EXTRA DIMENSIONS
LECTURE NOTES

Lecture notes are largely based on a lectures series given by Tim Tait on extra dimensions at TASI 2013 ...

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Contents

Chapter 1

Preface

This lecture notes are based on a TASI course given by Tim Tait. If you have any corrections please let me know at ajd268@cornell.edu. Useful References on the subject are earlier TASI notes by Csaki (2003, 2005), Sundum (2004), Kribs (2004), Cheng (2009), Gherghetta (2010), and Ponton (2011).

Chapter 2

Introduction

The motivations for extra dimensions are

1. The heirarchy problem
2. Dark matter
3. Grand unification
4. String theory

There are many issues associated with extra dimensions,

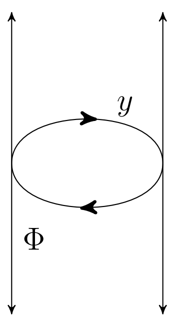
1. Flavor violation in $K^0 \leftarrow \bar{K}^o$ as well as $\mu \rightarrow e\gamma$
2. Proton decay ($\not{B}, \not{L}$)
3. Precision EW measurements
4. How to compactify the extra dimensions

First, we have to specify precisely what we mean by extra dimensions. To understand this consider a scalar field. It is given as a function of ordinary $4D$ (x^μ) as well as internal dimensions (y),

$$\Phi(x^\mu, y) \tag{2.1}$$

Lets consider a theory made up of a massless complex scalar field that lives in a flat circular compactified fifth dimension of radius r (as well our four dimensions).

Pictorally we have,



where $0 \leq y \leq 2\pi r$ and $\Phi(x^\mu, y) = \Phi(x^\mu, y + 2\pi RN)$ where N is an integer.

The action is given by,

$$S_5 = \int d^5x \mathcal{L}_5 \tag{2.2}$$

where $\mathcal{L}_5 = (\partial_M \Phi^*)(\partial^M \Phi)$, where $M = \underbrace{0, 1, 2, 3,}_{\mu} 5$.

Our strategy to understand the implications of this is to rewrite this as a 4 dimensional theory. The generating functional is given by,

$$Z[J] = \int \mathcal{D}\Phi \mathcal{D}\Phi^* \exp\left[i \int d^5x \mathcal{L}_5 + J\Phi\right] \tag{2.3}$$

Our source is also a function of the large and compactified dimension, $J = J(x^\mu, y)$. Our metric is given by,

$$\eta^{\mu\nu} = \begin{pmatrix} 1 & & & & \\ & -1 & & & \\ & & -1 & & \\ & & & -1 & \\ & & & & -1 \end{pmatrix} \tag{2.4}$$

The first thing we need to do is to write it in a separable way[1],

$$\Phi(x^\mu, y) = \sum_n f^n(y) \phi^n(x^\mu) \tag{2.5}$$

f^n are normally called the "wave functions" while ϕ^n are normally called the Kaluza-Klien modes. Of course we need to choose our basis f^n to be complete such that we can write out any possible field.

The action is given by,

$$S_5 = \int d^5x (\partial_M \Phi^*)(\partial^M \Phi) = \int d^4x \int_0^{2\pi R} dy \left\{(\partial_\mu \Phi^*)(\partial^\mu \Phi) - (\partial_y \Phi^*)(\partial^y \Phi)\right\} \tag{2.6}$$

$$= \sum_{m,n} \int d^4x \int dy \left\{f_m(y)^* f_n(y)(\partial_\mu \phi_m(x)^* \partial^\mu \phi_n(x)) - f_m^{*\prime}(y) f_n^\prime(y) \phi_m^*(x) \phi_n(x)\right\} \tag{2.7}$$

A good choice for the $\{f_n\}$ would satisfy,

$$\int dy f_m^* f_n = \delta_{mn} \tag{2.8}$$

(make the basis orthogonal). The other condition that we want to satisfy is,

$$\int_0^{2\pi R} dy f_m^{*\prime} f_n^\prime = M_n^2 \delta_{nm} \tag{2.9}$$

[1]The fact that we can do this is an assumption of the model but true for almost all physical examples.

The best we can do is set the basis to be diagonal. At this point we are assuming that such functions exist, but this always turns out to be the case.

