
Neutrino LECTURE NOTES

LECTURE NOTES BASED IN PART ON A LECTURES SERIES GIVEN BY PILAR HERNANDEZ AT TASI 2013, *Neutrinos*[1], AND ON NOTES WRITTEN BY EVGENY AKHMEDOV IN 2000, *Neutrino Physics*[2].

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Chapter 1

Preface

In this set of notes I go over the fundamentals of neutrino physics. I assume familiarity with the Standard Model (SM) and Quantum Field Theory. I have added in exercises in the text and solutions can be found in the appendix. These notes will likely be expanded until I lose interest in neutrinos. If you have any corrections please let me know at ajd268@cornell.edu.

Chapter 2

Introduction

2.1 Neutrino Experiments

The history of neutrinos is linked tightly to the history of the SM. The way which neutrinos were discovered and later on the way their masses were measured is a fascinating story but we will not go through this story. Instead interested readers can see the book by Frank Close called *Neutrinos*.

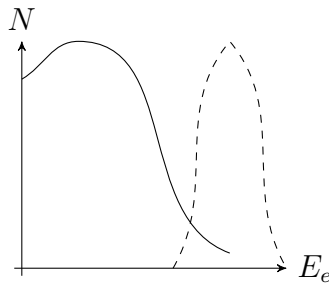
Neutrinos first appeared in physics in 1900 in β decay. There was a transition:



the energy of the electron theoretically was expected to be

$$E_e = (m_N - m_{N'}) c^2 \equiv Q \quad (2.2)$$

which suggested that the energy of the outgoing electron should have a Gaussian distribution:



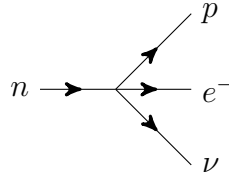
where the dashed distribution is the expected distribution while the experimental distribution is the solid line¹.

A solution to this puzzle was suggested by Pauli which was that there was an extra particle being emitted (Pauli's “dark matter”),

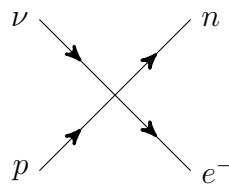
$$E_e = (m_N - m_{N'}) c^2 \equiv Q + \nu \quad (2.3)$$

¹The spectrum was expected to be a Gaussian and not a delta function due to potential kinetic energy of the nuclei.

This particle was called neutrino and this idea was taken seriously by Fermi who created the famous four-fermi interaction:



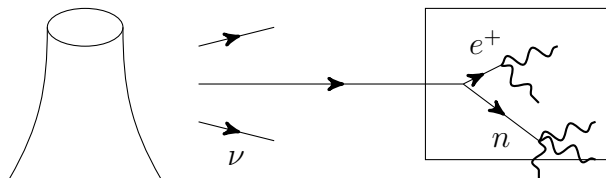
This reaction implied the existence of a related reaction:



Bethe-Peierls estimated this cross-section at the small value, $\sigma_\nu \propto 10^{-44} \text{cm}^2$. It was calculated that with a detector mass of about a ton and a neutrino flux of $\Phi \sim 10^{11} \nu/s$ you can get a few events per day using:

$$\sigma_\nu \times \Phi_\nu \times \frac{N_A}{A} \times Z \times M_{\text{det}} = \frac{\text{few events}}{\text{day}} \quad (2.4)$$

Reines and Cowen in 1956 first discovered the neutrinos in an experiment using a reactor:



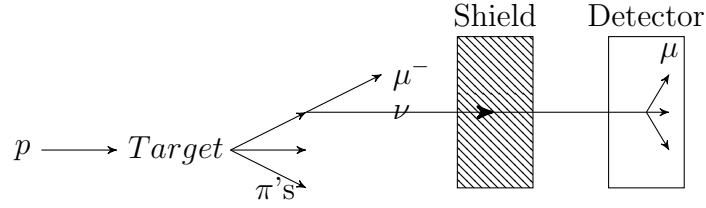
The reactor produced neutrinos which hit a target made of protons. Through the four-Fermi interaction an electron and neutron were produced. The electron quickly annihilated and the neutron underwent neutron capture². The delay between these two events is a very powerful discriminator which enables you to essentially eliminate the background.

Muons were discovered in cosmic rays in the 1930's in decays:

$$\pi^+ \rightarrow \mu^+ + \nu_\mu \quad (2.5)$$

To find if this neutrino was the same one as discovered from β decay, in 1962 Lederman, Schwarz, and Steinberger performed the first accelerator neutrino experiment:

²Neutron capture is when a neutron enters a nucleus and remains there to form a heavier nucleus.



The protons were accelerated into a target which produced pions. The pions decayed into muons and neutrinos. The muons were then stopped by a shield and the neutrinos passed through the shield and hit the detector. The detector only produced muons and not electrons. This is how flavor was discovered. If neutrinos from beta decay was the same as the ones arising from muons then they would have discovered both electrons and muons.

In all these experiments neutrinos could have had a mass. If they did they there would be kinematical effects. However, in all the experiments these kinematical effects were very small. The best current bounds on the neutrino masses are

$${}^3H \rightarrow {}^3He + e^- + \nu_e \Rightarrow "m_{\nu_e}" < 2.2\text{eV} \quad (2.6)$$

$$\pi^+ \rightarrow \mu^+ + \nu_\mu \Rightarrow "m_{\nu_\mu}" < 170\text{keV} \quad (2.7)$$

$$\tau^- \rightarrow 5\pi + \nu_\tau \Rightarrow "m_{\nu_\tau}" < 182\text{MeV} \quad (2.8)$$

2.2 Basic Properties

Neutrinos in the SM have the following properties

1. Form a $SU(2)_L$ doublet with the leptons:

$$\begin{pmatrix} \nu_{L_i} \\ \ell_{L_i} \end{pmatrix} \quad (2.9)$$

where $i = e, \mu, \tau$ with $SU(3) \times SU(2) \times U(1)$ charges of

$$(1, 2, -1) \quad (2.10)$$

Since the electric charge is $T_3 + \frac{Y}{2}$, the upper component of the doublet (which has $T_3 = 1/2$) has an electric charge, $Q_\nu = 0$.

2. Concerning Lorentz transformations neutrinos form left Weyl spinors:

$$\nu_L \equiv P_L \nu \quad (2.11)$$

where

$$P_L = \frac{1 - \gamma^5}{2} \quad (2.12)$$

The equation of motion for a free neutrino is

$$\not{p}\nu_L = 0 \quad (2.13)$$

This has a non-trivial solution if and only if the determinate of \not{p} is equal to zero. Finding the determinate gives:

$$p_0^2 - \mathbf{p}^2 = 0 \quad (2.14)$$

$$\Leftrightarrow E = \pm |\mathbf{p}| \quad (2.15)$$

The helicity operator is

$$\hat{h} = \frac{\boldsymbol{\sigma} \cdot \mathbf{p}}{|\mathbf{p}|} = \frac{p_0 - \not{p}}{|\mathbf{p}|} \quad (2.16)$$

Acting on the neutrino gives,

$$\hat{h}\nu_L = \frac{E}{|\mathbf{p}|}\nu_L \quad (2.17)$$

Inserting in the possible energies we found above we have,

$$\begin{cases} E > 0 & \hat{h} = -1 \\ E < 0 & \hat{h} = +1 \end{cases} \quad (2.18)$$

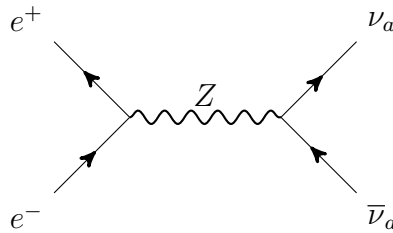
So ν_L can either be a particle with negative helicity or an antiparticle with positive helicity. The simple existence of neutrinos breaks Parity and charge conjugation since Parity sends left neutrinos to right neutrinos which don't exist and charge conjugation to left anti-neutrinos which again don't exist.

3. Neutrinos interact in the SM through:

$$\mathcal{L}_{SM} \supset \left(-\frac{g}{\sqrt{2}}\bar{\nu}_{L,a}\gamma_\mu W^\mu \ell_{L,a} + h.c. \right) - \frac{g}{2\cos\theta_W}\bar{\nu}_{L,a}\gamma_\mu Z^\mu \nu_{L,a} \quad (2.19)$$

where the sum over a runs over the generations.

It is due to the the neutral current interactions that we know there are three flavors of neutrinos. At LEP they did the experiment:



They then measured the ratio of the invisible width of the Z boson to theoretical width for a single neutrino. Since the Z boson couples equally to each flavor, the result should give the number of neutrino flavors. They measured:

$$\frac{\Gamma_{\text{inv}}(Z)}{\Gamma_{\nu_a\bar{\nu}_a}} = 2.984(8) \quad (2.20)$$

Thus the SM must have 3 families of neutrinos.

Chapter 3

Massive ν 's

3.1 Working in free space

We begin by studying massive neutrinos but ignoring any gauge interactions. In the SM the massive fermions are Dirac fermions which means that they are a four-component spinor which is the smallest representation of the Lorentz group + Parity. A free Dirac fermion has the Lagrangian:

$$\mathcal{L}_D = \bar{\Psi} (i\cancel{\partial} - m) \Psi \quad (3.1)$$

where

$$\Psi = \psi_L + \psi_R = P_L \Psi + P_R \Psi \quad (3.2)$$

so the mass term is of the form,

$$-m\bar{\Psi}\Psi = -m(\bar{\psi}_L\psi_R + \bar{\psi}_R\psi_L) \quad (3.3)$$

This implies immediately that for neutrinos in the SM, which only have the left component, we cannot write down such a mass term. If you want to write a Dirac mass for neutrinos you need to have a four-component representation of the neutrino and require a right-handed component for the neutrinos, ν_R . Since these neutrinos haven't been detected anywhere they can't interact with anything and are called sterile neutrinos.

