given by,

$$\frac{dy_i}{dt} = -H(T)T\frac{dy_i}{dT} = \sum_j \Gamma_{ij} y_j + \sum_{jk} \Gamma_{ijk} y_j y_k + ... \tag{6.32}$$

where Γ_{ij} (Γ_{ijk}) denotes the probability of convert j (j and k) into i and similarly and the ellipses denote higher order processes. To solve this equation we just need to one initial condition. To calculate all this we need the SM.

Solving this equation gives,

$$H \quad 75\% \tag{6.33}$$
$$He \quad 25\% \tag{6.34}$$
$$D \quad 10^{-5} \tag{6.35}$$
$$He^3 \quad 10^{-5} \tag{6.36}$$
$$H^3 \quad 10^{-5} \tag{6.37}$$
$$Li^7 \quad 10^{-10} \tag{6.38}$$

You can measure these number densities in areas where we have no stars and it agrees well with the predictions. All these number are related to one parameter,

$$\eta_B \equiv \frac{n_{baryons}}{n_{photons}} \tag{6.39}$$

Once you get this value you can predict all the other ratios. To get any heavier elements we need stars and supernova.

Note that all this calculation depends on the value of g_*. If for example we had 6 neutrinos then the universe would expand faster and we would get a different g_* and hence a different T_{freeze}. Therefore, just based on this data, you can't include more neutrinos into the SM (without forcing them to decoupling before BBN). Therefore, we can use BBN to put bounds on light degrees of freedom.

6.2 Baryogenesis

Today we have $\eta_B \sim 10^{-10}$. We'd like to understand where this number is coming from. This can be written as ,

$$\eta_B = \frac{n_B - n_{\bar{B}}}{s} \tag{6.40}$$

where $s \sim T^3$ is the entropy in the universe. Note that we don't really care about lepton assymetry because we can't measure it. One possibility is that this assymetry is due to initial conditions. This is more or less ruled out. The reason is that inflation made the universe empty and so would have erased any initial conditions.

Generating baryon assymetry requires three conditions.

1. Baryon number violation (e.g. $X \rightarrow p^+ X'$)

2. C and CP violation. This is needed because to make,

$$\Gamma(X \rightarrow p^+ X') \neq \Gamma(\bar{X} \rightarrow p^- \bar{X}') \tag{6.41}$$

   This is needed to stop the universe from erasing the baryon number violation generated with the above process.

3. We need to be out of equilibrium. Being out of equilibrium implies that,

$$\Gamma(X \rightarrow p^+ X') > \Gamma(p^+ X' \rightarrow X) \tag{6.42}$$

These are known as the Sakarov conditions.

We want to calculate η_B. Luckily we can separate the contributions of η_B into high energy physics and cosmological contributions,

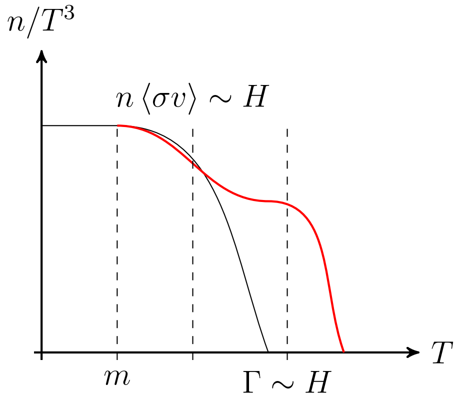
$$\eta_B = \epsilon_{HEP} \times \eta_{cosmology} \tag{6.43}$$

We have,

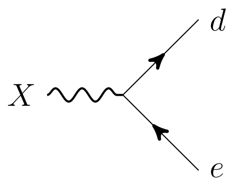
$$\epsilon_{HEP} = \frac{\Gamma(X \to p^+ e^-) - \Gamma(\bar{X} \to p^- e^+)}{\Gamma(X \to p^+ e^-) + \Gamma(\bar{X} \to p^- e^+)} \tag{6.44}$$

The $\eta_{cosmology}$ can be calculated by really understanding the out of equilibrium condition. This typically comes out to $\sim 10^{-3}$.

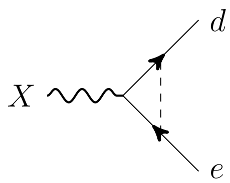
We now try to produce some model that can satisfy these conditions. This first mechanism for baryogenesis is known as out of equilibrium decay. Consider a particle that freezes out and decays afterwords. The freezing out process takes the rough form,



where the red line indicates a particle freezing out prior to equilibrium. The out of equilibrium particle can decay into the proton and since its out of equilibrium the proton won't convert back into it. We need some particle, X, that has the interaction,



We also need one more ingredient, which is CPV. We saw earlier that to get CPV you need two ampiltudes with a weak and strong phase to interfere. This can be given by a loop,

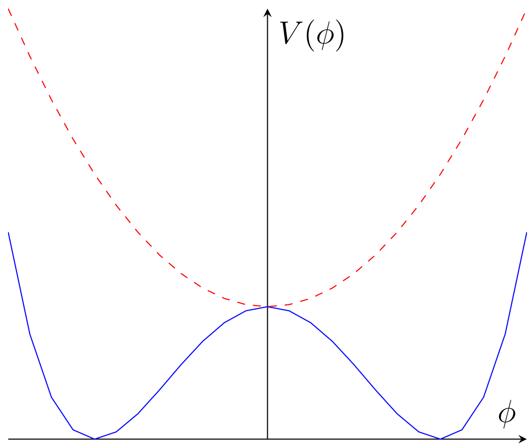


which will interfere with the other decay.

A second mechanism for baryogenesis is known as electroweak baryogenesis. This is done thruogh something called sphalarons. Spharalons are a nonperturbative effect that violate baryon number in the SM. This can be thought of as a tunneling between different one vacuum to another. Another way to think about it is that the term that makes this transitions is a total derivative and in a non-abelian theory total derivatives are important. This gives $\Delta B = \Delta L = 3$ so they are not conserved by $\Delta(B - L) = 0$.

The spharalons depend strongly on the temperature. At 0 temperature the probability to tunnel is exponentially suppressed and essentially zero. At $T \sim m_W$ this process is $\mathcal{O}(1)$ and so this process was in equilibrium with all the other processes.

The electroweak phase transition takes the Higgs potential to a quadratic:



To understand this we can use water as analogy. When water boils it forms bubbles. These bubbles expand but very few contract. This forms an out of equilibrium process. The same occurs in the early universe. If we can make some process that hit the walls of the bubble and back then there is an assymmetry in the number of particles that hit the wall and go out and stay in. So we can get baryon number violation out of equilibrium if the sphalarons are active on the wall. While, this is interesting as its baryogenesis within the SM, it turns out to not be a viable model.

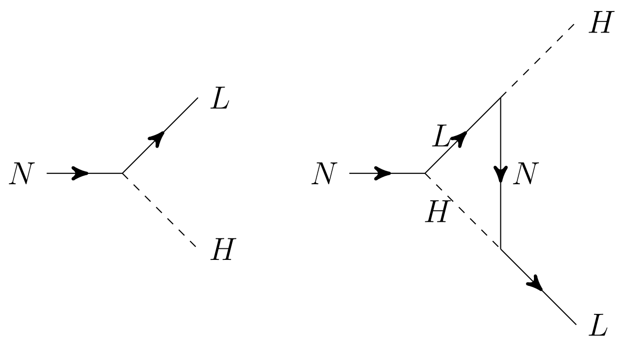
The previous two options are interesting but the most promising possibility is through leptogenesis. The idea is that suppose that some way we generate lepton number violatin. Then the sphalarons will convert this into baryon number violation such that $B - L$ is conserved. If we start with the condition that,

$$y_L(0) \neq 0 \quad y_B(0) = 0 \tag{6.45}$$

then one can show that we end up with,

$$y_B = \frac{12}{37} y_L(0) \tag{6.46}$$

This is useful since its much easier to produce lepton assymetry then it is to produce baryon assymetry. This is produce through a right handed neutrino,



The beauty of leptogenesis is if we assume a usual seesaw type I model we roughly predict the correct amount of baryon number violation.

6.3 Dark Matter

Dark matter has been observed in many ways. The most basic way is using rotation curves. You find a star and look its rotation curve. Based on th visible matter we expect a particular curve but instead we find something inconsistent with observation. [Q 42: Insert figure.]

To fix this we can add new dark matter. Similar experiments have done earlier in history. They initially tracked the rotation curves neptune and they found a deviation from theory. Using this they predicted a new planet which turned out to be jupiter. Now we have deviation and we predict dark matter.

The alternative to adding new matter is to change gravity. This is known as modifications to Newtonian dynamics (MOND). One can modify the gravitational equations,

$$F = G\frac{m_1 m_2}{r^2} \quad , \quad F = ma \tag{6.47}$$

It turns out that changing the first equation doesn't work, but one can actually modify the second equation,

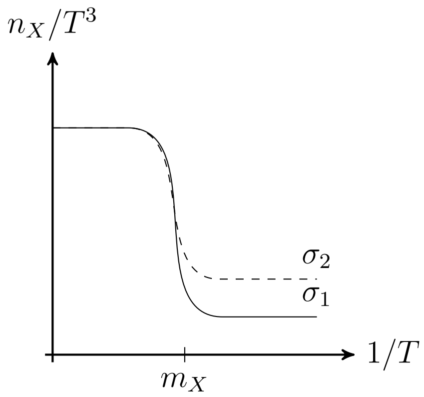
$$F = ma + \mu a^2 \tag{6.48}$$

This in fact has some predictions that dark matter doesn't have. Almost everyone agrees that MOND is ruled out. The reason is that its really hard to build a fundamental theory that will make this prediction. The reason is that one can prove that as long as the Lagrangian only depends on position and velocity, $L(x, \dot{x})$, then you can never produce an a^2 term. Another thing that rules out MOND is bullet cluster collisions.

For our purposes we assume that we have dark matter. Cosmology tells us that $\Omega_{DM} \sim 23\%$. Large scale structures tells us that we have cold DM. This means that the DM is non-relativistic. Hot dark matter would be very energetic and hence couldn't form galaxies and groups of galaxies. If this were true that DM would be completely uniform, but instead it clumps into galaxies. The observation of cold DM is crucial for particle physics since otherwise neutrinos would be a DM candidate.

In principle the DM could be baryonic. We now know this is not the case since we know the total number of baryons to form BBN which is lower then the amount of DM. The game is then to introduce a new particle(you could introduce more then one particle but we think about a single particle for simplicity), X. We want to know the properties of this particle. We need to introduce the new particle such that, $n_x = 23\%$. We focus on one possibility which lost some excitement recently, the WIMP (weakly interacting massive particle).