We can now do the integral over y:

$$S_5 = \int d^4x \sum_n \left\{ (\partial_\mu \phi_n^*)(\partial^\mu \phi_n) - M_n^2 \phi_n^* \phi_n \right\} = S_4 \tag{2.10}$$

So what we've learned is that the extra dimension looks like an infinite set of states that are all decended from the functions f_n. In practice we want to find the functions f_n and very importantly find the masses. Then we can find how the extra dimension looks like to a four dimensional observer. In our path integral we have,

$$\mathcal{D}\Phi^* \mathcal{D}\Phi \to \prod_n \mathcal{D}\phi_n^* \mathcal{D}\phi_n \tag{2.11}$$

[Q 1: How did we get this?]

Combining both the conditions from above we can write,

$$\int dy f_n^{*\prime} f_m^\prime = M_n^2 \int dy f_n^* f_m \tag{2.12}$$

$$-\int dy f_n^* (\partial_y^2 + M_n^2) f_m = 0 \tag{2.13}$$

where since this equation should hold for all f_n^* (which we assume is independent as always in QFT) we have,

$$(\partial_y^2 + M_m^2) f_m = 0 \tag{2.14}$$

Note that above we dropped the surface term this is justified in 4D QFT since we assume locality. However, one may worry that this is nontrivial in our case. Nevertheless since we are assuming periodic boundary conditions we necessarily have,

$$(\partial_y f_n^*) f_m \Big|_0^{2\pi R} = 0 \tag{2.15}$$

The equation of motion is trivial to solve,

$$f_n = \mathcal{N}_n \exp\left[i M_n y \right] \tag{2.16}$$

Orthonomality of the f_n basis gives $\mathcal{N}_n \equiv \frac{1}{\sqrt{2\pi R}}$.

Periodicity tells us what the eigenvalues are since,

$$f_n(y) = f_n(y + 2\pi R) \Rightarrow e^{i M_n 2\pi R} = 1 \tag{2.17}$$

This will hold as long as,

$$M_n = \frac{n}{R} \quad n = 0, \pm 1, \pm 2, ... \tag{2.18}$$

This result isn't very surprising. Since the mass term is just the momenta going around in the compact dimension[2] (equivalent to a particle in a box with periodic boundary condition) it must be discrete. A nice way to understand this is that we started with a massless $5D$ field. For an on-shell field the 5-momentum squared is,

$$\mathcal{P}_5^2 = 0 \tag{2.19}$$

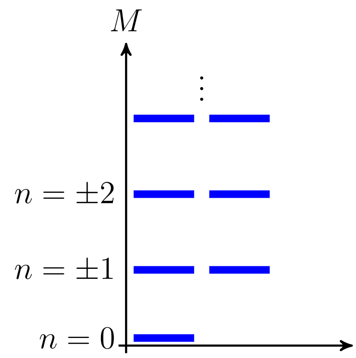
If we expand this out it looks like the usual four dimensional p^2 with an extra piece,

$$\mathcal{P}_5^2 = p_0^2 - p_1^2 - p_2^2 - p_3^2 - p_5^2 \tag{2.20}$$

leading to an on-shell condition of,

$$E^2 - \mathbf{p}^2 = p_5^2 \tag{2.21}$$

We have a massless particle moving in the speed of light, but if its moving in an extra dimension which an outside observer doesn't notice it will appear to have a mass, n/R. The spectrum is,



We call the $n = 0$ mode the "zero mode". All modes but the zero mode come in pairs.

2.1 Modifications

There are many possible modifications to this simple treatment. The possibly most obvious one is to add a bulk mass,

$$\mathcal{L}_5 = (\partial_M \Phi)^* (\partial^M \Phi) - M^2 |\Phi|^2 \tag{2.22}$$

When we substitute our decomposition of KK modes we will again get an integral of $f_m f_m$. Since that term is chosen to be diagonal, this term is still diagonal which gives the Lagrangian,

$$\mathcal{L}_4 = \sum_n (\partial_\mu \phi_n^*) \left(\partial^\mu \phi_n \right) - \frac{n^2}{R^2} + M^2 \Big) \phi_n^* \phi_n \tag{2.23}$$

[2]Recall that the mass arises from $\int dy f_m' f_n'$.

This will shift up all the modes of the theory. We call this a bulk mass because it doesn't depend on the extra dimension (in principle we could have had $M(y)$ since Lorentz invariance need not hold in this extra dimension).