While one can introduce a Dirac mass, this is not the only way that one could add mass to the neutrinos. Majorana realized that you can give mass to particles in a more *economical* way. Majorana fermions are two component Weyl fermions that have the mass term:

$$\mathcal{L}_{Majorana} = -\frac{m}{2} (\bar{\psi}_L\psi_L^c + h.c.) \quad (3.4)$$

where

$$\psi_L^c = C\bar{\psi}_L^T = C\gamma_0\psi_L^* \quad (3.5)$$

where

$$C = i\gamma_0\gamma_2 \quad (3.6)$$

A Dirac mass is a coupling between a fermion and its conjugate since

$$\bar{\psi}_L = \left(C^T C \bar{\psi}_L^T \right)^T \quad (3.7)$$

$$= (\psi_L^c)^T C \quad (3.8)$$

and thus a Dirac mass is of the form,

$$-m (\bar{\psi}_L \psi_R + \bar{\psi}_R \psi_L) = -m \left[(\psi_L^c)^T C \psi_R + (\psi_R^c)^T C \psi_L \right] \quad (3.9)$$

$$= -\frac{m}{2} \left[(\psi_L^c)^T C \psi_R + (\psi_R^c)^T C \psi_L + \psi_R^T C \psi_L^c + \psi_L^T C \psi_R^c \right] \quad (3.10)$$

$$= -\frac{1}{2} \begin{pmatrix} \psi_L^T & (\psi_R^c)^T \end{pmatrix} \begin{pmatrix} 0 & m \\ m & 0 \end{pmatrix} C \begin{pmatrix} \psi_L \\ \psi_R^c \end{pmatrix} + h.c. \quad (3.11)$$

Similarly one can rearrange the Majorana mass for left handed particles as:

$$\mathcal{L}_{Majorana} = -\frac{m}{2} (\bar{\psi}_L \psi_L^c + h.c.) \quad (3.12)$$

$$= -\frac{m}{2} (\psi_L^T C \psi_L + h.c.) \quad (3.13)$$

$$= -\frac{m}{2} \begin{pmatrix} \psi_L^T & (\psi_R^c)^T \end{pmatrix} \begin{pmatrix} m & 0 \\ 0 & 0 \end{pmatrix} C \begin{pmatrix} \psi_L \\ \psi_R^c \end{pmatrix} + h.c. \quad (3.14)$$

Exercise 1.

a) Show that the Majorana mass does not vanish identically in a trivial way. In other words that $P_R \psi_L^c = \psi_L^c$.

b) Show that the mass term is Lorentz invariant. Recall that

$$\psi \rightarrow e^{-\frac{i}{4} \omega_{\mu\nu} \sigma^{\mu\nu}} \psi \quad (3.15)$$

where

$$\sigma_{\mu\nu} = \frac{i}{4} [\gamma^\mu, \gamma^\nu] \quad (3.16)$$

c) Show that the equations of motion derived from this Lagrangian in momentum space give solutions that have the equation

$$E^2 = \mathbf{p}^2 + m^2 \quad (3.17)$$

Exercise 2. Write down a four-component spinor as

$$\bar{\Psi}_M = \psi_L + \psi_L^c \quad (3.18)$$

and show that this Lagrangian can be written as a Dirac Lagrangian

$$\mathcal{L}^{Majorana} = \frac{1}{2} \bar{\Psi}_M (i \not{\partial} - m) \Psi_M \quad (3.19)$$

Exercise 3. Show that a Dirac fermion is equivalent to two degenerate Majorana fermions. To do this you need to start with a Dirac field:

$$\Psi = \psi_L + \psi_R \quad (3.20)$$

You then define Majorana fields the same way we above,

$$\Psi_{M_1} = \psi_L + \psi_L^c \quad (3.21)$$

$$\Psi_{M_2} = \psi_R + \psi_R^c \quad (3.22)$$

The Dirac mass term can then be written in terms of these new fields as

$$-m\bar{\Psi}\Psi = -\begin{pmatrix} \bar{\psi}_{M_1} & \bar{\psi}_{M_2} \end{pmatrix} \begin{pmatrix} 0 & m \\ m & 0 \end{pmatrix} \begin{pmatrix} \psi_{M_1} \\ \psi_{M_2} \end{pmatrix} \quad (3.23)$$

3.2 Interacting Case

We know that for any fermion in the SM that carries charge we can't write down mass terms since the left and right handed fermions have different charges. Furthermore, we can't write down Majorana mass terms due to the conjugation in ψ_L^c .

3.2.1 Dirac mass

In the SM we have a Higgs field which has the quantum numbers: $(1, 2, \frac{1}{2})$. We also have $\tilde{\Phi} = i\sigma_2\Phi^*$ with quantum numbers: $(1, 2 - \frac{1}{2})$. The Higgs acquires a VEV:

$$\langle \Phi \rangle = \begin{pmatrix} 0 \\ v/\sqrt{2} \end{pmatrix}, \quad \langle \tilde{\Phi} \rangle = \begin{pmatrix} v/\sqrt{2} \\ 0 \end{pmatrix} \quad (3.24)$$

To add a Dirac mass we need right-handed neutrinos such that we can write a Yukawa interaction of the form,

$$\mathcal{L}_{Yuk} = -(\lambda_\nu \underbrace{\bar{L}\tilde{\Phi}}_{(1,1,0)} \nu_R + h.c.) \quad (3.25)$$

$\begin{matrix} (1,1,0) \\ \uparrow \\ \downarrow \\ (1,1,0) \end{matrix}$

Upon spontaneous symmetry breaking (SSB) this term gives a Dirac mass:

$$-(\bar{\nu}_L m_\nu \nu_R + h.c.) \quad (3.26)$$

with

$$\lambda_\nu \frac{v}{\sqrt{2}} \quad (3.27)$$

3.2.2 Majorana Masses

A Majorana mass term for a left-handed neutrino is of the form, $-m\nu_L^T C\nu_L$. In the SM prior to spontaneous symmetry breaking such a term is forbidden such it breaks $U(1)_Y$ symmetry.

The next suggestion would be to form such a term through spontaneous symmetry breaking. However, there are some important requirements

1. The Majorana mass term has to come from a term with two $SU(2)$ doublets
2. Both $SU(2)$ doublets need to be either unbarred or both barred to have a Majorana mass
3. The term must conserve hypercharge

These are very strong requirements. One can go ahead and try different terms and it is easy to see that most naive combinations are forbidden. To try to better understand the type of allowed terms consider the charges of the Majorana term. It has an isospin, $T_3 = 1$, and hypercharge -2 . This suggests that it can only arise from a term that transforms as a vector under isospin rotations. The lowest order term that transforms as a vector under isospin and has a neutrino-neutrino coupling is given by,

$$(L^T i\sigma_2 \sigma L) \quad (3.28)$$

Exercise 4. Show that this term transforms as a vector

However, we have yet to find a isospin singlet. To do this we have to take the product of this term with another term which transforms as a vector under $SU(2)$ and has hypercharge of 2. If we had such a particle, Δ , we could write:

$$-\mathcal{L}_{Yuk}^\Delta = f_\Delta (L^T i\sigma_2 \sigma L) \cdot \Delta + h.c. \quad (3.29)$$

Then when the electrically neutral component of the Higgs (the third component) gets a VEV we would get a Majorana mass. However, in the SM where we don't have such a mass we can still get a Majorana mass by taking a product of Higgs:

$$-\mathcal{L}_{Yuk} = -\alpha (L^T i\sigma_2 \sigma L) \cdot (H^T i\sigma_2 \sigma H) + h.c. \quad (3.30)$$

Note that this term cannot arise from loop in the SM due to conservation of lepton number.

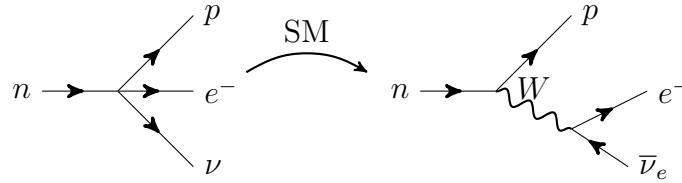
Matching dimensions we see that we have

$$[\alpha] = -1 \quad (3.31)$$

We can write

$$\alpha = \frac{\lambda}{\Lambda} \quad (3.32)$$

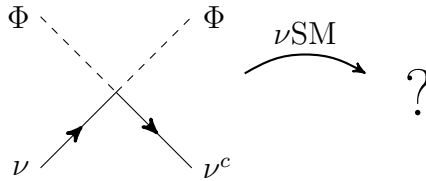
where Λ is a scale of new physics. A big question of new physics is if neutrinos are Majorana, *what the hell is this scale?* If neutrinos are Majorana then we are very close the situation that Fermi was in when he was describing β decay. The four-point Fermi interactions had dimensionful coupling which was an effective theory:



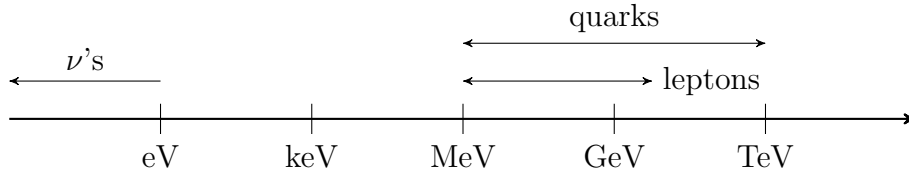
In this way you can understand that the Fermi coupling in the limit of low energy is just

$$G_F \sim \frac{1}{M_W^2} \quad (3.33)$$

Now with neutrinos we are in a very similar situation where we have the coupling



The good thing about having this possibility is that neutrinos are very light compared to the other particles in the SM and masses arising from non-renormalizable operators are naturally suppressed. Plotting fermion masses on a Logarithmic scale gives roughly,

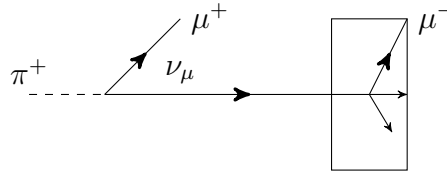


There is a gap of at least 6 orders of separating the neutrino masses from the remaining fermions. If there was a Dirac mass then it's hard to understand how this would come about because after all neutrinos are in the same doublet as the leptons. However, in the case of Majorana neutrinos if $\Lambda \gg v$ then the neutrino mass would be

$$m_\nu \sim m_F \frac{v}{\Lambda} \quad (3.34)$$

Having Dirac or Majorana masses has very different implications in terms of global symmetries. If neutrinos are Dirac fermions then Lepton number ($L_i \rightarrow e^{i\alpha} L_i$) is still a good symmetry. However, if neutrinos are Majorana then no such symmetry remains since a Majorana mass term breaks all possible symmetries.