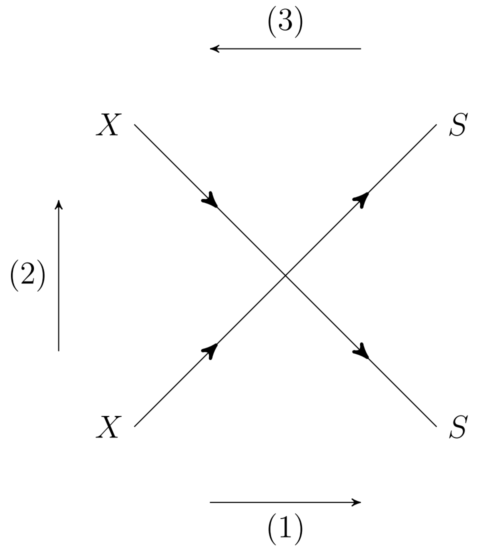
We make the assumption that the particle used to be in thermal equilibrium. This significantly constrains the allowed possibilities. Recall that we have,



We just need to observe the amount of particle we have today and then we can extract the cross section of the interaction. The amazing thing is that if $m_\chi \sim m_W$ and $g_\chi \sim g$ then $\Omega_\chi \sim \mathcal{O}(1)$. This is known as the WIMP miracle. This is referred to a miracle since we take the weak scale (where we suspect we have new physics) and we just happen to get roughly the correct abundance.

6.3.1 Hunting for DM

Consider an interaction of DM with some particle,



where the arrow indicates the array of time.

The first array direction is called annihilation,

$$XX \to SS \tag{6.49}$$

The second way is called direct detection,

$$XS \to XS \tag{6.50}$$

You put a detector very deep underground to reduce background and look for a signal. As of the time of this writing there is no clear indication that we found anything. Depending on how energetic the incoming dark matter particle is it may or may not probe the substructure of a nuclei. Due to the speed of the Earth around the Sun we should see more DM in the summer vs the winter. This should give DM oscillations above a SM background. One experiment is called DAMA/Libra which saw these oscillations, however its still not clear whether their systematics are too large.

The large way to look for dark matter is,

$$SS \to XX \tag{6.51}$$

which is done at the LHC (show up as missing energy signals).

Chapter 7

Grand Unified Theories

7.1 Introduction

In this chapter we describe grand unified theories in detail. Unlike the rest of the book this chapter is made using a compilation of references including the lectures by Yuval Grossman, a set of 3 lectures by Stuart Raby at PITP on SUSY GUTs, and *Supersymmetry - Theory,Experiment, and Cosmology* by Pierre Binétruy.

Unification is the bringing together of symmetry groups into a fewer groups. Electroweak unification is an example of this, where we can describe weak and electromagnetic physics using the group group, $SU(2) \times U(1)$. Grand unified theories (GUTs) are ones that try to unify the strong force as well.

The zeroth order motivation for GUTs is that when we run all 3 gauge couplings to higher energies with our current field content we find that they all meet at roughly the same energy scale.

Other motivations are the correct predictions of the Weinberg angle as well as the relative values of the bottom and tau Yukawa. The final important motivation for GUTs is charge quantization. In general the changes of all the particles could be completely arbitrary. This is because charges of Abelian symmetries are continuous. However, in Nature we see a very strange pattern between the charges. In particular they are all multiples of some fundamental charge. GUTs fix this because the charges come from a non-abelian symmetry. Non-abelian symmetries have quantized charges (e.g. the T_3 of all lepton and quark doublets can't be continuous but are fixed to be $\pm 1/2$).

GUTs can also help with other SM puzzles such as neutrino masses, baryogenesis, as well as flavor.

The striking prediction of all GUTs is proton decay. The reason for this is that grand unified theories put lepton and quarks in the same multiplets. This breaks lepton and baryon number explicitly. Proton decay experiments are ongoing. The most common is superK, which is now commonly used as a neutrino machine instead.

7.2 A little group theory

We want to consider the breaking of a Lie group into the SM Lie groups. This is a highly nontrivial process and apriori its not obvious how a group will break under spontaneous symmetry breaking. In order to find the breaking patterns without resorting to representations we can use the technique known as Dynkin diagrams. This is very simple to implement but confusing to justify.

[Q 43: Expand on this section and fix the reference...]

One strong indication that the SM arises from a larger gauge group is that the SM matter fields all add to zero. To see why this is important we recall that non-abelian group generators must be traceless. Furthermore, for hypercharge to be unbroken after the breaking of the gauge group, it must form a subgroup of the GUT group and hence appear as one of the GUT generators (in some choice of basis). Explicitly this says that if we denote the particles in representation n by $|i_n\rangle$ then we must have,

$$\langle i_n|Y|j_n\rangle = \delta_{i_n j_n} y_{j_n} \tag{7.1}$$

where Y denotes the hypercharge generator in the GUT group. However, if the SM fields make up a collection of complete representations (no new fields) then we have,

$$\sum_n \sum_{i_n} \langle i_n|Y|i_n\rangle = \sum_n \sum_{i_n} y_{n,i_n} = 0 \tag{7.2}$$

where the sum on the right is just sums over all particles in all representations. For the SM fields we have,

$$\sum_{n,i_n} y_{n,i_n} = \left(-\frac{1}{2} - \frac{1}{2} - 1 + 3\left(\frac{1}{6} + \frac{1}{6} + \frac{2}{3} - \frac{1}{3}\right)\right) \times 3 = 0 \tag{7.3}$$

as required. This also shows that one cannot include the Higgs field into the GUT group since doing so wouldn't allow hypercharge to be unbroken after breaking the GUT group unless one includes other new fields as well.

7.3 Coupling Unification

There are two ways to proceed to see if coupling constant unify. We could either start in the IR and run the couplings up to the GUT scale or we could start in the UV and then run the couplings down to scale of experiment. Here we proceed in the first way.

The running of couplings takes the form[1],

$$\frac{d\alpha}{d\log\mu} = -b_i\frac{\alpha^2}{2\pi}, \quad b_i = \left(\frac{11}{3}C_2(G) - \frac{2}{3}C(r)n_{weyl} - \frac{1}{3}C(r)n_{scalar}\right) \tag{7.4}$$

[1]Note the factor of two difference in fermionic and bosonic results. This should be expected and is due a Weyl fermion having two degrees of freedom.

where μ is the renormalization scale, $C_2(G) = N$ for $SU(N)$, $C(r) = \frac{1}{2}$ for $SU(N)$, n_{weyl} is the number of Weyl fermions, and n_{scalar} is the number of complex scalars.

We have (as is common in discussions of GUTs we denote the gauge groups of $U(1)_Y, SU(2)_L$, and $SU(3)_c$ as 1, 2, and 3 respectfully),

$$\text{QCD}: \quad C_2(G) = 3, C(r) = 1/2, n_{weyl} = (\overbrace{1+1+2}^{u_R, d_R, Q_L}) \times N_g, n_{scalar} = 0 \Rightarrow b_3 = 7 \tag{7.5}$$

$$SU(2)_L: \quad C_2(G) = 2, C(r) = 1/2, n_{weyl} = (\overbrace{3+1}^{Q_L, L}) \times N_g, n_{scalar} = 1 \Rightarrow b_2 = \frac{19}{6} \tag{7.6}$$

For hypercharge, due to its abelian nature the calculation is a bit different. In particular, the scale of hypercharge isn't fixed as for non-abelian gauge theories. However, if we assume it will arise from the spontaneous breaking of some non-abelian group under which the SM fields from a full representation then it turns out that we can uniquely rescale the hypercharge.

To see this we note that at the generators of the GUT group should all be normalized in the same way, Tr $\left[T^a T^b\right] = \frac{N}{2}\delta^{ab}$. Explicitly we must have,

$$\sum_n \sum_{i_n} \langle i_n | Y^2 | i_n \rangle = \sum_n \sum_{i_n} \langle i_n | T_3 | i_n \rangle \tag{7.7}$$

Now since hypercharge doesn't have a correct normalization in the SM case we can write, $y_\alpha = C y_{SM,\alpha}$ for particle α. Explicitly we have,

$$\sum_n \sum_{i_n} y_{i,n}^2 = \sum_n \sum_{i_n} t_{3,i_n}^2 \tag{7.8}$$

$$\sum_{SM} C^2 y_{SM}^2 = \sum_{SM} t_{3,SM}^2 \tag{7.9}$$

where the sum is now over all the charges of the SM fields.

Carrying out the sums we have,

$$\sum_{SM} y_{SM}^2 = \left(\frac{1}{4} + \frac{1}{4} + 1 + 3 \times \left(\frac{1}{36} + \frac{1}{36} + 1 + \frac{4}{9} + \frac{1}{9}\right) 3\right) \times 3 = 10 \tag{7.10}$$

$$\sum_{SM} t_{3,SM}^2 = \left(\frac{1}{4} + \frac{1}{4} + 3 \times \left(\frac{1}{4} + \frac{1}{4}\right)\right) \times 3 = 6 \tag{7.11}$$

which gives,

$$C = \sqrt{\frac{10}{6}} = \sqrt{\frac{5}{3}} \tag{7.12}$$

We see that when you go to the GUT scale the hypercharge is no longer arbitrary. It is uniquely given by $\sqrt{\frac{5}{3}}$ times the values which were conventionally assigned many years ago. We will check this relationship again once we choose a particular GUT group.

Alternatively one can think of scaling the couplings. To see this consider the Lagrangians in the SM and the broken GUT. We must have,

$$\mathcal{L}_{SM} = g' Y_{SM} B_\mu \, (...) \quad \mathcal{L}_{GUT} = g_{GUT} Y B_\mu \, (...) \tag{7.13}$$

i.e.

$$g_{GUT} Y = g' Y_{SM} = g' \sqrt{\frac{5}{3}} Y \tag{7.14}$$

which gives the relation,

$$g_{GUT} = \sqrt{\frac{5}{3}} g' \tag{7.15}$$

Now that we are equipped with this we are finally ready to consider the abelian as well as non-abelian running. First we note that the one-loop coefficient for an abelian coupling beta function is given by[2],

$$b_1 = -\frac{1}{C^2} \left(\frac{2}{3} \sum_f y_f^2 + \frac{1}{3} y_\phi^2 \right) \Rightarrow b_i = -\frac{41}{10} \tag{7.16}$$

where y_f and y_ϕ are the SM gauge charges. This gives,

$$\frac{d\alpha_3}{d \log \mu} = -7 \frac{\alpha_3^2}{2\pi} \tag{7.17}$$

$$\frac{d\alpha_2}{d \log \mu} = -\frac{19}{6} \frac{\alpha_2^2}{2\pi} \tag{7.18}$$

$$\frac{d\alpha_1}{d \log \mu} = \frac{41}{10} \frac{\alpha_1^2}{2\pi} \tag{7.19}$$

The solution to the equation, $\alpha' = \beta \alpha^2$ is given by,

$$\frac{1}{\alpha} = \frac{1}{\alpha_0} - \beta \log \frac{\mu}{m_Z} \tag{7.20}$$