Another modification is to have a bulk coupling, i.e., make this an interaction theory instead. For example we can have,

$$\mathcal{L}_5 = (\partial_M \Phi^*)(\partial^M \Phi) - M^2 \left|\Phi\right|^2 - \frac{\lambda}{4}(\Phi^*\Phi)^2 \tag{2.24}$$

Lets consider the dimensions. S_5 is dimensionless, $[d^5x] \sim \frac{1}{M^5} \Rightarrow [\mathcal{L}_5| \sim M^5$ and we have $[\partial_\mu] \sim M$. Looking at the kinetic term we have, $[\Phi] \sim M^{3/2}$. This leads to $[\lambda] \sim 1/M$. Therefore, the simple ϕ^4 theory which was renormalizable in four dimensions becomes nonrenormalizable in 5 dimension. This is a general theme in extra dimensional QFT, where they have to be thought of as effective field theories.

The four dimensional Lagrangian will have the diagonal kinetic term but now it also has,

$$\mathcal{L}_4 \supset \sum_{n,m,\ell,p} \left(-\frac{\lambda}{4}\right) \phi_n^* \phi_m \phi_\ell^* \phi_p \int dy f_n^* f_m f_\ell^* f_p \tag{2.25}$$

We still have the same normalization and expression for $f_m(y)$ so we can write the four dimensional coupling as,

$$\lambda_{m,n,p,\ell} = \frac{\lambda}{4}\frac{1}{(2\pi R)^2}\int_0^{2\pi R} dy \exp\left[\frac{i}{R}(m+p-n-\ell)\right] \tag{2.26}$$

This is only going to be nonzero if $m+p-n-\ell = 0$. Therefore we can write,

$$\lambda_{mn\ell p} = \frac{\lambda}{4(2\pi R)}\delta_{p+m,n+\ell} \tag{2.27}$$

This form also shouldn't be a complete surprise. We have translational invariance and hence momentum conservation. This delta function exactly enforces this constraint.

Chapter 3

Kaluza-Klien Theory

Here we again consider a 5 dimensional theory with a flat, circular, compactification. Our theory will just be $5D$ einstein gravity,

$$S_5 = \int d^5x \left\{ -\frac{1}{2} M_5^3 \sqrt{g} R\left[g\right] \right\} \tag{3.1}$$

[Q 2: Why is there an M_5?] We want to now think what the difference components of the metric look like for different components,

$$g^{\mu\nu} = \left(\begin{array}{c|c} \tilde{g}^{\mu\nu} & \tilde{A}^{\mu} \\ \hline \tilde{A}^{\mu} & \tilde{g}^{55} \end{array} \right) \tag{3.2}$$

where the $\tilde{g}^{\mu\nu}$ is a 4×4 block. $\tilde{g}^{\mu\nu}$ is commonly known as the graviton. The top right block composes of a vector field which looks like a gauge boson to a $4D$ observer. This field is known as a graviphoton. The last component ,$\tilde{g}^{55}$, it is often called the graviscalar. It is also the ruler which parameterizes how big the extra dimension is. For this reason it is often called the radion.

[Q 3: As an exercise we should carry out the next few manipulations in detail (for the zero modes). Its easier to carry this out using,

$$g^{\mu\nu} = \left(\begin{array}{c|c} \phi^{-1/3}(\tilde{g}^{\mu\nu} - \phi \tilde{A}^{\mu} \tilde{A}^{\nu}) & -\phi^{2/3} \tilde{A}^{\mu} \\ \hline -\phi^{2/3} \tilde{A}^{\mu} & -\phi^{2/3} \end{array} \right) \tag{3.3}$$

The answer is,

$$-\frac{1}{2}(2\pi R) M_5^3 \int d^4x \sqrt{g} \left[R_4\left[g\right] + \frac{1}{4} \phi F_{\mu\nu} F^{\mu\nu} - \frac{1}{6} \frac{\partial_\mu \phi)\partial^\mu \phi)}{\phi^2} \right] \tag{3.4}$$

].