Deciding whether we have Dirac or Majorana neutrinos is a fundamental question in neutrino physics. In principle Dirac and Majorana neutrinos leave different traces. Take for example a neutrino beam from pion decay. The following will always produce a muon at the detector,



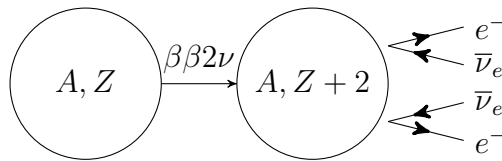
regardless of whether the neutrino is Dirac or Majorana. For a Dirac fermion the muon has to be of negative sign to conserve lepton number. However if a neutrino has a Majorana mass then it may spontaneously turn into an antineutrino and subsequently decay into a antimuon.

Unfortunately the rate for getting these negative muons is approximately equal to the rate of getting a position muons. The rate of getting a positive (R_+) or negative muon (R_-) are related by:

$$R_+ \sim R_- \left(\frac{m_\nu}{E} \right)^2 \tag{3.35}$$

because to make the transition from neutrino to antineutrino you need a helicity flip. Since m_ν is so small such a measurement is typically not practical.

The best method to look for Majorana neutrinos is in what's known as neutrinoless double beta decay ($\beta\beta 0\nu$):



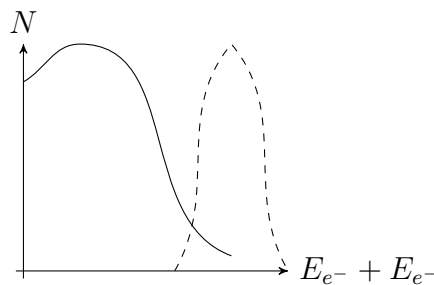
This process is very high order in the weak interaction and is very rare. However it has been measured to have rate

$$T_{\beta\beta 2\nu} \sim 10^{18} - 10^{20} y \tag{3.36}$$

If neutrinos are Majorana then such a process can take place without any outgoing antineutrinos since the two antineutrinos can annihilate one another. The rate has been measured to be roughly,

$$T_{\beta\beta 0\nu} > 10^{25} y \tag{3.37}$$

The $\beta\beta 0\nu$ decay has a larger phase space so such a measurement is not hopeless. One can then go ahead and measure the energy distribution. Depending on the type of mass the neutrinos have we have two possible distributions (notice the analogy to our earlier discussion):



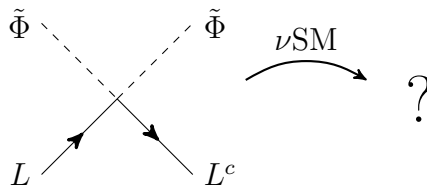
where now the dashed distribution holds if the neutrinos are Majorana while the solid distribution holds if neutrinos are Dirac.

3.3 Neutrino Masses

What we have learned from experiments was that

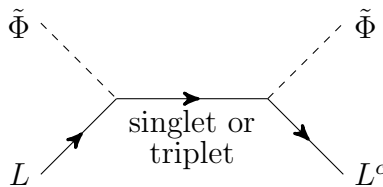
$$m_\nu \ll m_\ell, m_q \tag{3.38}$$

Neutrinos have a very different mixing pattern to the CKM mixing. There have been a lot of works in the literature to try to understand the mixing pattern however we haven't gotten very far in this aspect. On the other hand we do have a compelling explanation for why the neutrinos have small masses. This explanation is that neutrinos are Majorana and their masses arise from a non-renormalizable operator. This non-renormalizable operator represents some new physics, but what in particular is not yet understood,

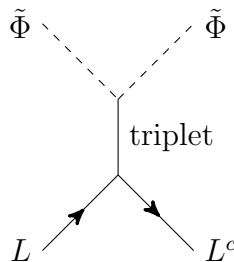


There are essentially three possibilities:

The particles could exchange a fermion,



The fermion could be a singlet fermion (type 1), triplet fermion (type 2). Alternatively, you could have a t-channel process through a triplet scalar boson in the t channel (type 3):

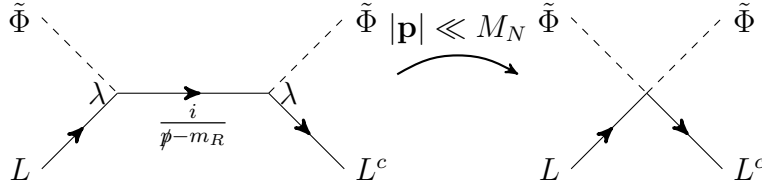


Of course one can consider the term coming from loop level. In this case there will be the non-renormalizability suppressed as well as loop suppression.

We focus only on type 1. Since the intermediate particles are singlets they are often called right-handed neutrinos. This gives the terms:

$$\mathcal{L}_{type\ 1} = \mathcal{L}_{SM} - \lambda \bar{L} \tilde{\Phi} N_R - \frac{1}{2} \bar{N}_R^c m_R N_R \quad (3.39)$$

where N_R is the new singlet fermion. At low energies we have,



The coupling in the four point interaction is:

$$\lambda^T \frac{1}{M_R} \lambda \quad (3.40)$$

The neutrino mass in this model is then,

$$m_\nu = \lambda^T \frac{v^2}{M_R} \lambda \quad (3.41)$$

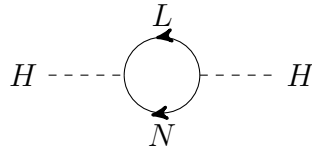
If we measure neutrino masses then how can measure the scale M_R ? The Yukawa couplings in the SM are:

If we expect the neutrino Yukawas to be of similar order as those in the SM then what are then if we take $M_R = \mathcal{O}(\text{GUT})$ then the neutrino Yukawas will be of the order of the top yukawa. If you take $M_R = \mathcal{O}(\text{TeV})$ then you get Yukawas of order of the electron Yukawa. So we have,

$$M_R \in [\text{TeV}, \text{GUT}] \quad (3.42)$$

The energy scale can be anything!

To constrain the couplings you can use natuarlness¹. The corrections to the Higgs mass due to these new fermions are given by,



¹This arguement fails if you have SUSY.

The corrections are

$$\delta m_H^2 \propto \frac{\lambda^2}{16\pi^2} M_R^2 \log \frac{M_R}{\mu} \quad (3.43)$$

Exercise 5. *Show that the corrections indeed take this form*

The predictions of type 1 seesaw models are the following:

1. $\beta\beta 0\nu$ provided $M_R \lesssim 100\text{MeV}$
2. Matter-antimatter asymmetry in the evolution of the universe which might explain the one we observe. However, this is only true if there is CP violation in the lepton sector which we haven't found yet.
3. New states which have not yet been observed.

The type 1 Lagrangian mass terms can be written as (see Eqs 3.14 and 3.11),

$$-\mathcal{L}_{\text{type 1}} = \frac{1}{2} \nu_L^T C m_L \nu_L + \frac{1}{2} \nu_R^T C m_R^* \nu_R + \bar{\nu}_L m_D^* \nu_R + h.c. = \frac{1}{2} n_L^T \mathcal{M} C n_L + h.c. \quad (3.44)$$

where $\nu_L \equiv (\nu_L^1, \nu_L^2, \dots, \nu_L^n)$, $\nu_R \equiv (\nu_R^1, \nu_R^2, \dots, \nu_R^n)$, $n_L \equiv (\nu_L, \nu_R^c)$,

$$\mathcal{M} \equiv \begin{pmatrix} m_L & m_D \\ m_D^T & m_R \end{pmatrix} \quad (3.45)$$

where m_R, m_L are complex symmetric $n \times n$ matrices and m_D is a complex $n \times n$ matrix.