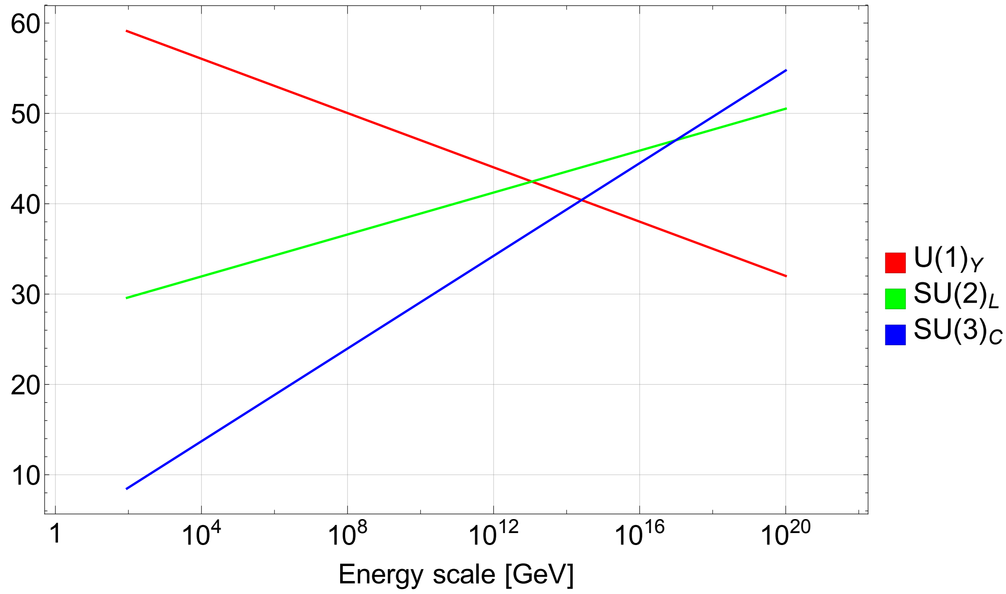
To track the couplings as a function of energy scale we need a starting point. Due to LEP these couplings are all best measured at the Z pole where they found,

$$\alpha_3(m_Z) = 0.118 \quad \alpha_e(m_Z) = 1/128 \quad \sin^2 \theta_w(m_Z) = 0.231 \tag{7.21}$$

$$\Rightarrow \alpha_3(m_Z) = 0.118 \quad \alpha_2(m_Z) = 0.0338 \quad \alpha_1(m_Z) = 0.0102 \times \frac{5}{3} \tag{7.22}$$

We can now plot the couplings as a function of energy. The result is shown below,

[2]To get the right results its important to note that above the EW scale the Higgs would count as two scalars.



The unification isn't perfect. However, its still quite good. In order to get this plot correct every factor of two and minus sign needs to be accounted for. Making a few mistakes along the way makes you appreciate how nontrivial this unification is. We see unification at $\sim 10^{14}$GeV.

Its important to note that in order to preserve grand unification we must have roughly the same matter structure. Otherwise the couplings won't unify. This assumption is often called the (SUSY) desert. This is not a severe constraint but if every time we go up by an order of magnitude in the energy scale we have a new set of particles then gauge unification will fail.

Now that we have the formulas for the SM case, its easy to generalize this to the Minimal Supersymmetric Model (MSSM)[3]. The only necessary ingredient is to modify 7.4 to allow for adjoint fermions. By staring at the equation its easy to see that we must have,

$$b_i = \frac{11}{3}C_2(G) - \frac{2}{3}C(G)n_{gaugino} - \frac{2}{3}n_{weyl} - \frac{1}{3}C(r)n_{scalar} \tag{7.23}$$

where $C(G) = C_2(G) = N$ for $SU(N)$. Since the number of gauginos charged under a representation is always going to be 1 we can combine the left to terms. Furthermore, the number of weyl fermions is always going to be equal to the number of complex scalars in SUSY. Therefore, we can also rewrite the final two terms to give,

$$b_i = 3C_2(G) - C(r)n_{chiral} \tag{7.24}$$

[3]While in this text we have assumed no knowledge of supersymmetry, we do assume a basic understanding of the MSSM for the calculation below.

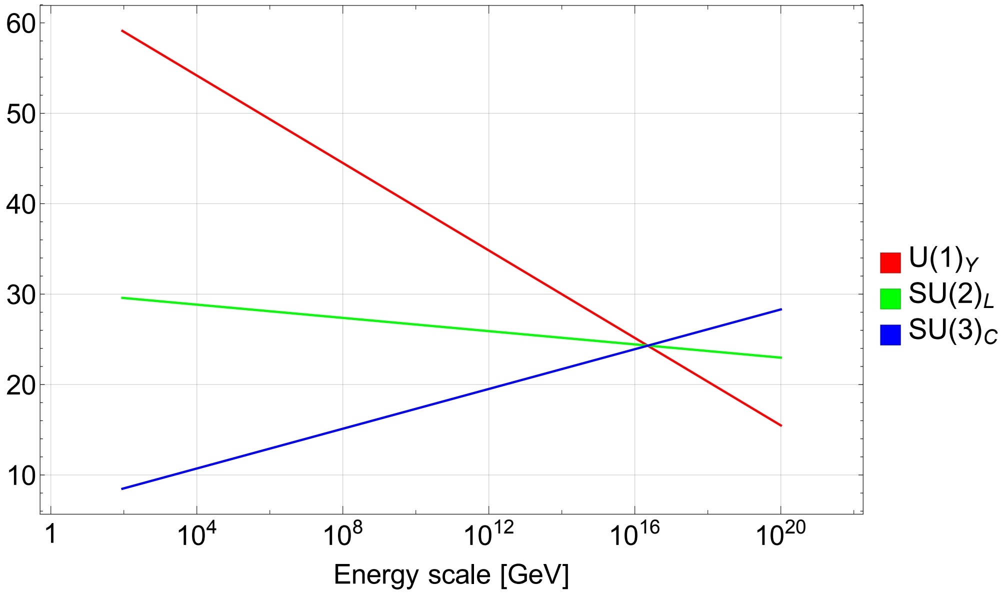
where n_{chiral} is the number of chiral multiplets. We have,

$$b_3 = 9 - \frac{1}{2}4 \times N_g = 3 \tag{7.25}$$

$$b_2 = 6 - \frac{1}{2}(1+3) \times N_g - \frac{1}{2}2 = -1 \tag{7.26}$$

$$b_1 = -\frac{3}{5}\left[\left(\left(\frac{1}{4}+\frac{1}{4}+1\right)+3\times\left(\frac{1}{36}+\frac{1}{36}+\frac{4}{9}+\frac{1}{9}\right)\right)\times N_g + 2\left(\frac{1}{4}+\frac{1}{4}\right)\right] = -\frac{33}{5} \tag{7.27}$$

The couplings as a function of energy scale are now shown below:



This is a highly nontrivial result! The couplings are no longer just close together at some high scale. They are exactly on top of one another.

While less motivated, in what follows we focus on non-supersymmetric GUTs. The discussion of SUSY GUTs is almost identical and we attempt to point out any significant differences as they become relevant.

7.4 Pati-Salam Model

It is well known that left-right symmetric models can produce the SM gauge structure,

$$SU(2)_L \times SU(2)_R \times U(1)_{B-L} \to SU(2)_L \times U(1)_Y \tag{7.28}$$

They also provide a natural explanation for baryon and lepton number conservation. In order to explain charge quantization one suggestion is to form the $U(1)_{B-L}$ from a $SU(4)_c$ group using the fact that $SU(3)\times U(1) \supset SU(4)$. This set of models are called Pati-Salam models and have the breaking pattern,

$$SU(4)_c \times SU(2)_L \times SU(2)_R \to SU(3)_c \times SU(2)_L \times SU(2)_R \times U(1)_{B-L} \tag{7.29}$$

The fundamental representation can be put into a fundamental of $SU(4)$:

$$4_R = \begin{pmatrix} q_R \\ R \end{pmatrix} \equiv \begin{pmatrix} \begin{pmatrix} u^c_{R,r} \\ d^c_{R,r} \end{pmatrix} \\ \begin{pmatrix} u^c_{R,g} \\ d^c_{R,g} \end{pmatrix} \\ \begin{pmatrix} u^c_{R,b} \\ d^c_{R,b} \end{pmatrix} \\ \begin{pmatrix} \nu^c_R \\ e^c_R \end{pmatrix} \end{pmatrix} \quad 4_L = \begin{pmatrix} q_L \\ L \end{pmatrix} \equiv \begin{pmatrix} \begin{pmatrix} u^c_{L,r} \\ d^c_{L,r} \end{pmatrix} \\ \begin{pmatrix} u^c_{L,g} \\ d^c_{L,g} \end{pmatrix} \\ \begin{pmatrix} u^c_{L,b} \\ d^c_{L,b} \end{pmatrix} \\ \begin{pmatrix} \nu^c_L \\ e^c_L \end{pmatrix} \end{pmatrix} \tag{7.30}$$

where r, g, b here denote the colors. Note that because we also have $SU(2)_L \times SU(2)_R$ invariance we are forced to use only right handed or only left fields to form multiplets. We can write these more succientl

Since $SU(3)_c \times U(1)_{B-L}$ is a subgroup we can write the generators of $SU(4)$ by,

$$X_{1-8} = \begin{pmatrix} \lambda_{1-8}/2 & 0 \\ 0 & 0 \end{pmatrix} \quad X_9 = \frac{1}{2} \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix} \tag{7.31}$$

(the 1/2 is required to preserve the generator normalization, $\mathrm{Tr}\left[T^a T^b\right] = 2\delta^{ab}$. In total $SU(4)$ has $4^2 - 1 = 15$ generators. Therefore, there are 6 other generators. These will not be part of the subgroup and hence have off-diagonal components.

To break $SU(4)$ but preserve $SU(3)$ we can introduce a heavy scalar in the adjoint representation, **15**, with the vacuum expectation value,

$$\langle \Sigma \rangle = \begin{pmatrix} 0 & 0 \\ 0 & v \end{pmatrix} \tag{7.32}$$

Furthermore, to break $SU(2)_R \times U(1)_{B-L}$ into hypercharge we need 2 Higgs doublets. They can be put into one irreducible representation,

$$\mathcal{H} = (H_d, H_u) \rightarrow (1, 2, \bar{2}) \tag{7.33}$$

If we only had one family of quarks and leptons and wrote down the most general renormalizable Lagrangian that coupled the Higgs to the quarks and leptons we would have a single coupling constant,

$$\lambda \mathbf{4}^\dagger_L \mathcal{H} \mathbf{4}_R \tag{7.34}$$

This needs to give the coupling to the top,bottom,tau, and neutrino at the GUT scale. This is known as Yukawa unification.

This not a Grand Unified Theory therefore even if we have a left-right symmetry you can still have two coupling constant at the high energy scale. So you can't make any predictions at the high energy scale from the coupling constants.

This model is primarily useful to acheive charge conservation as well as Yukawa unification, however it fails at explaining gauge unification since we are still left with 3 coupling constants.

7.5 Moving to simple groups

In the Pati-Salam model we take the SM and make it more complicated. The bulk of studies on unification focus on reducing the gauge content and use a simple group instead [4]. To understand which groups can be used to produce the SM as a low energy theory we need to use some group theory. The rank of the SM is, $2+1+1=4$ (recall that the rank of $SU(N)$ is $N-1$).