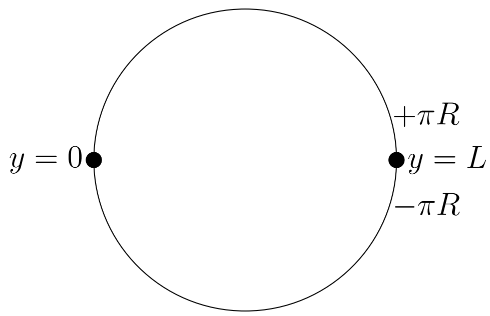
Chapter 4

Tools/Techniques

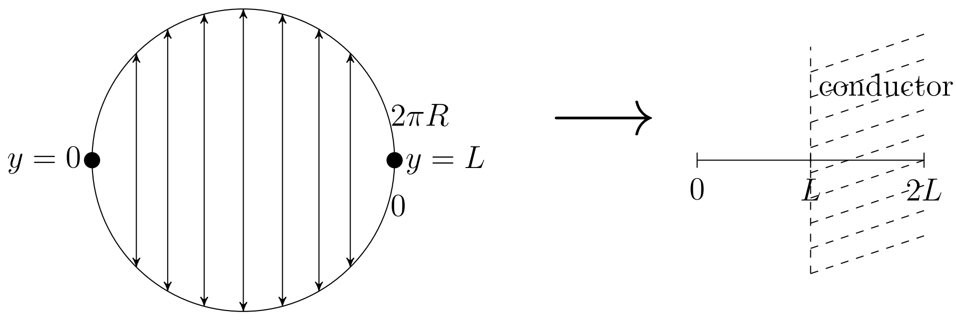
Our examples thus far, while enlightening, missed some key features which are exist in realistic extra dimensional models. One such feature is the noncircular topologies of extra dimensions. The problem is that fermions in circular spaces will always be vector like. Therefore, we need to find other ways to compactify the extra dimension. This brings us to singular or finite spaces.

The simplest working example is to simple end the extra dimension,

There is some "wall" that cuts off the dimension. This arises naturally in theories with orbifolds. An orbifold is just a 5 dimensional way to understand where this line segment can come from. The idea is to start with a circular dimension:



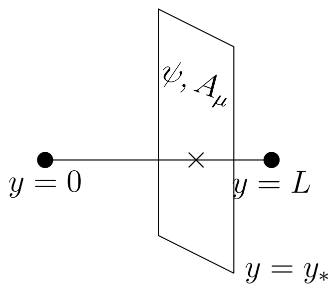
We now identify each part on the top segment with the bottom segment. If we unwrap this:



So the reason why the fields are shadowing from one side of the dimension to the other is because of the boundary condtions. The physical way to represent a charge is using a mirror charge in the conductor. While this appears very contrived, we can just think of it as a dimension with some set of boundary conditions.

This is a nice picture, but its best to think of general boundary conditions and not be prejudice for less aesthetic choices.

Another important concept is the idea of a "brane". A brane is a submanifold which is special. The idea is we have the extradimension. There may be some point along the extra dimension which is special,



The matter fields are confined to this brane which is at some special point in the extra dimension. [Q 4: It seems to bit peculiar to have fields confined to a brane. What is causing this?]. You might imagine that the entire SM lives on a brane. In fact the end points of the extra dimensions are necessarily made of branes, but occasionally we will also include branes in the middle of a extra dimension.

An example of an action for fermions in a brane and scalars in the bulk we have,

$$S = \int d^5x \left\{ \frac{1}{2}\partial_\mu \Phi \partial^\mu \Phi + \delta(y - y_*) \left[i\bar{\psi}\partial\!\!\!/ \psi + g\Phi\bar{\psi}\psi \right] \right\} \tag{4.1}$$

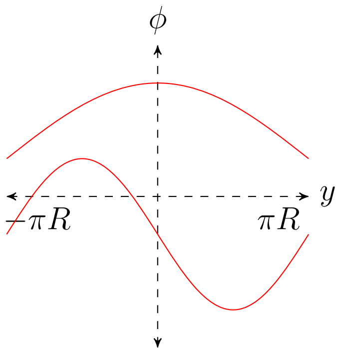
4.1 General KK Decompositions

We know that at least to a good approximation energy and momentum are conserved in the 4 large dimensions. We'd like to consider a background metric that preserves Lorentz invariance at least to the degree that energy and momentum are conserved. The canonical procedure is to just consider one that preserves lorentz invariance exactly to avoid having to worry about such subtleties. We write,

$$ds^2 = g_{MN} dx^M dx^\nu = e^{-2A(y)} \eta_{\mu\nu} dx^\mu dx^\nu - dy^2 \tag{4.2}$$

The flat extra dimension corresponds to $A(y) = 0$. $A(y) = ky$ refers to a warped extra dimension (also known as Randall Sundrum extra dimension).