Exercise 6. *Show that the matrix form shown above is consistent with the explicit form.*

We first consider the special case of a single generation. Diagonalization in this case is straightforward. Furthermore, for simplicity we consider the mass parameters to be real. In this case the matrix can be diagonalized by an orthogonal matrix:

$$U \equiv \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \quad (3.46)$$

We denote the diagonalized fields by χ_L :

$$n_L = U \begin{pmatrix} \chi_{1L} \\ \chi_{2L} \end{pmatrix} \quad (3.47)$$

We now find the angle that diagonalizes the mass matrix:

$$\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} = U^T \mathcal{M} U \quad (3.48)$$

$$= \begin{pmatrix} c_\theta & -s_\theta \\ s_\theta & c_\theta \end{pmatrix} \begin{pmatrix} m_L & m_D \\ m_D & m_R \end{pmatrix} \begin{pmatrix} c_\theta & s_\theta \\ -s_\theta & c_\theta \end{pmatrix} \quad (3.49)$$

$$= \begin{pmatrix} s_\theta^2 m_R + c_\theta^2 m_L - s_{2\theta} m_D & c_{2\theta} m_D + \frac{1}{2} s_{2\theta} (m_L - m_R) \\ c_{2\theta} m_D + \frac{1}{2} s_{2\theta} (m_L - m_R) & s_\theta^2 m_L + c_\theta^2 m_R + s_{2\theta} m_D \end{pmatrix} \quad (3.50)$$

This matrix is diagonal if

$$\tan 2\theta = \frac{2m_D}{m_R - m_L} \quad (3.51)$$

To calculate the mass eigenvalues we use the following relations

$$c_{2\theta} = \frac{m_R - m_L}{\sqrt{4m_D^2 + (m_R - m_L)^2}}, \quad s_{2\theta} = \frac{2m_D}{\sqrt{4m_D^2 + (m_R - m_L)^2}} \quad (3.52)$$

$$c_\theta^2 = \frac{1}{2} + \frac{1}{2} \frac{m_R - m_L}{\sqrt{4m_D^2 + (m_R - m_L)^2}}, \quad s_\theta^2 = \frac{1}{2} - \frac{1}{2} \frac{m_R - m_L}{\sqrt{4m_D^2 + (m_R - m_L)^2}} \quad (3.53)$$

which gives,

$$m_{1/2} = \frac{m_L + m_R}{2} \mp \sqrt{\left(\frac{m_L - m_R}{2}\right)^2 + m_D^2} \quad (3.54)$$

The masses are real but can be positive or negative. In terms of the mass eigenstates the Lagrangian takes the form,

$$-\mathcal{L}_m = \frac{1}{2} \chi_L^T \mathcal{M}_d \chi_L + h.c. \quad (3.55)$$

$$= \frac{1}{2} (m_1 \chi_{1L}^T C \chi_{1L} + m_2 \chi_{2L}^T C \chi_{2L}) + h.c. \quad (3.56)$$

Consider the Dirac spinors,

$$\chi_1 = \chi_{1L} + \eta_1 \chi_{1L}^c, \quad \chi_2 = \chi_{2L} + \eta_2 \chi_{2L}^c \quad (3.57)$$

where $\eta_{1/2}$ is positive if $m_{1/2} > 0$.

With these we have,

$$\bar{\chi} \chi = (\bar{\chi}_L + \eta \bar{\chi}_L^c) (\chi_L + \eta \chi_{1L}^c) \quad (3.58)$$

$$= (\chi_L^{cT} + \eta \chi_L^T) C (\chi_L + \eta \chi_{1L}^c) \quad (3.59)$$

$$= \eta (\chi_L^T C \chi_L + \chi_L^{cT} C \chi_{1L}^c) \quad (3.60)$$

$$= \eta (\chi_L^T C \chi_L + h.c.) \quad (3.61)$$

Thus we can alternatively write the mass term as two products of Majorana spinors:

$$-\mathcal{L}_m = \frac{1}{2} \left(\frac{m_1}{\eta_1} \bar{\chi}_1 \chi_1 + \frac{m_2}{\eta_2} \bar{\chi}_2 \chi_2 \right) \quad (3.62)$$

$$= \frac{1}{2} (|m_1| \bar{\chi}_1 \chi_1 + |m_2| \bar{\chi}_2 \chi_2) \quad (3.63)$$

3.3.1 Dirac Spinor

Consider now the special case of a Dirac spinor. In this case $m_D \equiv m$ and $m_L = m_R = 0$. This gives mass eigenvalues of $\pm m$ and a mixing angle of $\theta = \frac{\pi}{4}$. The mass eigenstates are,

$$\nu_L = \frac{1}{\sqrt{2}}(\chi_{1L} + \chi_{2L}), \quad \nu_R^c = \frac{1}{\sqrt{2}}(-\chi_{1L} + \chi_{2L}) \quad (3.64)$$

$$\Leftrightarrow \chi_{1L} = \frac{1}{\sqrt{2}}(\nu_L - \nu_R^c), \quad \chi_{2L} = \frac{1}{\sqrt{2}}(\nu_L + \nu_R^c) \quad (3.65)$$

This gives,

$$\chi_1 = \frac{1}{\sqrt{2}}(\nu_L - \nu_R^c - \nu_L^c + \nu_R), \quad \chi_2 = \frac{1}{\sqrt{2}}(\nu_L + \nu_R^c + \nu_L^c + \nu_R) \quad (3.66)$$

which gives,

$$\bar{\chi}_1 \chi_1 = \bar{\nu}_L \nu_R + \bar{\nu}_R^c \nu_L^c + \bar{\nu}_L^c \nu_R^c + \bar{\nu}_R \nu_L - (\dots) \quad (3.67)$$

where we omit the negative terms as they will cancel with the strictly positive $\bar{\chi}_2 \chi_2$ contribution. Thus we have,

$$-\mathcal{L}_m = \frac{m}{2} (\bar{\nu}_L \nu_R + \bar{\nu}_R^c \nu_L^c + \bar{\nu}_L^c \nu_R^c + \bar{\nu}_R \nu_L) \quad (3.68)$$

$$= m (\bar{\nu}_L \nu_R + \bar{\nu}_R \nu_L) \quad (3.69)$$

where we have used the identity, $\bar{\nu}_L^c = \nu_R^c$. This finally be rewritten in terms of a Dirac spinor:

$$-\mathcal{L}_m = m \bar{\nu}_D \nu_D \quad (3.70)$$

where $\nu_D \equiv \nu_L + \nu_R$.

Under a global $U(1)$ phase transformation we have,

$$\bar{\nu}_D \nu_D \rightarrow e^{i(L_L - L_R)} \bar{\nu}_L \nu_R + e^{-i(L_L - L_R)} \bar{\nu}_R \nu_L \quad (3.71)$$

Theres a conserved Lepton number only if

$$L_L = L_R \quad (3.72)$$

The same can be shown for $\bar{\chi}_2 \chi_2$.

For the Dirac case, the mass term is also particularly simple:

$$-\mathcal{L}_m = m (\bar{\nu}_L \nu_L + \bar{\nu}_R^c \nu_R^c + \bar{\nu}_L^c \nu_L^c + \bar{\nu}_R \nu_R) \quad (3.73)$$

$$= m \quad (3.74)$$

3.3.2 See-saw

Now consider another type-1 see-saw mechanism limit, $m_L \ll m_D \ll m_R$. In this case

$$m_1 \approx m_L - \frac{2m_D^2}{m_R}, \quad m_2 \approx m_R \quad (3.75)$$

Thus we have one eigenstate that is very heavy with a mass of approximately, m_R . The second eigenstate has a very small mass suppressed by the scale m_R . Furthermore, in this limit $\theta \approx \frac{m_D}{m_R} \ll 1$. The left handed components of the mass eigenstates are approximately given by,

$$\chi_{1L} \approx \nu_L - \frac{m_D}{m_R} \nu_R^c, \quad \chi_{2L} = \nu_R^c + \frac{m_D}{m_R} \nu_L \quad (3.76)$$

which gives full mass eigenstates,

$$\chi_1 \approx \nu_L + \eta_1 \nu_L^c - \frac{m_D}{m_R} (\nu_R^c + \eta_1 \nu_R) \quad (3.77)$$

$$\chi_2 \approx \nu_R^c + \eta_2 \nu_R + \frac{m_D}{m_R} (\nu_R + \eta_2 \nu_R^c) \quad (3.78)$$

The eigenstates are approximately just the left and right handed neutrinos but with a small admixture of order $\frac{m_D}{m_R}$.

We now consider the same limit but for the full n -generation case. We want to block diagonalize the matrix \mathcal{M} :

$$u_L = U \chi_L, \quad U^T \mathcal{M} U = U^T \begin{pmatrix} m_L & m_D \\ m_D^T & M_R \end{pmatrix} U \approx \begin{pmatrix} \tilde{m}_L & 0 \\ 0 & \tilde{M}_R \end{pmatrix} \quad (3.79)$$

U is an orthogonal $2n \times 2n$ matrix. Since we take the eigenvalues of M_R to be large this matrix is almost diagonal. We take as an ansatz U to be of the form,

$$U = \begin{pmatrix} 1 & \rho \\ -\rho^T & 1 \end{pmatrix} \Rightarrow U^T U = 1 + \mathcal{O}(\rho^T \rho) \quad (3.80)$$

and we furthermore, take ρ to be small.

Multiplying out the matrices gives,

$$U^T \mathcal{M} U = \begin{pmatrix} m_L - \rho m_D^T - m_D \rho^T & m_L \rho + m_D - \rho M_R \\ \rho^T m_L + m_D^T - M_R \rho^T & m_D^T \rho + \rho^T m_D + M_R \end{pmatrix} \quad (3.81)$$

This matrix is diagonal if $\rho = m_D m_R^{-1}$ and gives the mass matrices:

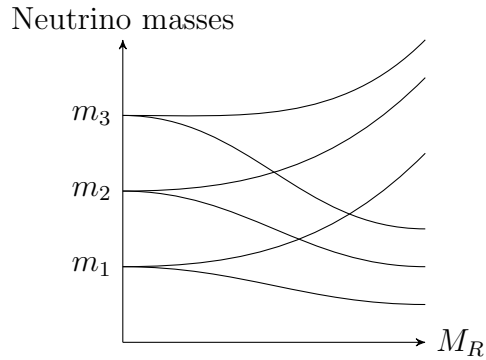
$$\tilde{m}_L \approx m_L - m_D M_R^{-1} m_D^T \quad (3.82)$$

$$\tilde{m}_R \approx M_R \quad (3.83)$$

Diagonalization of the effective mass matrix \tilde{m}_L gives n light Majorana neutrinos made up primarily of ν_L with a small admixture of order of $m_D M_R^{-1}$ of the ‘‘sterile’’ right handed

neutrinos. Diagonalization of the matrix \tilde{m}_R gives the masses for the heavy eigenstates composed primarily of the right handed neutrinos with a small admixture of the left handed neutrinos.