So we need a group that is rank 4 or above. The groups that are rank 4 are:

$$SU(5), SO(8), SO(9), Sp(8), F_4 \tag{7.35}$$

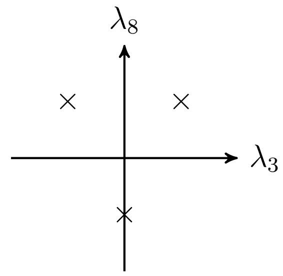
where recall that the rank of a $SU(N)$ group is $N-1$ while the rank of $SO(N)$ is $\frac{1}{2}N-1$ (even) and $\frac{1}{2}(N+1)-1$ (odd). All these groups except $SU(5)$ turn out not to work [Q 44: show]. There are a couple of groups that are also candidates for GUTs which aren't mentioned here, including the particularly popular $SO(10)$.

7.5.1 Example of simple Lie group breaking, $SU(3) \to SU(2)$

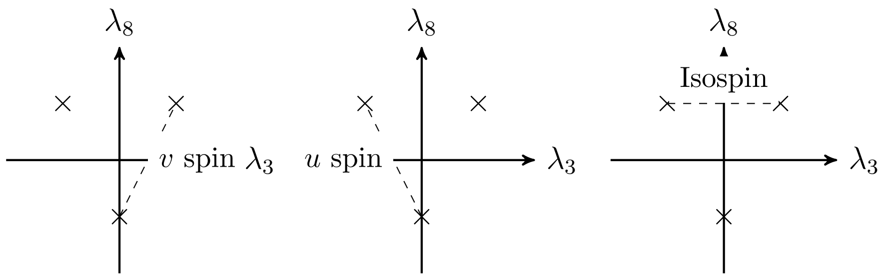
Lets consider an $SU(3)$ model. $SU(3)$ we has 2 roots which are usually called λ_3 and λ_8. For a physical example we consider a fundamental under $SU(3)_c$,

$$\mathbf{3} = \begin{pmatrix} u \\ d \\ s \end{pmatrix} \tag{7.36}$$

The weights form an equalateral triangle,



We can visualize the breaking to form a **2** in 3 different ways,



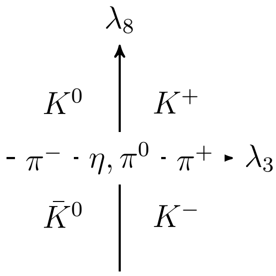
[4]Recall that for a group $\mathcal{G}$ with generators, T_g, $\mathcal{G}$ is simple if there does NOT exist a subset of generators, T_h, that obey $[T_h, T_g] = T_h \quad \forall$ generators of the group, T_g. In a practical sense it means the group can be written as a product of other groups.

The breaking pattern is,

$$\underbrace{\mathbf{3}}_{SU(3)} \to \underbrace{\mathbf{2}+\mathbf{1}}_{SU(2)} \tag{7.37}$$

From a group theory point of view all the breakings are equivalent. However, depending the direction we break $SU(3)$ we get a different doublet. Isospin is commonly used but you can also use the other breaking patterns.

Similary one can consider the octet:



[Q 45: Is it still correct to write λ_3, λ_8 as the axes?]

$\mathbf{8}$ breaks through,

$$\mathbf{8} \to \mathbf{3}+\mathbf{2}+\mathbf{2}+\mathbf{1} \tag{7.38}$$

So whenever we break $\mathbf{8}$ in SU(3) it will break into $\mathbf{3}+\mathbf{2}+\mathbf{2}+\mathbf{1}$ of $SU(2)$. Any irreducible representation under the group form a sum of irreducible representation under the broken group. In particular isospin is $\{\lambda_1, \lambda_2, \lambda_3\}$.

Note that any representation we have of $SU(3)$ we can write it as a matrix. For the fundamental we can write it in a block diagonal form,

$$\left(\begin{array}{c|c} 2\times 2 & \\ \hline & 1\times 1 \end{array}\right) \tag{7.39}$$

where the empty blocks are broken. Similarly for the adjoint we can write it as,

$$\left(\begin{array}{c|c|c|c} 2\times 2 & & & \\ \hline & 2\times 2 & & \\ \hline & & 3\times 3 & \\ \hline & & & 4\times 4 \end{array}\right) \tag{7.40}$$

If you look at the Gell-man matices,

$$\lambda_i = \begin{pmatrix} \sigma_i & 0 \\ 0 & 0 \end{pmatrix}, \lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \tag{7.41}$$

$$\lambda_6 = \begin{pmatrix} 0 & 0 \\ 0 & \sigma_x \end{pmatrix}, \lambda_7 = \begin{pmatrix} 0 & 0 \\ 0 & \sigma_y \end{pmatrix}, \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \tag{7.42}$$

then you can see this also. [Q 46: expand...]

Here we had a symmetry that was only an approximate symmetry which holds in the limit that all the quarks are degenerate. Now that we have the symmetric result we can begin to introduce corrections to this. The lowest order symmetry breaking term is the mass to the strange quark,

$$m_s \bar{s}s + \hat{m}(\bar{u}u + \bar{d}d) = \begin{pmatrix} \bar{u} & \bar{d} & \bar{s} \end{pmatrix} m_1 \begin{pmatrix} u \\ d \\ s \end{pmatrix} + \begin{pmatrix} \bar{u} & \bar{d} & \bar{s} \end{pmatrix} m_8 \begin{pmatrix} u \\ d \\ s \end{pmatrix} \tag{7.43}$$

where m_1 transforms as a fundamental under $SU(3)$ and m_8 transforms as a octet under $SU(3)$.

$$m = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} a + b \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \tag{7.44}$$

We are doing a spurion analysis. [Q 47: I don't understand what is going on here...]

In $SU(5)$ we have 24 (N^2-1) generators and each of them is 5×5. We want to break this 5×5 matrix into a 2×2 and 3×3:

$$\left(\begin{array}{c|c} 3\times 3 & 0 \\ \hline 0 & 2\times 2 \end{array}\right) \tag{7.45}$$

The breaking becomes,

$$\mathbf{5} \rightarrow (\mathbf{3}, \mathbf{1}) + (\mathbf{1}, \mathbf{2}) \tag{7.46}$$

where the (,) is the $SU(3)\times SU(2)$ charges. [Q 48: Why are the quarks $SU(2)$ singlets?] The first three generators are given by,

$$T_{1-3} = \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & \sigma_i \end{array}\right) \tag{7.47}$$

We want T_{4-11} to be the Gell-man matrices so we write,

$$T_{4-11} = \left(\begin{array}{c|c} \lambda_a & 0 \\ \hline 0 & 0 \end{array}\right) \tag{7.48}$$

So far we have T_3, T_6, and T_{11} that are diagonal. Since we have a rank 4 group we can have one more diagonal generator. This should correspond to $U(1)_Y$. So we have,

$$T_{12} = \frac{1}{\sqrt{60}} \begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & -3 \end{pmatrix} \tag{7.49}$$

For $T_{13} - T_{24}$ they are general traceless matrices that mix between the blocks.

The way we think about is that we have 24 generators. After you break the symmetry you have 12 broken generators ($T_{13} - T_{24}$ and 12 unbroken which correspond to the SM. We have 12 massive gauge bosons and 12 massless ones.

This was for a fundamental of SU(5). As another example for a **10** of $SU(5)$ we have,

$$\mathbf{10} \rightarrow (\mathbf{3}, \mathbf{2}) + (\bar{\mathbf{3}}, \mathbf{1}) + (\mathbf{1}, \mathbf{1}) \tag{7.50}$$

Recall that,

$$\mathbf{5} \rightarrow (\mathbf{3}, \mathbf{1}) + (\mathbf{1}, \mathbf{2}) \tag{7.51}$$

We can form a fundamental of $SU(5)$ through,

$$\mathbf{5} \equiv \begin{pmatrix} \nu_L \\ e_L \\ q^c_{R,r} \\ q^c_{R,g} \\ q^c_{R,b} \end{pmatrix} \tag{7.52}$$

Here these are right handed quarks (we can't fit the left handed quarks into a fundamental). For the rest of the chapter we will use the representation to label fields in the Lagrangian as is done here. This notation will become clear shortly if it isn't so already.

Note that here we are only using left handed Weyl fermions as is standard for GUTs. Every right handed Weyl fermion can be turned into a left handed using the charge conjugation operator,

$$\psi^c_R \equiv \epsilon \psi^*_R \tag{7.53}$$

The fields that we work with are,

$$Q(3,2)_{1/3}, u^c(3,1)_{-4/3}, d^c(3,1)_{2/3}, L(1,2)_{-}, E^c(1,1)_2 \tag{7.54}$$

At this point we don't know whether to use up type or down type quarks. Recall that we have a form for the hypercharge multiplet required by demanding it be traceless. When this operator acts on the multiplet it must give the hypercharges of the particles (with GUT normalization)

Recall that here we take the hypercharge operator to be diagonal and it must be traceless. To identify it with hypercharge we must be able to Hence when acting on a multiplet

We need the hypercharge of all the particles to add up to 0. This requirement arises from wanting the representation to break into a traceless hypercharge generator. The hypercharge of the lepton doublet is -1, the hypercharge of the conjugate up type quarks is $-4/3$, and $2/3$ for the down type. We have,

$$-2 \times 1 + 3\frac{2}{3} = 0 \tag{7.55}$$

if we use down type quarks.

Now lets repeat the above procedure for a **10**. We still need to fit Q, u and e_R. A convenient way to write a **10** is as a antisymmetric 5×5 matrix,

$$\left(\begin{array}{cc|ccc} 0 & e_R^c & u_{L,r} & u_{L,g} & u_{L,b} \\ . & 0 & d_{L,r} & d_{L,g} & d_{L,b} \\ \hline . & . & 0 & u_{R,r}^c & u_{R,g}^c \\ . & . & . & 0 & u_{R,b}^c \\ . & . & . & . & 0 \end{array}\right) \tag{7.56}$$

Now we do the hypercharg counting:

$$2 + 6\times\frac{1}{6} - 3\times\frac{4}{3} = 0 \tag{7.57}$$

as required.

The kinetic Lagrangian for the fermions takes the form,

$$\mathcal{L}_{fermions} = \bar{5}_\alpha^\dagger \bar{\sigma}_\mu D_\alpha^{\mu,\beta} \bar{5}_\beta + 10^{\alpha\beta\dagger}\bar{\sigma}_\mu (D^\mu)_{\gamma\delta}^{\alpha\beta} 10^{\gamma\delta} \tag{7.58}$$

where $D^\mu = \partial^\mu + i g_{GUT} T_A A_A^\mu$.

Notice that this model has no place for a right handed neutrino. However, if you use $SO(10)$ then the fundmental is a **16** can be used which is given by,

$$\underbrace{\mathbf{10}+\mathbf{5}+\mathbf{1}}_{SU(5)} \tag{7.59}$$

The singlet is proposed to be the right handed neutrino.