Lets consider the boundary conditions for an orbifold. If we have some field ϕ that lives in an extra dimension:



There are two types of boundary conditions (BC) that are very natural to apply:

$$\phi(-y) = \phi(y) \quad \text{“even”} \tag{4.3}$$
$$\phi(-y) = -\phi(y) \quad \text{“odd”} \tag{4.4}$$

The first case corresponds to the conductor we discussed earlier. The BCs on the even field are known as Neumann BC (“+”) and for odd fields they are called Dirichlet (“-”),

$$\partial_y \phi \Big|_{0,L} = 0 \quad \text{(Neumann)} \tag{4.5}$$
$$\phi \Big|_{0,L} = 0 \quad \text{(Drichlet)} \tag{4.6}$$

These conditions let us make the connection with orbifolds but now we want to generalize these conditions. The simplest generization is to mix these two conditions, e.g. Neumann on one side and Drichlet on the other. We can have,

$$(+,+);(+,-);(-,+);(-,-) \tag{4.7}$$

Note these are still not the most general boundary conditions you can use. You can still have for instance,

$$(\partial_y - \alpha)\phi \Big|_{y_0} = 0 \tag{4.8}$$

Its important to note that not all BCs lead to sensible theories.

Chapter 5

Scalars, fermions, gauge fields (gravity)

5.1 Scalars

We now repeat the procedure above but explore different sets of BCs. Consider the action of now a real scalar (using a complex or real scalar doesn't make a big difference here)

$$\int d^5x\sqrt{g}\left\{\frac{1}{2}(\partial_M\Phi)(\partial^M\Phi) - \frac{1}{2}M^2\Phi^2\right\} \tag{5.1}$$

We want this to turn into something that looks like,

$$\int d^4x\sum_n\left\{\frac{1}{2}(\partial_\mu\phi_n)(\partial_\mu\phi_n) - \frac{1}{2}m_n^2\phi^2\right\} \tag{5.2}$$

We again do the mapping through separation of variables,

$$\Phi(x,y) = \frac{e^{A(y)}}{\sqrt{L}}\sum_n\phi_n(x)f_y(y) \tag{5.3}$$

We pull out an $e^{A(y)}/\sqrt{L}$, where L is the size of the extra dimension for convenience. Pulling out the length is convenient because this helps us regain the expected dimensions of our objects, $[\phi_n]\sim M, [f_n]\sim 1$. The equation of motion for the field is,

$$\partial_M\left[\sqrt{g}g^{MN}\partial_N\Phi\right] + \sqrt{g}M^2\Phi = 0 \tag{5.4}$$

As mentioned above Lorentz invariance restricts the metric to the form,

$$g_{MN} = \begin{pmatrix} e^{-2A}\eta_{\mu\nu} & 0 \\ 0 & -1 \end{pmatrix} \quad g^{MN} = \begin{pmatrix} e^{2A}\eta^{\mu\nu} & 0 \\ 0 & -1 \end{pmatrix} \tag{5.5}$$

which implies that[1],

$$\sqrt{g} = (e^{-2A\times 4})^{1/2} = e^{-4A(y)} \tag{5.6}$$

[1] Recall that the determinant of a diagonal matrix is the product of its diagonal elements

Inserting these relations into the above we have,

$$e^{-2A}\partial_\mu\partial^\mu\Phi + \partial_y(e^{-4A}\partial_y\Phi) + M^2\Phi = 0 \tag{5.7}$$

where $\partial_\mu\partial^\mu = \eta_{\mu\nu}\partial^\nu\partial_\mu$ (as opposed to contraction with $g_{\mu\nu}$).

Inserting in our KK mode expansion:

$$\sum_n f_n''\phi_n - 2A'f_n\phi_n' - (A'' + A'^2 - 4A)f_n\phi_n - e^{-2A}f_n\partial_\mu\partial^\mu\phi - M^2 f_n\phi_n = 0 \tag{5.8}$$

But our KK decoposition (satifying equations 2.8 and 2.9) implies that,

$$\partial^\mu\partial_\mu\phi_n = -m_n^2\phi_n \tag{5.9}$$

Furthermore, we make the assumption that every mode obeys the differential equation (if we can find such a solution then of course a linear combination also obeys in the equation),

$$f_n'' - 2A'f_n' + (-A'' - A'^2 + 4A + e^{-2A}m_n^2 - M^2)f_n = 0 \tag{5.10}$$

In flat space this is very easy since $A' = 0$ and we end up with the same results we found earlier.

For convenience we define the differential operator,

$$Df_n \equiv (\partial_y^2 - 2A'\partial_y + 4A - A'^2 - A'' + e^{-2A}m_n^2 - M^2)f_n \quad (= 0) \tag{5.11}$$

Then if we construct the object that we get by combining this antisymmetrically,

$$\int dy e^{-2A}\left\{f_m(Df_n) - f_n(Df_m)\right\} = 0 \tag{5.12}$$

Rearranging we have,

$$\int dy\partial_y\left[e^{-2A}(f_m f_n' - f_n f_m')\right] + (m_n^2 - m_m^2)\int dy f_n f_m = 0 \tag{5.13}$$

Since the first part is a total derivative we can write it as a surface term,

$$\left[e^{-2A}(f_m f_n' - f_n f_m')\right]_0^L + (m_n^2 - m_m^2)\int dy f_n f_m = 0 \tag{5.14}$$

What this does is that it suggests a class of boundary conditions. As long as we can set the first square bracket to zero, we have an implied orthogonality condition on the f_m's (just as we would like).