The neutrino masses as a function of the right handed neutrino Majorana mass can be calculated. At very small right handed neutrino Majorana masses, the left and right neutrinos just have a Dirac mass. However, as the right neutrino mass increases, there is splitting and we get 3 very heavy neutrinos and 3 very light:



Chapter 4

Lepton Mixing

Whether neutrinos are Dirac or Majorana the Yukawa coupling will still not be diagonal in flavor. The Lagrangian will include one of the two terms:

$$\text{Majorana: } \mathcal{L} \supset -\frac{v^2}{2\Lambda} \bar{\nu}_{L,a} (\lambda_\nu^{(M)})_{ab} \nu_{L,b}^c + h.c. \quad \text{Dirac: } \mathcal{L} \supset -\frac{v}{\sqrt{2}} \bar{\nu}_{L,a} (\lambda_\nu^{(D)})_{ab} \nu_{R,b} + h.c. \quad (4.1)$$

Its not hard to show that the Majorana mass matrix must be symmetric $n \times n$ complex matrix. In the case of Dirac mass $\lambda^{(D)}$ is a symmetric $n \times n_R$ complex matrix (the number of right handed neutrinos need not be equal to the number of left handed neutrinos). To go from the flavor basis to the mass basis you can always write $\lambda_\nu^{(M)}$ as a diagonal matrix:

$$\lambda_\nu^{(M)} = V_\nu^\dagger \text{diag}(m_{\nu_L}) V_\nu \quad (4.2)$$

$$\lambda_\nu^{(D)} = V_\nu^\dagger \text{diag}(m_{\nu_L}) U_\nu \quad (4.3)$$

In the Dirac case the unitary matrix on the right and on the left are different.

First we consider the Dirac mass case. You can always go to the mass basis and rotate your fields:

$$\nu'_R = U_\nu \nu_R \quad (4.4)$$

$$\ell'_R = U_\ell \ell_R \quad (4.5)$$

$$\nu'_L = V_\nu \nu_L \quad (4.6)$$

$$\ell'_L = V_\ell \ell_L \quad (4.7)$$

Now in terms of the primed fields the masses are diagonal, however the gauge interactions are no longer diagonal. The only interaction which remains non-diagonal is the charge current interactions which in terms of neutrinos looks like

$$\mathcal{L}_{SM} \supset -\frac{g}{\sqrt{2}} \bar{\nu}'_L \underbrace{V_\nu^\dagger V_\ell}_{U_{PMNS}} \gamma_\mu \ell'_L W_\mu + h.c. \quad (4.8)$$

The U_{PMNS} matrix is completely analogous to the CKM matrix. An interesting exercise is to compute how many physical parameters are in the PMNS matrix. A typical unitary matrix has

$$\frac{n(n \pm 1)}{2} \quad (4.9)$$

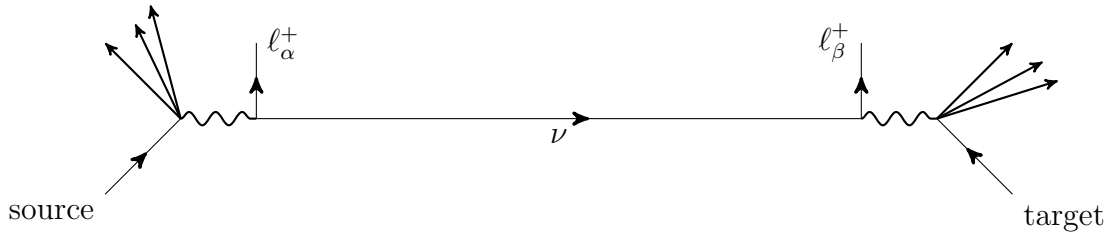
physical parameters where the plus is for phases and the minus is for angles. However this is not the correct answer.

Exercise 7. *Perform this exercise for both Dirac and Majorana neutrinos.*

When you produce neutrinos they are produced in a flavor eigenstate, say ν_ℓ for example. The neutrino is a linear combination of mass eigenstates:

$$|\nu_\ell\rangle = \sum_i U_{i\ell} |\nu_i\rangle \quad (4.10)$$

where $U_{i\ell}$ are entries from the PMNS matrix. When neutrinos propagate in space, the mass eigenstates are the ones that propagate. When the flavor is again measured after a while there is a probability to measure it as any of the flavor eigenstates. Thus lepton number for each generation is broken, though total lepton number is still conserved. Pictorially we have,



In the beginning we have,

$$\begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix} = U_{PMNS}^* \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix} \quad (4.11)$$

At some point farther away these states will pick up phases:

$$|\nu_i(\mathbf{p})\rangle \rightarrow e^{-iE_i t} |\nu_i(\mathbf{p})\rangle \quad (4.12)$$

where $E_i = \sqrt{\mathbf{p}^2 + m_i^2}$. Due to this interference between states we have a non-vanishing probability to find a new type of neutrino. This gives rise to the phenomena of neutrino oscillations. We now answer the question “if you produce a flavor α what is the probability that you will detect a flavor β a given distance away?” We label this probability $P(\ell_\alpha \rightarrow \ell_\beta)$.

4.1 Neutrino Oscillations in Vacuum

We derive this result using simple quantum mechanics and treat the neutrinos as plane waves. At some initial time you produce a neutrino is a flavor state;

$$|\nu_\alpha(t_0)\rangle = \sum_i U_{\alpha,i}^* |\nu_i(\mathbf{p})\rangle \quad (4.13)$$

These states are eigenstates of the Hamiltonian and are on shell:

$$\hat{H} |\nu_i(\mathbf{p})\rangle = E_i |\nu_i(\mathbf{p})\rangle \quad (4.14)$$

$$E_i = \sqrt{m_i^2 + \mathbf{p}^2} \quad (4.15)$$

We evolve the initial state through the evolution operator:

$$|\nu_\alpha(t)\rangle = \sum_i U_{\alpha,i}^* e^{-iE_i(t-t_0)} |\nu_i(\mathbf{p})\rangle \quad (4.16)$$

The probability is simply

$$P(\nu_\alpha \rightarrow \nu_\beta) = |\langle \nu_\beta | \nu_\alpha(t) \rangle|^2 \quad (4.17)$$

$$= \left| \sum_i U_{\beta,i} U_{\alpha,i}^* e^{-iE_i(t-t_0)} \right|^2 \quad (4.18)$$

The difference of the energies is,

$$E_j - E_i = \sqrt{|\mathbf{p}|^2 + m_j^2} - \sqrt{|\mathbf{p}|^2 + m_i^2} \quad (4.19)$$

$$\approx \frac{m_j^2 - m_i^2}{2|\mathbf{p}|} \quad (4.20)$$

Furthermore,

$$L = t - t_0 \quad (4.21)$$

Inputting in these results gives the master formula:

$$P(\nu_\alpha \rightarrow \nu_\beta) = \sum_{ij} U_{\beta,j}^* U_{\alpha,j} U_{\beta,i} U_{\alpha,i}^* e^{i\Delta m_{ji}^2 L/2|\mathbf{p}|} \quad (4.22)$$

This equation assumes a few things. Firstly that the neutrino mass eigenstate each starts off with the same momenta. Further you may wonder how the neutrino states can be a good approximation of plane waves if you initially say that these states are localized at the source and detector. This derivation also did not incorporate the two key complications:

1. The uncertainty in the momentum of the source and detector.

2. Coherence over macroscopic distances.

One can go ahead and make this more rigorous but that is beyond the scope of these notes.

The master formula gives the “probability of appearance” for $\alpha \neq \beta$ and the “probability of disappearance” for $\alpha = \beta$. One can show that the master formula can be written in a slightly more familiar form using the Unitarity of the matrix:

$$\begin{aligned}
 P(\nu_\alpha \rightarrow \nu_\beta) &= \delta_{\alpha\beta} - 4 \sum_{i < j} \text{Re} [U_{\alpha,i}^* U_{\beta,i} U_{\alpha,j} U_{\beta,j}^*] \sin^2 \frac{\Delta m_{ij}^2 L}{4E} \\
 &\quad - 2 \sum_{i < j} \text{Im} [U_{\alpha,i}^* U_{\beta,i} U_{\alpha,j} U_{\beta,j}^*] \sin \frac{\Delta m_{ij}^2 L}{2E}
 \end{aligned} \tag{4.23}$$

Exercise 8. Prove equation 4.23

Solution 1.

To compute the corresponding formula for neutrinos we just need to take the complex conjugate of the product of matrix. This gives,

$$\begin{aligned}
 P(\bar{\nu}_\alpha \rightarrow \bar{\nu}_\beta) &= \delta_{\alpha\beta} - 4 \sum_{i < j} \text{Re} [U_{\alpha,i}^* U_{\beta,i} U_{\alpha,j} U_{\beta,j}^*] \sin^2 \frac{\Delta m_{ij}^2 L}{4E} \\
 &\quad + 2 \sum_{i < j} \text{Im} [U_{\alpha,i}^* U_{\beta,i} U_{\alpha,j} U_{\beta,j}^*] \sin \frac{\Delta m_{ij}^2 L}{2E}
 \end{aligned} \tag{4.24}$$

Thus to the extent that the imaginary term does not vanish we will have a difference in the oscillation probability for neutrinos and antineutrinos. In order to avoid this term the phases in the imaginary parts need to add to zero. It is a simple exercise to show that in the case that $\Delta m = 0$ for one set of the neutrinos this term vanishes.