7.6 Normalization of hypercharge (g')

Recall that T_{12} is the generator that gives us hypercharge. We can write it as,

$$T_{12} = \frac{1}{\sqrt{60}}\begin{pmatrix} -2 & & & & \\ & -2 & & & \\ & & 3 & & \\ & & & 3 & \\ & & & & 3 \end{pmatrix} \tag{7.60}$$

The 15 arises from the requirement that the generator be properly normalized through, $\mathrm{Tr} T^a T^b = 2\delta^{ab}$.

In the SM Lagrangian we have terms that go as,

$$\mathcal{L} \sim g' Y \,(...) \tag{7.61}$$

We know that as we go to GUTs this Lagrangian must go to,

$$\mathcal{L} \sim g_{GUT} T_{12} \,(...) \tag{7.62}$$

where g_{GUT} is the GUT coupling. This implies that,

$$g'Y = g_{GUT}T_{12} \tag{7.63}$$

We just need to do this for the one particle. For e_L we have,

$$Y = \frac{1}{2} \quad T_{12}^e = \frac{3}{\sqrt{60}} \tag{7.64}$$

So,

$$g_{GUT} = g'\frac{Y}{T_{12}} = g'\frac{\sqrt{60}}{6} = g'\sqrt{\frac{5}{3}} \tag{7.65}$$

Therefore, the GUT gauge coupling is related to what we called hypercharge gauge coupling by $\sqrt{5/3}$.

7.7 $\sin^2\theta_w$

Now that we are equipped with a sensible definition for hypercharge we have a concrete prediction for the Weinberg angle which is defined by,

$$\sin^2\theta_w = \frac{g'^2}{g^2 + g'^2} \tag{7.66}$$

We know that at the GUT scale $g' = \sqrt{\frac{3}{5}}g_{GUT}$ and $g = g_{GUT}$ which gives,

$$\sin^2\theta_w = \frac{3/5}{1+3/5} = \frac{3}{8} = 0.375 \tag{7.67}$$

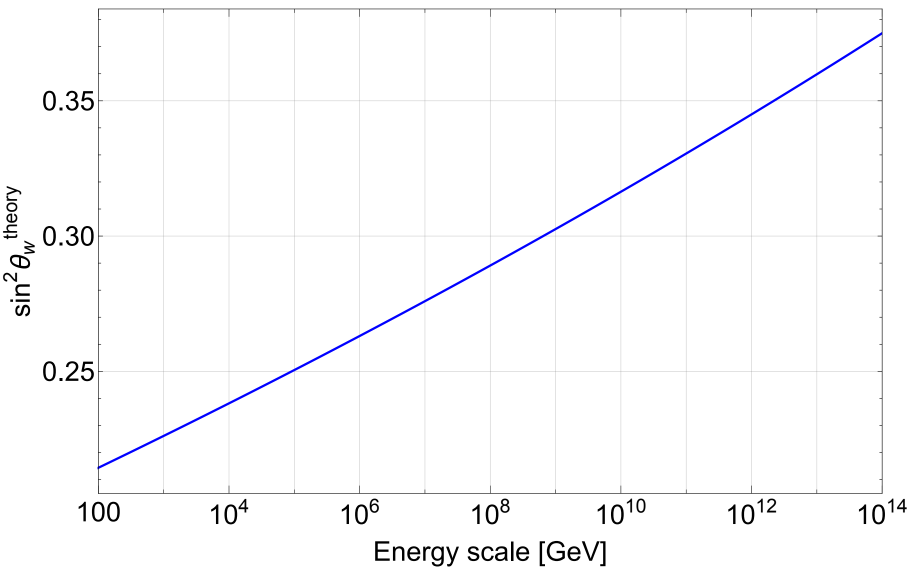
In practice what we measure is ≈ 0.23. Now that we have the formula calculated earlier for the beta function its easy to run this down the electroweak scale,

$$\sin^2\theta_w^{theory}(m_Z) = \frac{(3/5)\alpha_1}{(3/5)\alpha_1 + \alpha_2} \tag{7.68}$$

where,

$$\alpha_i = \left(\alpha_G - \beta_i \log\frac{\mu}{M_{GUT}}\right)^{-1} \tag{7.69}$$

with $\beta_1 = \frac{41}{10}\frac{1}{2\pi}$ and $\beta_2 = -\frac{19}{2\pi}$. The running is shown below:



We find,

$$\sin^2\theta(m_Z^2) \approx 0.21 \tag{7.70}$$

This is close to the true value but not exactly equal to. If you use supersymmetry the prediction becomes 0.23 as expected.

7.8 Charge Quantization

Charge in the SM is given by,

$$Q = T_3 + Y \tag{7.71}$$

Changing the coupling g' effectively changes the hypercharge giving,

$$Q = T_3 + \sqrt{\frac{5}{3}} T_{12} \tag{7.72}$$

The fundamental has the charges,

$$\mathbf{5} \rightarrow \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1/2 \\ -1/2 \end{pmatrix} + \begin{pmatrix} 1/3 \\ 1/3 \\ 1/3 \\ -1/2 \\ -1/2 \end{pmatrix} = \begin{pmatrix} 1/3 \\ 1/3 \\ 1/3 \\ 0 \\ -1 \end{pmatrix} \tag{7.73}$$

As promised electric charge is necessarily quantized. In particular, the charge of the electron is a multiple of the charge of the downtype quarks as expected. Similarly the right handed lepton has a charge which is a multiple of the left handed quark doublet.

7.9 Gauge Bosons

SU(5) has 24 generators and the SM has $8+3+1=12$ generators. By taking SU(5) down to the SM we have 12 massive degrees of freedom.

To see how to discuss the gauge bosons recall that in $SU(2)$ we like to write our gauge bosons as,

$$W^a T^a = \frac{1}{2}(W_1\sigma_1 + W_2\sigma_2 + W_3\sigma_3) \tag{7.74}$$

We can write the analogous matrices for SU(5):

$$G^A T^A \sim \left(\begin{array}{ccc|cc} g_{1-8} & g_{1-8} & g_{1-8} & X_1 & Y_1 \\ g_{1-8} & g_{1-8} & g_{1-8} & X_2 & Y_2 \\ g_{1-8} & g_{1-8} & g_{1-8} & X_3 & Y_3 \\ \hline \bar{X}_1 & \bar{X}_2 & \bar{X}_3 & B+W_3 & W_1 \\ \bar{Y}_1 & \bar{Y}_2 & \bar{Y}_3 & W_2 & B-W_3 \end{array} \right) \tag{7.75}$$

where the massless gauge bosons are the SM bosons and the X and Y are the massive. They form an $SU(2)$ doublet and transform as,

$$\begin{pmatrix} X \\ Y \end{pmatrix} \rightarrow (3,2)_{5/3} \tag{7.76}$$

This gives,

$$Q(X) = \frac{1}{2} + \frac{1}{2}\frac{5}{3} = \frac{4}{3} \quad , \quad Q(Y) = -\frac{1}{2} + \frac{1}{2}\frac{5}{3} = \frac{1}{3} \tag{7.77}$$

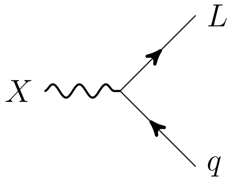
The crucial point is that the massive gauge bosons are off-diagonal. We have gauge interactions for the fermions which will inevitable break lepton and baryon number [5],

$$\mathbf{5}^\dagger \left(\partial_\mu - i g_{GUT} G^A_\mu T^A\right)\gamma^\mu \mathbf{5} = \begin{pmatrix} \bar{d}^c & \bar{L} \end{pmatrix} \begin{pmatrix} g_{1-8} & X, Y \\ X, Y & W, B \end{pmatrix}_\mu \gamma^\mu \begin{pmatrix} d^c \\ L \end{pmatrix} = \bar{d}^c X_\mu \gamma^\mu L + ... \tag{7.78}$$

$$\mathrm{Tr}\left\{\mathbf{10}^\dagger \not{D} \mathbf{10}\right\} = \mathrm{Tr}\left\{\mathbf{10}^\dagger \left(\not{\partial} - i g_{GUT} G^a_\mu T^a\right)\mathbf{10}\right\} \tag{7.79}$$

[Q 49: I'm not sure I properly write the kinetic term of a **10**.] we see that it connects quarks to leptons. In general such objects are called leptoquarks.

We have the interactions,



[5]Here we have an implicit γ_0 in the dagger. We avoid using the "bar" notation as it is often used in this context to be part of the definition of multiplets with conjugate charges.

In order to see the couplings of the gauge bosons to the matter fields we consider the kinetic term.

Then we have,

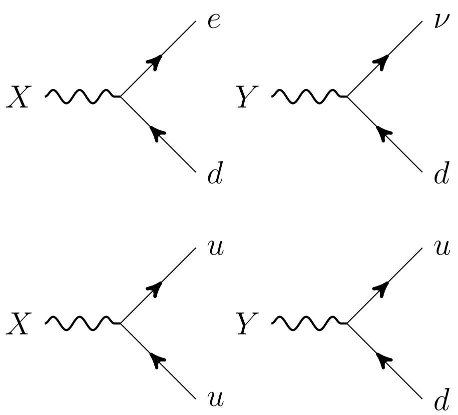
$$\mathbf{5}^\dagger \not{D}\mathbf{5} = ed^cX + \nu d^cY + ... \tag{7.80}$$

$$\mathbf{10}^\dagger \not{D}\mathbf{10} = Q\begin{pmatrix} X \\ Y \end{pmatrix} e^c + u^c e^c Y + ... \tag{7.81}$$

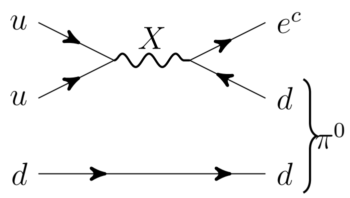
Note that from $\mathbf{5}^\dagger \not{D}\mathbf{5}$ we can actually assign baryon numbers for X and Y of $B = -1/3, L = -1$. Therefore on its own, its insufficient to get proton decay. Similarly for the $\mathbf{10}$ we can assign $B = 2/3, L = 0$. Therefore, only by combining both of these kinetic terms can we get proton decay.

Its important to note that a consistent $B-L$ assignment can be made for both kinetic terms (2/3). Therefore, we see that we can still have a $B-L$ symmetry in grand unified theories.

We now have the interactions,



which can combined to give proton decay,



or equivalently,

$$p^+ \to e^+\pi^0 \tag{7.82}$$

There are quite a few analogous diagrams as well.

Proton decay is a very robust prediction of this framework. Integrating out the X gives a contribution to the amplitude of $1/m_X^2$ and hence $\Gamma \propto 1/m_X^4$. The only other scale is the proton mass. Therefore, the lifetime of the proton is,

$$\tau \sim \frac{m_X^4}{m_p^5} \times \# \tag{7.83}$$

Experiment has measured,

$$\tau_{exp} \gtrsim 10^{31} - 10^{33} y \tag{7.84}$$

From the comparison of these numbers we get,

$$m_X \gtrsim 10^{15}\text{GeV} \tag{7.85}$$

Therefore, we expect to be right on the threshold of probing GUTs.