All the BCs we have considered so far have actually satisfies this criterion (that the piece in square brackets was equal to zero). In the case of the periodic BCs we required that the end points were equal to one another. The orbifold BCs just asked that each term cancels separately (it just requires that either the field or its derivative is zero at the boundary).

5.2 Fermions

The tool we need is to generalize the γ matrices, which we denote, Γ^M. As long as they satisfy,

$$\left\{\Gamma^M, \Gamma^N\right\} = 2g^{MN} \tag{5.15}$$

then can construct representations of the Lorentz group where the generators are the analoges of the sigma matrices,

$$M^{MN} = \frac{i}{2}\left[\Gamma_N, \Gamma_M\right] \tag{5.16}$$

In this case we need a set of 5 objects. If we think in terms of Weyl fermions, we already have a complete basis for 2×2 matrices,

$$\sigma^\mu = \{1, \boldsymbol{\sigma}\} \tag{5.17}$$

This tells you that fermions in $5D$ (even if they are massless) are four-component.

If we choose our first four gamma matrices to be the usual matrices then there is only one such matrix that has all the right properties,

$$\Gamma^A = \left\{\gamma^0, \gamma^1, \gamma^2, \gamma^3, -i\gamma_5\right\} \tag{5.18}$$

In other words γ_5 really is γ_5!

The vielbeins of the field are

$$g_{MN} = e^A_M e^B_N \eta_{AB} \tag{5.19}$$

In our background,

$$e^\mu_\alpha = e^{A(y)}\delta^\mu_\alpha \quad e^y_5 = +1 \tag{5.20}$$

with all other components being zero. The covariant deriavative is given by,

$$D_M = \partial_M + \frac{1}{8}\omega_{M,AB}\left[\Gamma^A, \Gamma^B\right] \tag{5.21}$$

$$= \begin{cases} \partial_\mu - \frac{1}{2}e^{-A(y)}A'(y)\gamma_\mu\gamma_5 \\ \partial_5 \end{cases} \tag{5.22}$$

[Q 5: This extra term is called the torsion?] We see that we have an extra term to worry about as far as the derivatives in terms of the large dimensions. We have all the ingredients to write down a generally covariant action,

$$\int d^5x\sqrt{g}\left\{\frac{1}{2}\bar{\Psi}e^\mu_A\Gamma^A D_\mu\Psi - \frac{i}{2}D_\mu\Psi^\dagger\Gamma^0 e^\mu_A\Gamma^A\Psi - M\bar{\Psi}\Psi\right\} \tag{5.23}$$

[Q 6: Derive this equation.]

Again what we'd like to do is integrate this by parts and write the equations of motion. We can write this in terms of the $4D$ chiral projectors,

$$\Psi_L = P_L \Psi \quad \Psi_R = P_R \Psi \tag{5.24}$$

where,

$$P_{L,R} = \frac{1}{2}(\mathbb{1}_{4\times 4} \mp \gamma_5) \tag{5.25}$$

If we were really living in a $5D$ space that was not compactified then this would be an insane thing to do because these are related to each other intimately under 5D Lorentz transformations, but since we know we will compactify the extra dimension this separates the things that are used as the Lieb Watson in 4D. [Q 7: What is that???]

Doing this we derive the equations of motion,

$$ie^{A(y)}\gamma^\mu \partial_\mu \Psi_L + \left((\partial_5 - \frac{1}{2}A') - M\right)\Psi_R = 0 \tag{5.26}$$

$$ie^{A(y)}\gamma^\mu \partial_\mu \Psi_R + \left((-\partial_5 - \frac{1}{2}A') - M\right)\Psi_L = 0 \tag{5.27}$$

We make the Kaluza-Klien decomposition,

$$\Psi_{L,R}(x,y) = \frac{e^{3A(y)/2}}{\sqrt{L}} \sum_n \psi^n_{L,R}(x) f^n(y)_{L,R} \tag{5.28}$$

These equations can be solved in general but instead lets look at some interesting cases,