For only two generations there is one angle in the mixing matrix and its given by,

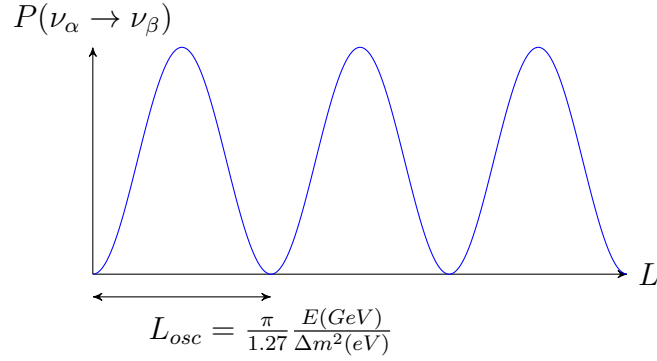
$$U = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \tag{4.25}$$

Plugging in to the result above for disappearance¹

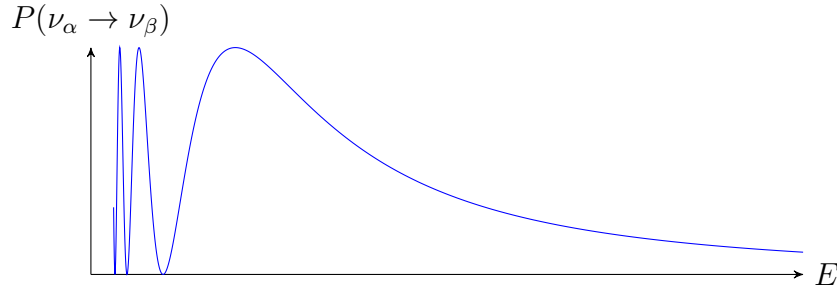
$$P(\nu_\alpha \rightarrow \nu_\beta) = \sin^2 2\theta \sin^2 \frac{\Delta m^2 L}{4E} = \sin^2 2\theta \sin^2 \left(\frac{1.27 \Delta m^2 (eV) L (km)}{E (GeV)} \right) \tag{4.26}$$

$$P(\nu_\alpha \rightarrow \nu_\alpha) = 1 - P(\nu_\alpha \rightarrow \nu_\beta) \tag{4.27}$$

¹It helps to note that for $i > j$ we can only have $i = 1, j = 2$.



In a real experiment we don't have the pleasure of varying L but we can still vary the energy:



The position of the first maxima is given by

$$E_{max} = \frac{1.27 \Delta m^2 (eV^2) L (km)}{\pi/2} \quad (4.28)$$

If you want to do a neutrino experiment for which you want to get the angle θ and Δm^2 then you should choose

$$\frac{E}{L} \sim \Delta m^2 \quad (4.29)$$

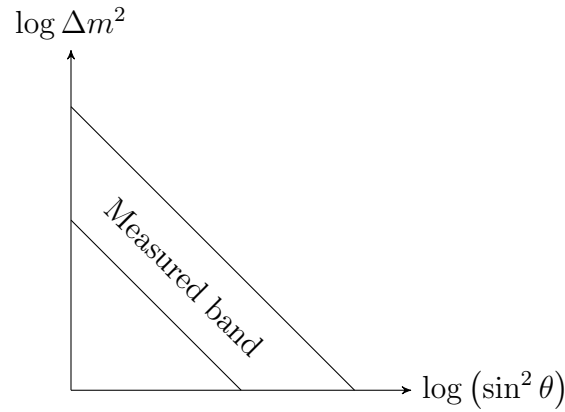
If

$$E/L \gg \Delta m^2 \quad (4.30)$$

Then the probability is approximately:

$$P(\nu_\alpha \rightarrow \nu_\beta) \approx \sin^2 2\theta \Delta m^2 \times \left(\frac{1.27 L (km)}{E (GeV)} \right) \quad (4.31)$$

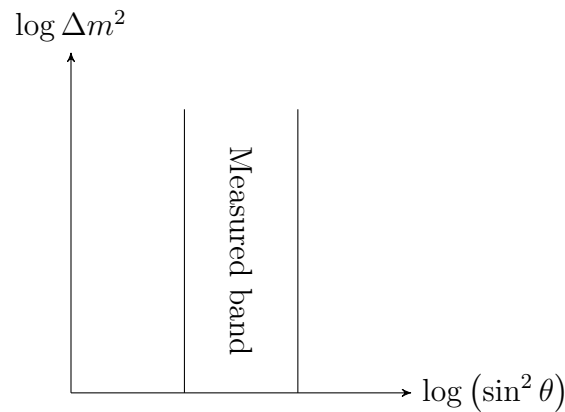
Any measurement of the probability of disappearance can't disentangle the contribution of Δm^2 and $\sin^2 2\theta$. Typical measurement plot is given by,



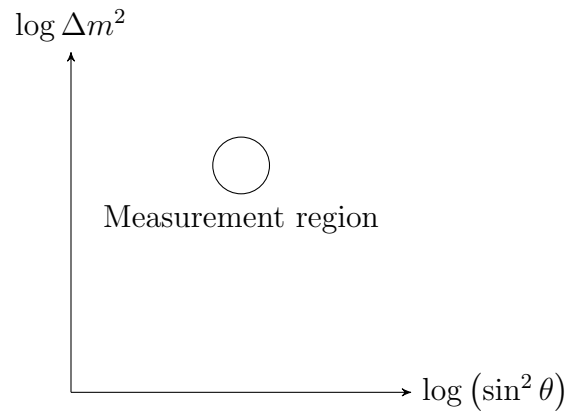
If $EL \ll \Delta m^2$ then the $\sin^2 \Delta m^2$ oscillates very quickly and averages out to $1/2$. Then the probability is roughly:

$$P(\nu_\alpha \rightarrow \nu_\beta) \approx \frac{1}{2} \sin^2 2\theta \quad (4.32)$$

In this limit the length is so large that you lose your coherence and all you get is the mixing angle. This gives the following plot:



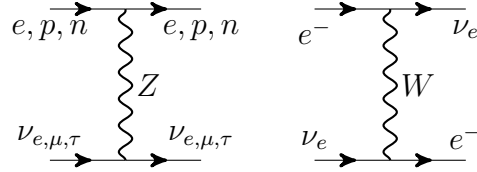
Finally in the optical region where $E/L \sim \Delta m^2$ we have:



This discussion referred to neutrinos propagating in vacuum. However, most neutrino experiments don't pass through the vacuum. The effect of matter can be significant as we will see.

4.2 Neutrino Oscillations in Matter

A neutrino passing through the Earth can undergo the following processes:



where the second diagram only involves electrons since muons and taus aren't for in normal matter. The neutral current interactions is harmless as it effects each neutrino flavor equally. On the other hand since only electrons are found on Earth. The electron neutrino will be effected more strongly then the other flavors as the neutrino beam passes through matter.

To calculate this effect we use the Fermi approximation:

$$H_{CC} \sim \frac{G_F}{\sqrt{2}} \langle \bar{e} \gamma_\mu (1 - \gamma_5) \nu_e \bar{\nu}_e \gamma_\mu (1 - \gamma_5) e \rangle_{matter} \quad (4.33)$$

$$= \frac{G_F}{\sqrt{2}} \underbrace{\langle e^- \gamma_\mu (1 - \gamma_5) e \rangle}_{\delta_{\mu 0} \frac{N_e}{2}} \bar{\nu}_e \gamma_\mu (1 - \gamma_5) \nu_e \quad (4.34)$$

where the second line can be achieved using properties of gamma matrices. The first term is just the number density of electrons in matter.

The extra term in the Hamiltonian is

$$H_{CC} = \bar{\nu}_L \gamma_0 V_m \nu_L \quad (4.35)$$

where

$$V_m = \begin{pmatrix} N_e - N_n/2 & 0 & 0 \\ 0 & N_n/2 & 0 \\ 0 & 0 & N_n \end{pmatrix} \quad (4.36)$$

and N_n is the number density of neutrons.

The Lagrangian in the presence of these extra terms is:

$$\bar{\nu}_L (i \not{\partial} - M_\nu - \gamma_0 V_m) \nu_L \quad (4.37)$$

The equations of motion dropping terms of order $V_m^2, V_m M_\nu^2$ lead to the dispersion relation:

$$E^2 - \mathbf{p}^2 = M_\nu^2 \pm 2E V_m \quad (4.38)$$

We have a new “effective mass” for the neutrino. Let us consider the case of constant matter density \Rightarrow constant V_m . Diagonalizing this matrix gives mass eigenstates $\tilde{m}_1, \tilde{m}_2, \dots, \tilde{m}_n^2$ where n is the number of families. For two families you find the mixing angle and mass differences of

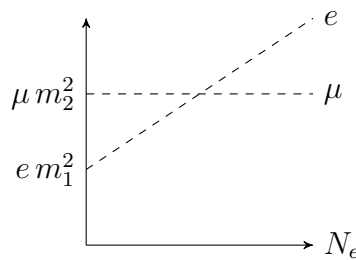
$$\sin^2 2\tilde{\theta} = \frac{(\Delta m^2 \sin 2\theta)^2}{(\Delta \tilde{m}^2)^2} \quad (4.39)$$

$$\Delta \tilde{m}^2 = \sqrt{(\Delta m^2 \cos 2\theta \mp 2EV_e)^2 + (\Delta m^2 \sin 2\theta)^2} \quad (4.40)$$

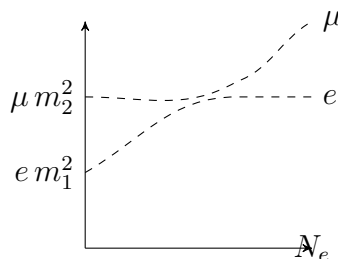
where the *plus – minus* corresponds to neutrinos or antineutrinos. Notice that for $\Delta m^2 \cos 2\theta = 2EV_e$, $\sin^2 2\tilde{\theta} = 1$. This is the famous MSW resonance. This condition can be met for neutrinos or antineutrinos but not both.