7.10 Yukawa Unification

Thus far we have discussed SM fermions and the gauge bosons but we have yet to discuss the scalar sector. The first scalar we will add is just the SM Higgs. We know that we need masses for the down-type quarks. We want to make a singlet out of **10** and **5**.

$$\bar{\mathbf{10}} \otimes \mathbf{5} = ... \tag{7.86}$$

Its not hard to imagine (or one could check) This can be made into a singlet by multiplying by a **5**. We therefore introduce a Higgs doublet, $\mathbf{5}_H$, which gives

$$Y_1 \mathbf{5}_H \bar{\mathbf{10}} \mathbf{5} = Y_1 \left(H \bar{q} d + H \bar{L} E + ... \right) \tag{7.87}$$

where $\mathbf{5}_H$ contains the SM Higgs. We can also give masses to the up-type quarks through,

$$Y_2 \mathbf{5}_H \bar{\mathbf{10}} \mathbf{10} = Y_2 \left(\tilde{H} \bar{Q} u + ... \right) \tag{7.88}$$

We see that we have a crucial prediction for the masses of the quarks. At the GUT scale all the down type quarks and leptons have a given mass, $\sim Y_1 v$, and all the up type quarks also have the same mass, $\sim Y_2 v$.

Running these down we have the prediction [Q 50: show this.],

$$m_\tau \approx \frac{m_b}{3} \tag{7.89}$$

This doesn't work as well for the 1st and 2nd generation where we predict,

$$m_\mu \approx \frac{m_s}{3}, m_e \approx \frac{m_d}{3} \tag{7.90}$$

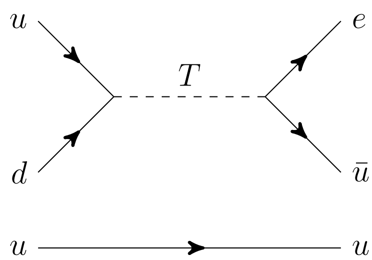
However, one can play some model building tricks to fix this.

7.10.1 Doublet-Triplet Splitting

The Higgs multiplet has an inherent problem. It contains an unwanted triplet under $SU(3)_c$ which we don't see in Nature. Since this triplet is in the same multiplet as the SM Higgs (which to fit the data, we need to have a mass around 126 GeV), it generically have a mass at roughly the EW scale. This is a big problem as it induces proton decay. To see this we recall are Yukawa interaction,

$$\mathbf{5}_H^\dagger \mathbf{105} = \begin{pmatrix} T & H \end{pmatrix}^* \begin{pmatrix} \begin{pmatrix} 0 & u & u \\ & 0 & u \\ & & 0 \end{pmatrix} & Q \\ & \begin{pmatrix} 0 & e \\ & 0 \end{pmatrix} \end{pmatrix} \begin{pmatrix} T \\ H \end{pmatrix} = T^* \begin{pmatrix} 0 & u & u \\ 0 & 0 & u \\ 0 & 0 & 0 \end{pmatrix} d + T^* Q e + \ldots \tag{7.91}$$

These terms are bad. They are baryon and lepton violating and their combination results in proton decay through,



This diagram is roughly given by $\sim y_u^2/m_T^2$, where we define the up-type to be denoted by y_u. By dimensional analysis gives a rate of,

$$\Gamma_{decay} = \frac{y^4 m_p^5}{m_T^4} \tag{7.92}$$

To obey the bound $\tau \gtrsim 10^{34} y$ we must have,

$$m_T \gtrsim 10^{12} \mathrm{GeV} \tag{7.93}$$

With this in mind we now calculate the mass of the triplet in the simplest model. First we need to introduce a GUT breaking scalar. Since we need 12 massive gauge bosons, we need at least 12 potential Goldstone bosons. This can be accomplished by introducing a field in the adjoint representation of the group, a $\mathbf{24}_\Sigma$. The VEV of Σ should preserve $SU(3) \times SU(2)$. This requires,

$$\langle \Sigma \rangle = \frac{v}{\sqrt{60}} \begin{pmatrix} 3 & & & & \\ & 3 & & & \\ & & -2 & & \\ & & & -2 & \\ & & & & -2 \end{pmatrix} \tag{7.94}$$

By ensuring $SU(3) \times SU(2)$ isn't broken we also gaurentee that the fermions will remain massless until the EWSB scale.

A $\mathbf{5}$ and $\mathbf{24}$ produce the potential,

$$V = g \mathbf{5}_H^\dagger \left| \mathbf{24}_\Sigma \right|^2 \mathbf{5}_H - m^2 \left| \mathbf{5}_H \right|^2 + \lambda \left| \mathbf{5}_H \right|^4 + \ldots \tag{7.95}$$

The $\mathbf{5}_H$ is made up of the SM doublet and a new color triplet Higgs,

$$\mathbf{5}_H = \begin{pmatrix} T \\ H \end{pmatrix} \tag{7.96}$$

where $H = (1,2)_1, T = (3,1)_{-1/3}$. The masses of the Higgses are,

$$m_H^2 \sim 3^2 \langle \Sigma \rangle^2 + m^2 \tag{7.97}$$
$$m_T^2 \sim 2^2 \langle \Sigma \rangle^2 + m^2 \tag{7.98}$$

To have m_T at around the GUT scale we need $\langle \Sigma \rangle$ to be around the GUT scale. Therefore, the only way to have a light Higgs is to get $\langle \Sigma \rangle^2$ and m^2 to cancel to very high precision. This is called the doublet triplet splitting problem. For SUSY this problem turns out to be much more severe.

7.11 $SO(10)$

One might wonder why we bothered to put the matter fields into different representations. The number of matter fields in the SM for each generation is,

$$\overbrace{3 \times 4}^{quark} + \underbrace{3}_{lepton} = 15 \tag{7.99}$$

We could have put all the SM fields of one generation into a **15**. While this is appealing, it can be very problematic when trying to build a Higgs sector for this theory as it will inevitably give the same Yukawa for all the matter fields, including the neutrinos. Neutrinos will form extremely massive Majorana spinors, which horribly contradicts reality. Furthermore, we can't just use the seesaw mechanism since there are no right handed neutrinos (keep in mind that adding a field here necessarily means a whole new mutliplet with a new set of particles that we don't observe.

While this idea of putting all the fields in the same multiplet is a big nuisance in $SU(5)$, its completely natural in $SO(10)$. $SO(10)$ has the subgroups,

$$SU(5) \times U(1) \tag{7.100}$$
$$SU(4) \times SU(2) \times SU(2) \tag{7.101}$$

[Q 51: Why do I get $SU(4) \times SU(2)$ using Dynkin diagrams?]

The possibility of breaking through $SU(5) \times U(1)$ is intriguing as all the good features we mentioned earlier will still hold here. The only difference is that we have an extra $U(1)$ symmetry. This $U(1)$ can be identified with $B - L$. Since we don't see an additional massless gauge boson in Nature, this symmetry must be broken. However, the scale of this breaking can be anything.

All the SM can fit neatly into **16**, where now we have room for an additional field, the right handed neutrino. A yukawa interaction will need to involve two **16**'s. To find the possible Higgs representations we need to know the possible charges of a product of matter fields,

$$\mathbf{16} \times \mathbf{16} = \mathbf{10} + \mathbf{126} + \mathbf{120} \tag{7.102}$$

Therefore, the simplest possibility is to have a Higgs of **10** and use a two Higgs doublet model,

$$\mathbf{10} = \mathbf{5} + \bar{\mathbf{5}} \tag{7.103}$$

The terms in the Lagrangian in terms of the $SU(5)$ reps are

$$\mathcal{L} = \mathbf{161610}_H = \bar{\mathbf{5}}\mathbf{105}_H + \mathbf{10} \times \mathbf{105}_H + \mathbf{1}\bar{\mathbf{5}}\mathbf{5}_H \tag{7.104}$$

where **1** is the right handed neutrino. We see that the SM particles get masses,

$$m_u = m_\nu = -\lambda v_u \quad m_d = m_e = -\lambda v_d \tag{7.105}$$

However, there can be an additional mass for the right handed neutrino arising from the GUT breaking Higgs. Suppose the GUT breaking scalar is a $\mathbf{1\bar{2}6}$ (with a scalar ϕ) such that it has a Yukawa with the right handed neutrinos.

$$\lambda_N \bar{\nu}_R^c \nu_L \phi \tag{7.106}$$

This won't break the SM gauge symmetry if it has $U(1) = 10$ and $B - L = -2$. When the scalar gets a VEV it will break $B - L$ but since $B - L$ is even it will leave matter parity,

$$R_m = (-1)^{3(B-L)} \tag{7.107}$$

invariant. Matter parity is equivalent to the much more well known R parity,

$$R_p = (-1)^{3(B-L)+2s} \tag{7.108}$$

Therefore, GUTs actually provide a natural motivation for R_p which is commonly imposed ad hoc in the MSSM.

Furthermore, we now have the neutrino mass matrix,

$$\begin{pmatrix} 0 & \lambda v_u \\ \lambda v_u & \langle\phi\rangle \end{pmatrix} \tag{7.109}$$

To get neutrino masses at around the eV scale we need $\langle\phi\rangle \sim 10^{13}$GeV, which is conveniently very close to the GUT scale. This was the original motivation for the famous see-saw mechanism.

An alternative to the breaking pattern discussed above is to use the Pati Salam models. Recall that Pati Salam theories are made using $SU(4)_c$. $SU(4)_c$ is homomorphic to $SO(6)$.[6] Furthermore, the other two groups in Pati Salam are given by $SU(2) \times SU(2)$ which is homomorphic to $SO(4)$. Therefore, Pati Salam is equivalent to,

$$SO(6) \times SO(4) \tag{7.110}$$

[6] Recall that a group homomorphism is a mapping $f : G \to H$ such that for every element of g: $f(g_1 \cdot g_2) = f(g_1) \cdot f(g_2)$. In contract an group isomorphism is a mapping that can go both from $G \to H$ and from $H \to G$.

Its easy to show (or guess) that this is a subgroup of $SO(10)$. The spinor representation of $SO(10)$ contains one family of quarks and leptons. In fact if you take $SO(10)$ and break it up into Pati salam you get (recall that the notation here is $(SU(4)_c, SU(2)_L, SU(2)_R)$),

$$\mathbf{16} \rightarrow (4, 2, 1) + (\bar{4}, 1, \bar{2}) \tag{7.111}$$

For the Higgses we have through SSB,

$$\mathbf{10} \rightarrow (6, 1, 1) + (1, 2, \bar{2}) \tag{7.112}$$

7.12 Orbifold GUTs

Current four dimensional solutions of the doublet-triplet splitting problem aren't pretty. They either require very large representations for the scalar sector or significant fine-tuning as shown above. Its hard to imagine these theories coming out of theory of quantum gravity such as string theory. The mainstream solution to this problem is to consider orbifold GUTs. The idea that the GUT is a GUT in 5 or 6 dimensions but when we go to 4 dimensions you have broken the unification symmetry down the SM.

