

Goldstone Bosons

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I. GENERAL DERIVATION

Here we derive the Goldstone boson theorem. The derivation is based on lecture notes given by Yuval Grossman. Consider a Lagrangian of the form

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - V(\phi) \quad (1.1)$$

where $V(\phi)$ is some arbitrary potential which is a function of $\phi_1, \phi_2, \dots, \phi_n$ (we assume that ϕ_i are all real however since complex fields can always be written as linear combinations of real fields this discussion is still very general). Now consider the expansion of $V(\phi_1, \phi_2, \dots, \phi_n)$ about some point $\phi = \mathbf{v}$.

$$V(\phi_1, \dots, \phi_n) = V(\mathbf{v}) + \frac{\partial V}{\partial \phi_i} \Big|_{\phi=\mathbf{v}} (\phi_i - v_i) + \frac{1}{2} \frac{\partial^2 V}{\partial \phi_i \partial \phi_j} \Big|_{\phi=\mathbf{v}} (\phi_i - v_i)(\phi_j - v_j) + \dots \quad (1.2)$$

It's clear from above that the potential includes a term $\frac{1}{2} \frac{\partial^2 V}{\partial \phi_i \partial \phi_j} \Big|_{\phi=\mathbf{v}} \phi_i \phi_j$. This term is quadratic in the fields and forms the mass matrix, M_{ij}^2 (note that if one were working in the mass basis this term would be $\frac{1}{2} M_i^2 \phi_i^2$).

Suppose further that $V(\phi)$ and hence \mathcal{L} are invariant under an arbitrary gauge transformation $\phi_k \rightarrow e^{i\epsilon_a (T^a)_{kj}} \phi_j$ or

$$\delta \phi_k = i\epsilon_a(x) (T^a)_{kj} \phi_j \quad (1.3)$$

Since V is only a function of the fields and not their derivatives we have

$$\delta V = \frac{\partial V}{\partial \phi_k} \delta \phi_k = \frac{\partial V}{\partial \phi_k} (i\epsilon_a(x) (T^a)_{kj} \phi_j) \quad (1.4)$$

Now V was initially gauge invariant. Elitzur's theorem tells us that since V initially respected the symmetry, gauge symmetry will still hold after shifting the fields[1]. Thus we still have $\delta V = 0$. Thus

$$0 = \frac{\partial V}{\partial \phi_k} (\epsilon_a(x) (T^a)_{kj} \phi_j) \quad (1.5)$$

$$= \epsilon_a(x) \left\{ \frac{\partial^2 V}{\partial \phi_\ell \partial \phi_k} (T^a)_{kj} \phi_j + \frac{\partial V}{\partial \phi_k} ((T^a)_{kj} \delta_{j,\ell}) \right\} \quad (1.6)$$

$$= \epsilon_a(x) \left\{ \frac{\partial^2 V}{\partial \phi_\ell \partial \phi_k} (T^a)_{kj} \phi_j + \frac{\partial V}{\partial \phi_k} (T^a)_{k\ell} \right\} \quad (1.7)$$

However this is true for all possible transformations. Since ϵ_a is something we choose and arbitrary we can write

$$\frac{\partial^2 V}{\partial \phi_\ell \partial \phi_k} (T^a)_{kj} \phi_j = \frac{\partial V}{\partial \phi_k} (T^a)_{k\ell} \quad (1.8)$$

Now suppose that $V(\phi_1, \dots, \phi_n)$ is a minima at some point $\phi = \mathbf{v}$. We define the vacuum expectation values (VEV) of a field to be at this minima since systems always go to the configurations of lowest energy. We evaluate this expression at this point. Since $V(\mathbf{v})$ is a minima we know that $\frac{\partial V}{\partial \phi_i} \Big|_{\phi=\mathbf{v}} = 0$. Thus

$$\frac{\partial V}{\partial \phi_k} (T^a)_{k\ell} \Big|_{\phi=\mathbf{v}} = 0 \quad (1.9)$$

$$\frac{\partial^2 V}{\partial \phi_\ell \partial \phi_k} \Big|_{\phi=\mathbf{v}} (T^a)_{kj} v_j = 0 \quad (1.10)$$

Now we use our earlier identification that $\left. \frac{\partial^2 V}{\partial \phi_\ell \partial \phi_k} \right|_{\phi=\mathbf{v}} \equiv M_{\ell k}$ is the mass matrix:

$$M_{\ell k}^2 (T^a)_{kj} v_j = 0 \quad (1.11)$$

$$M^2 (T^a) \mathbf{v} = 0 \quad (1.12)$$

Now here comes the key connection. If the VEV is invariant under the action of the all the generators of the transformation then:

$$\mathbf{v} = e^{i\epsilon_a T_a} \mathbf{v} = \mathbf{v} + i\epsilon_a T_a \mathbf{v} \quad (1.13)$$

then we have $T_a \mathbf{v} = 0$. In this case 1.11 is trivially satisfied. Note further that if $\mathbf{v} = 0$ (our VEV is zero) then M^2 there are not restrictions on M^2 (which will correspond to no Goldstone bosons as expected). However if the VEV is not invariant under this transformation, i.e.

$$T_b \mathbf{v} \neq 0 \quad (1.14)$$

for some T_b then the only way equation 1.11 is satisfied is if $T_b v$ is an eigenvector of M^2 with eigenvalue 0. So the mass matrix has an 0 eigenvalue for each generator that acting on the VEV is nonzero.

II. NOTES

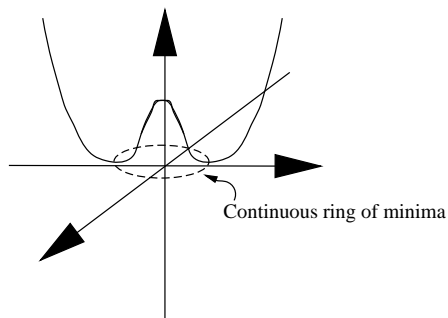
To derive this equation we made the following assumptions:

1. The potential was invariant under some transformation given by the generators T^a
2. The potential was at a minima at some point v

This equation is very general.

III. $SO(2)$ SYMMETRY BREAKING

For concreteness consider the case of $V(\phi_1, \phi_2) = \sum_i -\frac{m^2}{2} \phi_i^2 + \frac{\lambda}{4!} \phi_i^4$. $V(\phi_1, \phi_2)$ has a “ring” of minima at $\phi_1^2 + \phi_2^2 = v^2$:



Consider the expansion about the minima of $V(\phi)$. We can expand at any min we wish. Let's expand about $\mathbf{v} = (v, 0)^T$. Further assume we had a $SO(2)$ symmetry,

$$\mathcal{L} = \sum_i \frac{1}{2} \partial_\mu \phi_i \partial^\mu \phi_i - \frac{m^2}{2} \phi_i^2 + \frac{\lambda}{4!} \phi_i^4 \quad (3.1)$$

The generator is:

$$T = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \quad (3.2)$$

In this case we have

$$T\mathbf{v} = \begin{pmatrix} \cos \theta v \\ -\sin \theta v \end{pmatrix} \neq 0 \quad (3.3)$$

Thus we must have one massless Goldstone boson. We still have one massive degree of freedom. Initially we had two massive degree's of freedom. By breaking the symmetry we broke one generator but gained a massless degree of freedom.

[1] S. Elitzur. Impossibility of spontaneously breaking local symmetries. *Phys Rev D*, 1975.