We now consider the mass eigenstates as a function of the density for the case that $\theta \rightarrow 0$. In this limit we have the mass matrix:

$$\begin{pmatrix} m_1^2 & 0 \\ 0 & m_2^2 \end{pmatrix} + \begin{pmatrix} 2EV_e & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} m_1^2 + 2EV_e & 0 \\ 0 & m_2^2 \end{pmatrix} \quad (4.41)$$



The two states cross. This is known as a level crossing. This level crossing is not allowed if you have mixing. Instead what will happen is:



The existence of the level crossing implies $\nu_e \rightarrow \nu_\mu$ conversion.

[Q 1: Last ten minutes of lecture 2 was a discussion of neutrinos propagating in matter which I omit.] [Q 2: First 30 minutes of lecture 3 was different experiments]

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Appendix A

Useful Identities

In deriving relations for Weyl fermions there are many useful relations That we attempt to make clear in this section. We have,

$$\psi \xrightarrow{\hat{C}} \psi^c = C\bar{\psi}^T \quad (\text{A.1})$$

with $C \equiv i\gamma_2\gamma_0$. C obeys the following properties:

$$C^\dagger = C^T = C^{-1} = -C \quad (\text{A.2})$$

which also implies

$$C^{-1} = -C \quad (\text{A.3})$$

$$C^* = C \quad (\text{A.4})$$

and

$$C\gamma_\mu C^{-1} = -\gamma_\mu^T \quad (\text{A.5})$$

We also have,

$$(\psi^c)^c = \psi, \quad \bar{\psi}^c = \psi^T C, \quad \bar{\psi}_1\psi_2^c = \bar{\psi}_2^c\psi_1, \quad \bar{\psi}_L = (\psi_L^c)^T C \quad (\text{A.6})$$

and

$$(\chi_L^T C \chi_L)^\dagger = \chi_L^{cT} C \chi_L^c \quad (\text{A.7})$$

Appendix B

Solutions

Solution 2. a) Applying the projection operator onto the conjugate of the left fermion gives:

$$P_R \psi_L^c = P_R i \gamma_2 P_L \Psi^* \quad (\text{B.1})$$

$$= i \gamma_2 \Psi^* \quad (\text{B.2})$$

$$\neq 0 \quad (\text{B.3})$$

b) We now prove Lorentz invariance:

$$\bar{\psi}_L \psi_L^c = \psi_L^\dagger \gamma_0 (-i \gamma_2 \psi^*) \quad (\text{B.4})$$

$$\rightarrow \psi_L^\dagger \exp\left(+\frac{i}{4} \omega_{\mu\nu} \sigma^{\mu\nu}\right) \gamma_0 (-i \gamma_2) \exp\left(-\frac{i}{4} \omega_{\mu\nu} \sigma^{\mu\nu}\right) \psi_L \quad (\text{B.5})$$

but $\sigma^{\mu\nu}$ is a product of gamma matrices and hence commutes with γ_0 and γ_2 . Thus we have,

$$\bar{\psi}_L \psi_L^c \rightarrow \psi_L^\dagger \gamma^0 (-i \gamma_2) \psi_L \quad (\text{B.6})$$

$$= \bar{\psi} \psi^c \quad (\text{B.7})$$

c) The Majorana Lagrangian is given by (see exercise below)

$$\mathcal{L} = \frac{1}{2} (\bar{\Psi} i \not{\partial} \Psi - m \bar{\Psi} \Psi^c) \quad (\text{B.8})$$

with

$$\Psi = \begin{pmatrix} \psi_\alpha \\ \bar{\psi}^{\dot{\alpha}} \end{pmatrix} = \Psi^c \quad (\text{B.9})$$

The mass term is given by

$$-\frac{m}{2} \bar{\Psi} \Psi^c = -\frac{m}{2} \begin{pmatrix} \bar{\psi}_{\dot{\alpha}} & \psi^\alpha \end{pmatrix} \gamma^0 \begin{pmatrix} \psi_\alpha \\ \bar{\psi}^{\dot{\alpha}} \end{pmatrix} \quad (\text{B.10})$$

$$= -\frac{m}{2} (\psi\psi + \bar{\psi}\bar{\psi}) \quad (\text{B.11})$$

The kinetic term is given explicitly as,

$$\frac{1}{2}\bar{\psi}i\partial\psi = i\bar{\psi}\bar{\sigma}^\mu\partial_\mu\psi \quad (\text{B.12})$$

and so

$$\mathcal{L} = i\bar{\psi}\bar{\sigma}^\mu\partial_\mu\psi - \frac{m}{2}(\psi\psi + \bar{\psi}\bar{\psi}) \quad (\text{B.13})$$

We now find the Euler-Lagrange equations of motion

$$\frac{\partial\mathcal{L}}{\partial\psi^\beta} = -m\psi_\beta \quad (\text{B.14})$$

$$\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi^\beta)} = -i\bar{\psi}_{\dot{\alpha}}(\bar{\sigma}^\mu)^{\dot{\alpha}\beta} \quad (\text{B.15})$$

and so

$$m\psi_\beta = -i\partial_\mu\bar{\psi}_{\dot{\alpha}}(\bar{\sigma}^\mu)^{\dot{\alpha}\beta} \quad (\text{B.16})$$

Taking the Fourier Transform using

$$\psi_\alpha = \int \tilde{\psi}e^{-ip\cdot x} \frac{d^4p}{(2\pi)^4} \quad , \quad \bar{\psi}_{\dot{\alpha}} = \int \bar{\tilde{\psi}}_{\dot{\alpha}}e^{ip\cdot x} \frac{d^4p}{(2\pi)^4} \quad (\text{B.17})$$

which gives,

$$m\tilde{\psi}_\beta(p) = \bar{\tilde{\psi}}_{\dot{\alpha}}(-p)(p\cdot\bar{\sigma})^{\dot{\alpha}\beta} \quad (\text{B.18})$$

taking the Hermitian conjugate:

$$m\bar{\tilde{\psi}}_{\dot{\beta}}(p) = \tilde{\psi}_\alpha(-p)(p\cdot\bar{\sigma})_{\dot{\beta}}^\alpha \quad (\text{B.19})$$

$$= \tilde{\psi}_\alpha(-p)(p\cdot\sigma)_{\dot{\beta}}^{\alpha\beta} \quad (\text{B.20})$$

$$m\bar{\tilde{\psi}}_{\dot{\beta}}(-p) = \tilde{\psi}^\alpha(p)(p\cdot\sigma)_{\alpha\dot{\beta}} \quad (\text{B.21})$$

Inserting this relation into Eq B.18:

$$m^2\tilde{\psi}_\beta(p) = \tilde{\psi}^\alpha(p)(p\cdot\sigma)_{\alpha\dot{\alpha}}(p\cdot\bar{\sigma})^{\dot{\alpha}\beta} \quad (\text{B.22})$$

$$m^2\delta_{\dot{\beta}}^\alpha = (p\cdot\sigma)_{\dot{\alpha}}^\alpha(p\cdot\bar{\sigma})^{\dot{\alpha}\beta} \quad (\text{B.23})$$

$$m^2 = p^2 \quad (\text{B.24})$$

where we in the last step we took the trace of both sides.

Solution 3. The Majorana fermion is given by

$$\Psi_M = \psi_L + \psi_L^c = P_L\Psi + P_R\Psi = \begin{pmatrix} \psi_\alpha \\ \frac{\psi}{\dot{\alpha}} \end{pmatrix} \quad (\text{B.25})$$

Calculating the Majorana kinetic term in four-spinor form:

$$\frac{1}{2}\bar{\Psi}_M i \not{\partial} \Psi_M = \frac{1}{2} (\bar{\psi} \bar{\sigma}^\mu i \partial_\mu \psi + \bar{\psi} \sigma^\mu i \partial_\mu \psi) \quad (\text{B.26})$$

$$= i \bar{\psi} \sigma^\mu i \partial_\mu \psi \quad (\text{B.27})$$

The mass term is given by,

$$- \frac{m}{2} \bar{\Psi}_M \Psi_M = - \frac{m}{2} (\psi^2 + \bar{\psi}^2) \quad (\text{B.28})$$

so the four spinor notation is equivalent to:

$$\mathcal{L} = \bar{\psi} \sigma^\mu i \partial_\mu \psi - \frac{m}{2} (\psi^2 + \bar{\psi}^2) \quad (\text{B.29})$$

which is the Majorana Lagrangian we found earlier.