For the circle we have the boundary conditions,

$$\psi_{L,R}\Big|_{y=0} = \psi_{L,R}\Big|_{y=L} \tag{5.29}$$

In this case nothing distinguishes left from right. This condition is going to lead to,

$$f^n_L = f^n_R \tag{5.30}$$

In the interval we have,

$$\Psi_L\Big|_{y=0} = 0 \tag{5.31}$$

(or equivalently that f^n_L vanishes on the boundary). One of our KK equations then tells us that,

$$\left[(\partial_5 - \frac{1}{2}A') - M\right]\Psi_R\Big|_{y=0} = 0 \tag{5.32}$$

This equation of motion implies that,

$$\partial_y \Psi_R\Big|_{y=0} = (\frac{1}{2}A' + M)\Psi_R\Big|_{y=0} \tag{5.33}$$

This is example of one of those hybrid boundary conditions. Its implied by just setting $\Psi_L = 0$ at the end point.

Our coupled equations now take the form,

$$(\partial_y + M - \frac{1}{2}A')f_L^n = m_n e^{A(y)} f_R^n \tag{5.34}$$

$$(\partial_y - M - \frac{1}{2}A')f_R^n = -m_n e^{A(y)} f_L^n \tag{5.35}$$

We lie to impose,

$$\frac{1}{L}\int_0^L dy f_{L/R}^m f^n f_{L/R} = \delta^{nm} \tag{5.36}$$

People normally write down what they are doing with $(\pm, \pm)$.

$$+ : \Psi_R\Big|_{y=0} = 0 \tag{5.37}$$

$$- : \Psi_L\Big|_{y=0} = 0 \tag{5.38}$$

So we still have the $+$ and $-$ notations for fermions but here they mean different things then they did for scalars. The boundary conditions of the other modes are actually determined from these two due to the equation of motions above.

Despite the fact that these equations have included in them a bulk mass for the fermions, we still have zero mass particles. Consider the case of $(+, +)$. Lets assume that such a mode exists, in that case we have,

$$(\partial_y + (M - \frac{1}{2}A'))f_L^0 = 0 \tag{5.39}$$

This equation is easy to solve,

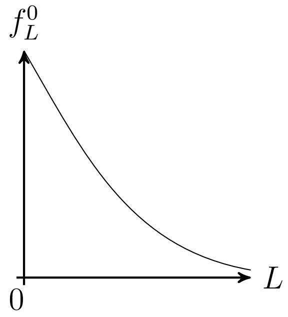
$$f_L^0(y) = \mathcal{N}_L^0 \exp\left[\frac{1}{2}A(y) - My\right] \tag{5.40}$$

We could also look for a zero model corresponding to f_R. However, in this case the boundary conditions will lead to a $f_R^0 = 0$ ([Q 8: show]). We see we can get Weyl fermions from four component five dimensional ones.

If we repeat this exercise for $(-, -)$ we have,

$$f_R^0(y) = \mathcal{N}_R^0 \exp\left[\frac{1}{2}A(y) + My\right] \tag{5.41}$$

Pictorically we have (for $M > 0$),



[Q 9: **Exercise:** In flat space, find the KK modes for a $(+,-)$.]

5.3 Gauge fields

We can write out an action for the gauge fields,

$$S_A = \int d^5x\sqrt{g}\left\{-\frac{1}{4}g^{MN}g^{PL}F_{MP}F_{NL}\right\} \tag{5.42}$$

To do any explicit computations we need to fix the gauge and add a gauge fixing action, S_{GF}. Expanding out with $g^{MN} = \mathrm{diag}(e^{2A(y)}\eta_{\mu\nu}, -1)$ as given for the scalar case we have,

$$S_A = \int d^4x \int dy - \frac{1}{4}e^{-4A}\left\{F_{\mu\nu}F^{\mu\nu} + 2g^{55}g^{\mu\nu}F_{5\mu}F_{5\nu} + \cancel{g^{55}g^{55}F_{55}^2}\right\} \tag{5.43}$$

The cross term is given by,

$$F_{5\mu}F_5^{\mu} = (\partial_y A_\mu)^2 + (\partial_\mu A_5)^2 - 2\partial_y A_\mu \partial^\mu A_5 \tag{5.44}$$