Consider a $SO(3)$ gauge theory which is defined on a circle (by this we mean that the fields have periodic boundary conditions in the fifth dimension) ,

$$\overbrace{\mathcal{M}_4}^{Minkowski} \otimes S_1 \tag{7.113}$$

The gauge bosons are given by,

$$A_M(x_\mu, y) \equiv A^a_M(x_\mu, y)T^a \tag{7.114}$$

where $M = \{0, 1, 2, 3, 5\}$ and y goes from 0 to $2\pi R$. The field strength tensor is given by,

$$F_{MN}(x, y) = F^a_{MN}T^a = \partial_M A_N - \partial_N A_M + i\,[A_M, A_N] \tag{7.115}$$

The Lagrangian is given by,

$$\mathcal{L}_5 = -\frac{1}{2g_5^2 k}\mathrm{Tr}F_{MN}F^{MN} \tag{7.116}$$

where k is the normalization of the generators,

$$\mathrm{Tr}T^aT^b = k\delta^{ab} \tag{7.117}$$

The gauge bosons transform under gauge transformations as,

$$A_M(x_\mu, y) \rightarrow UA_MU^\dagger - iU\partial_MU^\dagger \tag{7.118}$$

where $U = \exp\left(i\theta^a(x_\mu, y)T^a\right)$.

Lets consider the special case that $\partial_5 A_\mu(x,y) = 0$. In that case we have,

$$F_{\mu 5} = \partial_\mu A_5 + i\left[A_\mu, A_5\right] \equiv D_\mu A_5 \tag{7.119}$$

This is just equation to the covariant derivative of an adjoint field! We can define a new field

$$\tilde{\Phi} \equiv A_5 \frac{\sqrt{2\pi R}}{g_5} \tag{7.120}$$

Note that g_5 is a dimensionful coupling[7], $\left[g_5^{-2}\right] = E^1$ so we now rewrote it in terms of a new dimensionless coupling.

The Lagrangian becomes,

$$\mathcal{L}_5 = \frac{1}{2\pi R}\left[-\frac{1}{4g^2 k}\mathrm{Tr}\left(F_{\mu\nu}F^{\mu\nu}\right) + \frac{1}{2k}\mathrm{Tr}\left(D_\mu \tilde{\Phi} D^\mu \tilde{\Phi}\right)\right] \tag{7.121}$$

In this limit the theory reduces to a theory of a four dimensional gauge theory and an additional massless four dimensional scalar field in the adjoint representation of the gauge group. The resulting theory has a smaller symmetry then the initial one.

In general A_M will be a function of all dimensions but can be written through a Fourier decomposition,

$$A_M(x,y) = \frac{1}{\sqrt{\pi R}}\sum_n \left[a_M^n(x)\cos\frac{ny}{R} + b_M^n(x)\sin\frac{ny}{R}\right] \tag{7.122}$$

One can show that the field equations for A_M will be proportional to the $5D$ D'Alebertian [8],

$$\Box_5 A_M = \left(\partial_\mu\partial^\mu + \partial_y\partial^y\right)A_M = 0 \tag{7.123}$$

The $\partial_y\partial^y$ acting on A_M will given an effective mass term for the different modes,

$$\sum_n \left(\partial_\mu\partial^\mu - \frac{n^2}{R^2}\right)\left(a_M^n\cos\frac{ny}{R} + b_M^n\sin\frac{ny}{R}\right) = 0 \tag{7.124}$$

Only the $n = 0$ mode is massless (in practice this is the gauge boson we know and love). All the other modes give what are called Kaluza Klien modes of the gauge field. Its important to understand what happened. We started with a gauge field with continuous momenta and ended up with a field that has a different mass dependent on its momenta. These different modes can be interpreted as an infinite number of new particles with masses, n^2/R^2.

Which modes exist for each field will depend on the boundary conditions and any restrictions we apply to our extra dimension. In 4D we impose Lorentz symmetry. We continue to make that assumption in the extra dimensions, however now we will also consider some stronger assumptions about the geometry of spacetime. In particular we can assume invariance under transformations such as parity and translations. As is done for Lorentz invariance, this does not mean the fields must be invariant under these transformations, but just transform in a well-defined way so we can form invariants.

[7] Recall that the action is dimensionless and so $S = \int d^5x\mathcal{L} \Rightarrow [\mathcal{L}] = Energy^5$

[8] This is easy to show if we assume $g = \mathrm{diag}(1,-1,-1,-1,-1)$ and we can just follow the same steps from the $4D$ derivation.

7.12.1 Fermions

Now lets consider fermions in $5D$. In 5 dimensions we are forced to use four component fermions. The reason being that we need new degrees of freedom and the Pauli matrices already form a complete basis. The γ_M matrices obey,

$$\{\gamma_M, \gamma_N\} = 2g_{MN} \tag{7.125}$$

For simplicity we assume the metric,

$$g = \mathrm{diag}\,(1, -1, -1, -1, -1) \tag{7.126}$$

The equation of motion of the fermions is given by,

$$i\gamma_M \partial^M \psi = 0\,, \quad \psi = \begin{pmatrix} \psi_1 \\ i\sigma_2 \psi_2^* \end{pmatrix} = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} \tag{7.127}$$

This is analogous to what we do when we form Dirac spinors in $4D$ however the difference here is that this is the smallest representation we can have for the spinors. When we break this down the four dimensions, we will get two Weyl spinors. This is why a $5D$ theory is a vector-like gauge theory, not a chiral theory (as the SM). We need to break this symmetry and get a chiral theory. We can use orbifolding to break the chirality of the theory.

Lets define the matrix,

$$\gamma_5 = \begin{pmatrix} -\mathbb{1}_{2\times 2} & 0 \\ 0 & \mathbb{1}_{2\times 2} \end{pmatrix} \tag{7.128}$$

If we don't do anything then these fields are also periodic,

$$\psi_{L,R} = \frac{1}{\sqrt{\pi R}} \sum_n a_n^{L/R} \cos\frac{ny}{R} + b_n^{L/R} \sin\frac{ny}{R} \tag{7.129}$$

Lets now imagine that we have a symmetry under parity which acts as,

$$P : \psi\,(x_\mu, y) \to \psi(x_\mu, -y) = -\gamma_5 \psi(x_\mu, y) \tag{7.130}$$

This is equivalent to demanding the left (right) chiral fields be even (odd) under this transformation. Requiring the fields to satisfy this condition is known as "orbifolding". This conditions says that,

$$\frac{1}{\sqrt{\pi R}} \sum_n a_n^L \cos\left(\frac{ny}{R}\right) - b_n^L \sin\left(\frac{ny}{R}\right) = \frac{1}{\sqrt{\pi R}} \sum_n a_n^L \cos\left(\frac{ny}{R}\right) + b_n^L \sin\left(\frac{ny}{R}\right) \tag{7.131}$$

$$\frac{1}{\sqrt{\pi R}} \sum_n a_n^R \cos\left(\frac{ny}{R}\right) - b_n^R \sin\left(\frac{ny}{R}\right) = -\frac{1}{\sqrt{\pi R}} \sum_n a_n^R \cos\left(\frac{ny}{R}\right) + b_n^R \sin\left(\frac{ny}{R}\right) \tag{7.132}$$

which is satisfied if,

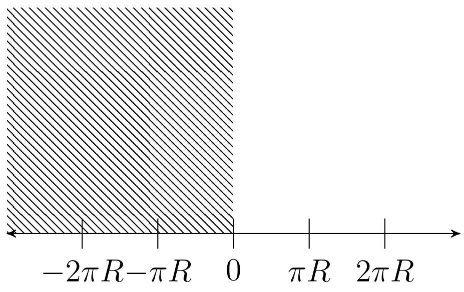
$$b_n^L = 0 \qquad a_n^R = 0 \tag{7.133}$$

Only the ψ_L will have a massless mode while the ψ_R will only have massive modes! By making R very small we can rid of half the degrees of freedom and we can have a chiral gauge theory.

Lets now define orbifolding more carefully. First of all lets take our circle and define it as follows as the real line, moded with the translation group:

$$S^1 = \mathbb{R}^1/\mathcal{T} \tag{7.134}$$

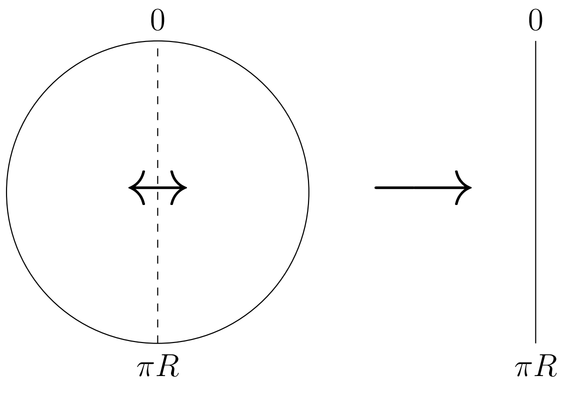
where $\mathcal{T}: \quad y \to y + 2\pi R$. Diagramatically,



where the shaded line is mapped to the unshaded region. We defined a circle through a space group operation. We then define parity as,

$$P: \quad y \to -y \tag{7.135}$$

Modding out by this identification we reduce the circle produced by modding out the translation into a line segement:



0 and πR are fixed points of the Parity operation. Z_2 parity operation has fixed points.

We define,

$$S^1/Z_2 \tag{7.136}$$

as an orbifold. We define our degrees of freedom consistent with the translation and parity operation. Note if we take a point somewhere between 0 and $-\pi R$ then under a translation that goes between πR and $2\pi R$. If we take that point and do a parity operation on it, then it gets translated over to be between $-\pi R$ and $-2\pi R$. Finally doing one more translation we get between 0 and πR. The next effect of this combination, $\mathcal{TPT}$ is equivalent to the Parity operation, i.e.,

$$\mathcal{TPT} = \mathcal{P} \tag{7.137}$$

Now lets represent this parity operation on fields. For the time being lets think of Φ to being in some representation of $SO(3)$. Parity takes the form:

$$\mathcal{P}: \quad \Phi(x,y) \to \Phi(x,-y) = P\Phi(x,y) \tag{7.138}$$

(P here represents a phase) where, $\mathcal{P}^2 = \mathbb{1}$ implies that $P^2 = 1$ and $P = \pm 1$.

The situation is similar for translations. Typically we start with periodic functions. But since this is a gauge theory we have periodic functions that come back to the same function up to a finite gauge transformation,

$$\mathcal{T}: \quad \Phi(x,y) \to \Phi(x,y+2\pi R) = T\Phi(x,y) \tag{7.139}$$

The finite gauge transformation can quite generally be written in the form,

$$T = \exp\left(i \oint \langle A_5 \rangle \, dy\right) \tag{7.140}$$

This is known as a Wilson line. Instead of the functions being purely periodic they are periodic up to the possibility of a phase. T is an element of the $SO(3)$ group and hence we also have $T^2 = 1$.