Solution 4. Based on the identification above we have,

$$\Psi_{M_1} = \begin{pmatrix} \psi_\alpha \\ \bar{\psi}^{\dot{\alpha}} \end{pmatrix}, \quad \Psi_{M_2} = \begin{pmatrix} \chi_\alpha \\ \bar{\chi}^{\dot{\alpha}} \end{pmatrix} \quad (\text{B.30})$$

Writing it out it is easy to see that the Dirac mass term is the same as an offdiagonal mass term:

$$- m \Psi \Psi = - \begin{pmatrix} \bar{\psi}_{M_1,L} & \bar{\psi}_{M_2,L} \end{pmatrix} \begin{pmatrix} 0 & m \\ m & 0 \end{pmatrix} \begin{pmatrix} \psi_{M_1,R} \\ \psi_{M_2,R} \end{pmatrix} \quad (\text{B.31})$$

Solution 5.

$$L^T i \sigma_2 \boldsymbol{\sigma} L \xrightarrow{SU(2)} L^T e^{-\frac{i}{2} \boldsymbol{\sigma}^T \cdot \boldsymbol{\theta}} i \sigma_2 \boldsymbol{\sigma} e^{-\frac{i}{2} \boldsymbol{\sigma} \cdot \boldsymbol{\theta}} L \quad (\text{B.32})$$

but we have,

$$e^{-\frac{i}{2} \boldsymbol{\sigma} \cdot \boldsymbol{\theta}} \sigma_2 = \left(\cos \frac{\theta}{2} - i \frac{\boldsymbol{\theta} \cdot \boldsymbol{\sigma}^T}{\theta} \sin \frac{\theta}{2} \right) \sigma_2 \quad (\text{B.33})$$

$$= \sigma_2 \left(\cos \frac{\theta}{2} + i \frac{\boldsymbol{\theta} \cdot \boldsymbol{\sigma}^T}{\theta} \sin \frac{\theta}{2} \right) \quad (\text{B.34})$$

Thus,

$$L^T i \sigma_2 \bar{\boldsymbol{\sigma}} L \xrightarrow{SU(2)} L^T i \sigma_2 \underbrace{e^{\frac{i}{2} \boldsymbol{\sigma} \cdot \boldsymbol{\theta}} \boldsymbol{\sigma} e^{-\frac{i}{2} \boldsymbol{\sigma} \cdot \boldsymbol{\theta}}}_{\boldsymbol{\sigma}'} L \quad (\text{B.35})$$

$\boldsymbol{\sigma}$ is both a vector and a matrix. Thus it can be transformed in two ways. We can write,

$$e^{\frac{i}{2} \boldsymbol{\sigma} \cdot \boldsymbol{\theta}} \boldsymbol{\sigma} e^{-\frac{i}{2} \boldsymbol{\sigma} \cdot \boldsymbol{\theta}} = e^{-\frac{i}{2} \mathbf{S} \cdot \boldsymbol{\theta}} \boldsymbol{\sigma} \quad (\text{B.36})$$

where \mathbf{S} is the 3 dimensional analogue to the Pauli matrices (the triplet representation of $SU(2)$ generators). So

$$L^T i \sigma_2 \bar{\boldsymbol{\sigma}} L \xrightarrow{SU(2)} e^{-\frac{i}{2} \mathbf{S} \cdot \boldsymbol{\theta}} (L^T i \sigma_2 \boldsymbol{\sigma} \cdot \boldsymbol{\theta} L) \quad (\text{B.37})$$

Solution 6. We take the Yukawa interaction to be of the form,

$$\Delta\mathcal{L} = -\lambda\phi\bar{\psi}\psi + h.c. \quad (\text{B.38})$$

The amplitude is

$$i\mathcal{M} = - \int \frac{d^4\ell}{(2\pi)^4} \text{Tr} \left\{ \frac{i(\ell + m)}{\ell^2 - m_1^2 + i\epsilon} \frac{i(\not{p} + \ell + m_2)}{(\ell - p)^2 - m_2^2 + i\epsilon} \right\} (-i\lambda)^2 \quad (\text{B.39})$$

$$= -\lambda^2 \int \frac{d^4\ell}{(2\pi)^4} \frac{1}{D} \text{Tr} [\not{\ell}(\not{p} - \not{\ell}) + m_1 m_2] \quad (\text{B.40})$$

The denominator is given by

$$\frac{1}{D} \equiv \int dx \frac{1}{[(\ell - px)^2 - \Delta + i\epsilon]^2} \quad (\text{B.41})$$

where $\Delta \equiv -p^2 x(1-x) - (m_1^2 - m_2^2)x + m_1^2$. After shifting momenta the trace becomes:

$$\text{Tr}[\dots] \rightarrow \text{Tr}[(\not{\ell} - \not{p}x)(\not{p} - \not{\ell} + \not{p}x)] = 4\ell^2 - 4p^2 x(1+x) \quad (\text{B.42})$$

In order to further simplify the amplitude we need two integrals:

$$\int \frac{d^4\ell}{(2\pi)^4} \frac{\ell^2}{[\ell^2 - \Delta + i\epsilon]^2} = i \int_0^1 \frac{d^4\ell_E}{(2\pi)^4} \frac{-\ell_E^2}{(\ell_E^2 + \Delta)^2} = -\frac{i}{16\pi^2} \left(\Lambda^2 - \Delta \log \frac{\Lambda}{\Delta} + \text{finite} \right) \quad (\text{B.43})$$

$$\int \frac{d^4\ell}{(2\pi)^4} \frac{1}{[\ell^2 - \Delta + i\epsilon]^2} = \frac{i}{16\pi^2} \log \frac{\Lambda}{\Delta} \quad (\text{B.44})$$

So the total amplitude is

$$\mathcal{M} = \lambda^2 \int \frac{dx}{4\pi^2} \left(\Lambda^2 - \Delta \log \frac{\Lambda}{\Delta} - p^2(1-x)x \log \frac{\Lambda}{\Delta} \right) \quad (\text{B.45})$$

In the limit that $m_2 \gg m_1$, Δ takes the particularly simple form, xm_2^2 where we now identify m_2 as M_R . The amplitude is,

$$\mathcal{M} = \delta m_H^2 = \lambda^2 \left(\frac{\Lambda^2}{4\pi^2} + \frac{M_R^2}{16\pi^2} \log \frac{\Lambda^2}{M_R^2} \right) \quad (\text{B.46})$$

The first term is a quadratic divergence which we assume will be taken care of by some high energy theory and so the Higgs corrections are

$$\delta m_H^2 = \lambda^2 \frac{M_R^2}{16\pi^2} \log \frac{\Lambda^2}{M_R^2} \quad (\text{B.47})$$

as required.

Solution 7. To show this we consider each component individually.

We have,

$$\nu_L^T C m_L \nu_L = \begin{pmatrix} \nu_L^{1,T} & \dots & \nu_L^{n,T} \end{pmatrix} \begin{pmatrix} m_L^{1,1} & \dots & m_L^{1,n} \\ \vdots & \ddots & \vdots \\ m_L^{n,1} & \dots & m_L^{n,n} \end{pmatrix} C \begin{pmatrix} \nu_L^1 \\ \vdots \\ \nu_L^n \end{pmatrix} \quad (\text{B.48})$$

$$= \begin{pmatrix} \nu_L^T & \nu_L^c \end{pmatrix} \begin{pmatrix} m_L & 0 \\ 0 & 0 \end{pmatrix} C \begin{pmatrix} \nu_L \\ \nu_L^c \end{pmatrix} \quad (\text{B.49})$$

To write the terms together in the matrix format, we consider the hermitian conjugate for the right-handed neutrinos. This needed because we want to write the mass matrix in terms of only left handed fields and leave the right handed fields in the hermitian conjugate part. We have,

$$(\nu_R^T C m_R^* \nu_R)^\dagger = \nu_R^\dagger m_R^T C^\dagger \nu_R^* \quad (\text{B.50})$$

$$= \nu_R^\dagger \gamma_0 m_R^T C \gamma_0 \nu_R^* \quad (\text{B.51})$$

$$= \bar{\nu}_R m_R \nu_R^c \quad (\text{B.52})$$

$$= (\nu_R^c)^T m_R C \nu_R^c \quad (\text{B.53})$$

$$= n_L^T \begin{pmatrix} 0 & 0 \\ 0 & m_R \end{pmatrix} C n_L \quad (\text{B.54})$$

We use a similar trick for the Dirac term:

$$(\bar{\nu}_L m_D^* \nu_R)^\dagger = \bar{\nu}_R m_D^T \nu_L \quad (\text{B.55})$$

$$= (\nu_R^c)^T m_D^T C \nu_L \quad (\text{B.56})$$

$$= \frac{1}{2} \left((\nu_R^c)^T m_D^T C \nu_L + (\nu_R^c)^T m_D^T C \nu_L \right) \quad (\text{B.57})$$

$$= \frac{1}{2} \left((\nu_R^c)^T m_D^T C \nu_L + (\nu_L)^T m_D C \nu_R^c \right) \quad (\text{B.58})$$

$$= \frac{1}{2} n_L^T \begin{pmatrix} 0 & m_D \\ m_D^T & 0 \end{pmatrix} n_L \quad (\text{B.59})$$

where we have used the identity,

$$(\psi_L)^T m C \phi_L = (\psi_L)_i^\alpha m_{i,j} C_{\alpha\beta} (\phi_L)_j^\beta \quad (\text{B.60})$$

$$= -(\phi_L)_i^\beta m_{i,j}^T C_{\beta\alpha}^T (\psi_L)_j^\alpha \quad (\text{B.61})$$

$$= \phi_L m^T C \psi_L \quad (\text{B.62})$$

and used the relation, $\psi_\alpha \phi_\beta = -\phi_\beta \psi_\alpha$.

In summary we can write,

$$- \mathcal{L}_m = n_L^T \begin{pmatrix} m_L & m_D \\ m_D^T & m_R \end{pmatrix} C n_L \quad (\text{B.63})$$

Solution 8.