We will want to gauge fix the action by fixing the value of the total divergence, $\partial_\mu A^\mu$, as is usually done. For this reason it is convenient to integrate the cross term by parts,

$$\int d^5x e^{-2A(y)}\partial_y A_\mu \partial^\mu A_5 = \int d^5x \partial_y(e^{-2A}A_5)\partial_\mu A^\mu \tag{5.45}$$

which gives the action,

$$\begin{aligned} S_A = \int d^5x - \frac{1}{4}\Big\{F_{\mu\nu}F^{\mu\nu} + e^{-2A(y)}(\partial_\mu A_5)(\partial^\mu A_5) - 2\partial_5\left[e^{-2A(y)}A_5\right](\partial_\mu A^\mu) \\ + e^{-2A(y)}(\partial_5 A^\mu)(\partial_5 A_\mu)\Big\} \end{aligned} \tag{5.46}$$

The gauge-fixing action is given by,

$$S_{GF} = \int d^5x \left(\frac{-1}{2\xi}\right)\left[\partial_\mu A^\mu - \xi\partial_5(e^{-2A}A_5)\right]^2 \tag{5.47}$$

This term has been engineered to cancel the cross term between A_5 and $\partial^\mu A_\mu$ in the action.

After this is carried out the equations of motion are now decoupled,

$$\left[\eta^{\mu\nu}\partial^2 - \left(1-\frac{1}{\xi}\right)\partial^\mu\partial^\nu\right]A_\nu - \partial_5\left(e^{-2A}\partial_5 A^\mu\right) = 0 \tag{5.48}$$

$$\partial^2 A_5 - \xi\partial_5^2\left[e^{-2A}A_5\right] = 0 \tag{5.49}$$

The mass of A_5 is now ξ dependent, showing that its a Goldstone boson.

Again we define two convenient boundary conditions. Due to the gauge fixing condition, $\partial_\mu A^\mu = \xi\partial_5\left(e^{-2A}\partial_\mu A_5\right)$ the boundary conditions are restricted. Without breaking $4D$ Lorentz symmetry (i.e. not giving A_μ different boundary conditions for each μ) we have two choices for each boundary:

$$(+) \quad \partial_5 A_\mu\Big|_{0/L} \Rightarrow A_5\Big|_{0/L} = 0 \tag{5.50}$$

$$(-) \quad A_\mu\Big|_{0/L} = 0 \Rightarrow \partial_5\left[e^{-2A}A_5\right]\Big|_{0/L} = 0 \tag{5.51}$$

We can again decompose our fields in KK modes,

$$A_\mu = \frac{1}{\sqrt{L}}\sum_n A_\mu^n(x)f^n(y) \tag{5.52}$$

$$A_5 = \frac{1}{\sqrt{L}}\sum_n A_5^n(x)f_5^n(y) \tag{5.53}$$

Substituting these into the equation of motion we get the equation for the modes themselves,

$$\partial_y(e^{-2A}\partial_y f_A^n) + m_n^2 f_A^n = 0 \tag{5.54}$$

$$\partial_y^2(e^{-2A}f_5^n) + m_n^2 f_5^n = 0 \tag{5.55}$$

where, analogously to before, we have used $\partial_\mu\partial^\mu A_M^n = m_n^2 A_M^n$. If we have f_A^n for $m_n \neq 0$ then we can construct a solution for f_5^n by,

$$f_5^n = \frac{1}{m_n}\partial_y f_A^n \tag{5.56}$$

To see this is a solution just plug it into the above,

$$\partial_y^2(e^{-2A}\partial_y f_A^n) + m_n^2\partial_y f_A = 0 \tag{5.57}$$

which is just the first equation after an integration. Therefore, this is always a solution.

For the $(+,+)$ boundary conditions, the $m=0$ solution always exists and its,

$$f_A^0(y) = 1 \quad (\Rightarrow f_5^0(y) = 0) \tag{5.58}$$

(a constant). The fact that this is a constant is nontrivial statement. What this implies is that this field couples the same way to every other field. That's important because if we think back to gauge theory, gauge invariance requires that the (massless!) vector field has the same gauge coupling to every field out there. In this case the fields are different modes but the same should hold nevertheless if gauge theory is to be preserved.

Its also important to note that we can use this idea to break a symmetry. If choose any other boundary condition $f_5^n \neq 0$ and the symmetry is broken. This idea is used in grand unification to break a simple group into the SM as an alternative to "ordinary" spontaneous symmetry breaking.

We now make a few quick remarks regarding the spin-2 graviton. Earlier we saw that we had $h^{\mu\nu}, A^{\mu}, \phi$. Here the graviton eats the other components to gain a mass.

Chapter 6

Models

6.1 Large extra dimensions

The basic idea is that you have a few extra dimensions (has to be more than 1) and the SM lives on a brane