Due to $\mathcal{TPT} = \mathcal{P}$ we also have [9],

$$(\mathcal{PT})^2 = \mathbb{1} \quad \Rightarrow \quad (PT)^2 = 1 \tag{7.141}$$

where we can define,

$$\mathcal{P}' \equiv \mathcal{PT} \tag{7.142}$$

We can now think of imposing the group,

$$S^1/Z_2 = S^1/Z_2 \otimes Z_2' \tag{7.143}$$

We now take our fields and assume they obey,

$$\mathcal{P}: \quad \Phi(y) \to \Phi(-y) = P\Phi(y) \tag{7.144}$$
$$\mathcal{P}' \quad \Phi(y) \to \Phi(-y+2\pi R) = P'\Phi(y) \tag{7.145}$$

We can either do a parity operation around point zero or around a point πR. We have parity eigenstate possibilities,

$$(P, P') = (\pm, \pm) \tag{7.146}$$

[9] Here we assume that you can simultaneously diagonalize $\mathcal{P}$ and $\mathcal{T}$.

Our components (i.e., the Fourier decomposition modes) will differ depending on these choices,

$$\xi_m(+,+)(y) \approx \cos\frac{my}{R} \tag{7.147}$$

$$\xi_m(+,-)(y) \approx \cos\left[\left(m+\frac{1}{2}\right)y/R\right] \tag{7.148}$$

$$\xi_m(-,+)(y) \approx \sin\left[\left(m+\frac{1}{2}\right)y/R\right] \tag{7.149}$$

$$\xi_m(-,-)(y) \approx \sin\left[(m+1)y/R\right] \tag{7.150}$$

To understand why this is take for example the $(+,-)$ mode. It should be even under $\mathcal{P}$ and odd under $\mathcal{P}'$. We have,

$$\xi_m(+,-) \xrightarrow{\mathcal{P}} \cos\left[-\left(m+\frac{1}{2}\right)y/R\right] = \xi_m(+,-) \tag{7.151}$$

$$\xi_m(+,-) \xrightarrow{\mathcal{P}'} \cos\left[-\left(m+\frac{1}{2}\right)y/R + 2\pi\left(m+\frac{1}{2}\right)\right] = \xi_m(+,-) = -\xi_m(+,-) \tag{7.152}$$

as required. Note that any function which is positive under the boundary conditions, the wavefunction in the fifth direction does not vanish on that boundary. So e.g. at the zero boundary, $\xi_m(+,\pm)$ has nonzero wavefunction.

We see that the 0 boundary and the πR boundary are totally different. We will think of them soon when we consider the $SO(3)$ gauge theory as different branes with different boundary conditions.[Q 52: Define branes carefully.]

Further, note the Kaluza Klien masses for the states are,

$$M_{KK}^2 = \begin{cases} ++ & m^2/R^2 \\ +- & \left(m+\frac{1}{2}\right)/R^2 \\ -+ & \left(m+\frac{1}{2}\right)/R^2 \\ -- & (m+1)^2/R^2 \end{cases} \tag{7.153}$$

Therefore, only the ++ boundary condition has massless modes. We will now consider how we can use these boundary conditions to break the gauge symmetry.

As an example lets consider $SO(3)$ gauge theory. We will take the Wilson line, T, to be given by,

$$T = \exp\left(i\pi T^3\right) = \begin{pmatrix} -1 & & \\ & -1 & \\ & & 1 \end{pmatrix} \tag{7.154}$$

where T^3 is the generator in the adjoint representation. This is equivalent to saying that only the third component of the generator receives a vacuum expectation value. Doing the integral from 0 to $2\pi R$ we are assuming,

$$\langle A_5 \rangle = \frac{1}{2R} \tag{7.155}$$

[Q 53: Clarify this.] Lets choose certain parities of the fields.

The A^a_μ fields appear in the field strength,

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i\left[A_\mu, A_\nu\right] \tag{7.156}$$

We need parity to be a symmetry of the theory. $F_{\mu\nu}$ can only be a parity eigenstate only if

$$A^a_\mu \xrightarrow{\mathcal{P}} A^a_\mu \tag{7.157}$$

A_5 appears in the field strength,

$$F_{\mu 5} = \partial_\mu A_5 - \partial_5 A_\mu - i\left[A_\mu, A_5\right] \tag{7.158}$$

Under the parity operator ∂_5 changes sign and A^a_μ does not which implies that $\partial_\mu A_5$ has to change sign as well. This can only be the case if $A_5 \xrightarrow{\mathcal{P}} -A_5$.

Under the second parity operation we have (remember that T is real and A_μ are in the adjoint representation),

$$A'_\mu = TPA_\mu PT \tag{7.159}$$

For A^3_μ which transforms as,

$$A^{3\prime}_\mu = TPA^3_\mu T^3 PT \tag{7.160}$$

the transformation is simple since A^3_μ is $\mathcal{P}$ invariant and trivially $\mathcal{T}$ invariant as well.

For A^3_5 we still have,

$$TT^3T = T^3 \tag{7.161}$$

but its also odd under $\mathcal{P}$. This results in A^3_5 also being odd under $\mathcal{P}'$.

To consider the off diagonal component we define the quantities,

$$W^\pm_\mu = \frac{1}{\sqrt{2}}\left(A^1_\mu \mp iA^2_\mu\right) \qquad\qquad T^\pm = T_1 \pm iT_2 \tag{7.162}$$

then,

$$TT^\pm T = \pm T^\pm \tag{7.163}$$

One can show that,

$$W^\pm_\mu = (+,-) \tag{7.164}$$

$$W^\pm_5 = (-,+) \tag{7.165}$$

[Q 54: check.]

We have broken up the theory so the different gauge bosons have different parities. The only gauge boson that has a massless mode is A^3_μ. Therefore, we've broken down $SO(3)$ down to $SO(2) \approx U(1)$. The lightest gauge boson has mass $\left(m + \frac{1}{2}\right)/R$.

Lastly we note that an infinitesimal gauge transformations for the fields must be consistent with the boundary conditions. For example A^3_μ has $++$ boundary conditions and so under a gauge transformation is,

$$\delta A^3_\mu = \partial_\mu \theta^3_{++} \tag{7.166}$$

Similarly,

$$\delta A_\mu^{1/2} = \partial_\mu \theta_{+-}^{1/2} \tag{7.167}$$

The $A_\mu^{1/2}$ fields vanish at the πR fixed point. We say that in bulk we have an $SO(3)$ gauge theory. We call the $y = 0$ brane an $SO(3)$ brane since there all gauge fields transform in the same way. [Q 55: Aren't the off-diagonal gauge fields still massive?] We only have $SO(2)$ symmetry on the right brane.

0 πR

$SO(3)$ $SO(2)$

7.12.2 SU(5) SUSY Orbifold GUTs

We consider an orbifolding,

$$\mathcal{M}_4 \otimes S^1/Z_2 \otimes Z_2' \tag{7.168}$$

We now think of an $N = 1$ supersymmetry is $5D$. The gauge sector of superfield in five dimensions contains,

$$V_5 = \{A_M, \lambda, \lambda', \sigma\} \tag{7.169}$$

where λ and λ' are two Weyl spinors and σ is a scalar where all fields are in the adjoint representation of $SU(5)$. This can be thought of as $N = 2$ supersymmetry in $4D$. The gauge multiplet in $4D$ is,

$$V = \{A_\mu, \lambda\} \tag{7.170}$$

and a chiral field,

$$\Sigma = \left\{ \frac{\sigma + A_5}{\sqrt{2}}, \lambda' \right\} \tag{7.171}$$

So out of the $5D$ adjoint multiplet we get a $4D$ $N = 1$ gauge multiplet and a $4D$ $N = 1$ chiral multiplet. these two together form a $N = 2$ representation. These will go in the bulk. We will also put our higgses in the bulk. The Higgs make up two multiplets:

$$\mathcal{H} = \{H_5, H_5^c\} \tag{7.172}$$

Together these form an $N = 1$ $5D$ multiplet. We also introduce,

$$\bar{\mathcal{H}} = \{\bar{H}_{\bar{5}}, \bar{H}_{\bar{5}}^c\} \tag{7.173}$$

These also go in the bulk. Pictorally we will have,

$SU(5)$

0 πR

$SU(5)$ brane SM brane

(V, Σ)

$(\mathcal{H}, \bar{\mathcal{H}})$

We now want to do orbifolding in the bulk. We want to choose our parities such that on the $SU(5)$ gauge degrees of freedom we have,

$$\mathcal{P} \to (+++++) \tag{7.174}$$

$$\mathcal{P}' \to (---++) \tag{7.175}$$

This will break the $SU(5)$ down the SM. Explicity we have,

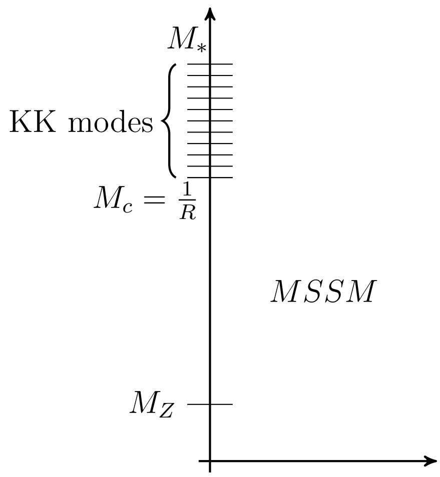
V	$V(8,1,0)$	$++$	$\Sigma(8,1,0)$	$--$
	$V(1,3,0)$	$++$	$\Sigma(1,3,0)$	$--$
	$V(1,1,0)$	$++$	$\Sigma(1,1,0)$	$--$
	$V(3,\bar{2},-5/3)$	$+-$	$\Sigma(3,2,5/3)$	$-+$
	$V(\bar{3},2,5/3)$	$+-$	$\Sigma(\bar{3},\bar{2},-5/3)$	$-+$
H_5	$T(3,1,-2/3)$	$+-$	$T^c(\bar{3},1,2/3)$	$-+$
	$H(1,2,+1)$	$++$	$H^c(1,\bar{2},-1)$	$--$
	$\bar{T}(\bar{3},1,2/3)$	$+-$	$\bar{T}(3,1,-2/3)$	$-+$
	$\bar{H}(1,2,-1)$	$++$	$\bar{H}^c(1,2,1)$	$--$

The massless states of the theory are the $V(8,1,0), V(1,3,0), V(1,1,0)$ as well as the pair of Higgs doublets, $H(1,2,1)$ and $\bar{H}(1,2,-1)$. The colour triplet get masses at the compactification scale.

Notice that we've now done GUT symmetry breaking through orbifolding. We didn't need to introduce a new Higgs sector to the theory. We've also done doublet-triplet splitting through orbifolding. All this also comes naturally from string theory as well.

7.12.3 Gauge coupling unification and proton decay

We have two scales in the theory,



where M_c is the compactification scale and M_* is the cutoff. The $4D$ GUT scale doesn't make sense in this theory. At the scale M_* is where we defined the theory and where the gauge couplings unify.

Bibliography