EFFECTIVE FIELD THEORIES LECTURE NOTES

Lecture notes are largely based on a lectures series given by Iain Stewart at MIT which is freely available on "http://ocw.mit.edu".
The notes are also partly based on the book, *Heavy Quark Physics*, by Wise and Manohar.

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Contents

Chapter 1

Preface

The course assumes a background roughly equivalent to two QFT courses as well as a Standard Model course.

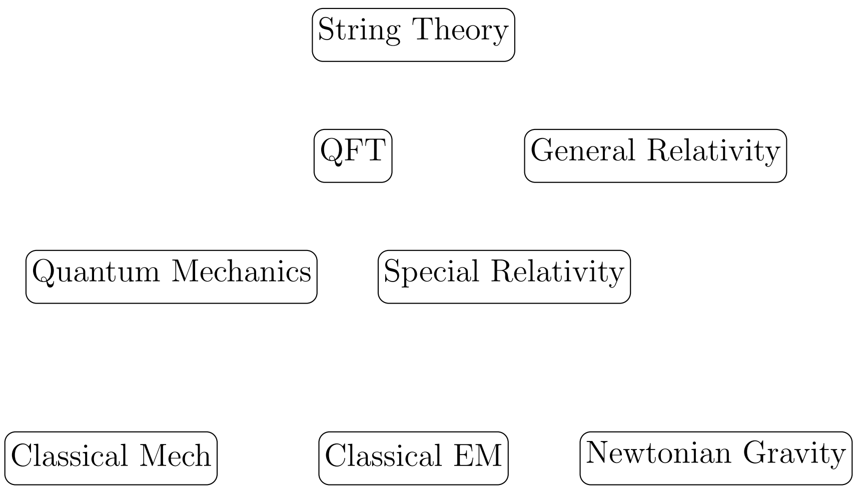
I have added in exercises in the text and solutions can be found in the appendix. These notes will likely be expanded until I lose interest in neutrinos. If you have any corrections please let me know at ajd268@cornell.edu.

Chapter 2

The Big Picture

2.1 Interesting Physics at All Scales

How we typically learn physics from the bottom up of the pyramid:



and then you keep synthesizing and putting them together. We'll do the opposite in this course. We will take QFT and we will get more and more specific. The reason to do that is that as we move up in the pyramid it becomes harder to compute. As an example to compute the energy spectrum of Hydrogen in QM is straight forward however this is a very difficult task in QFT.

By focusing in on particular cases, you are able to compute ina simpler fashion. We try to find the simplest framework that captures the essential physics, but in a a manner that can be corrected in principle to arbitrary precision. In other words, what we want to do is take QFT and expand it in a convenient regime.

2.2 Describing a Physical System

1. Determine the relevant degrees of freedom (this can be harder then it sounds).

2. Identify the symmetries. Sometimes you will have a theory without a symmetry to begin with, however the expansion will haev such a symmetry.

3. Find the expansion parameters and get a 1st order description

For QFT this equates to,

1. Determine the fields

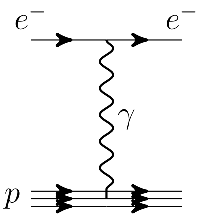
2. Find the interactions

3. What kind of power counting can you do

For regular SM you still do steps 1 and 2, however step 3 is kind of new. In an Effective Field Theory (EFT), power counting is just as important as something like gauge symmetry. The power counting to be consistent is actually necessary for an EFT to be consistent.

The key principle is that to describe physics at some scale, m^2, we don't need to know the detailed dynamics at scales, $\Lambda^2 \gg m^2$. We don't need to know the field content or anything else about the dynamics.

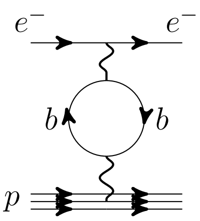
An example of this is that we don't need to know about bottom quarks to describe Hydrogen.



The binding energy of Hydrogen is given by

$$E_0 = \frac{1}{2} m_e \alpha^2 \left[1 + \mathcal{O}\overbrace{\left(\frac{m_e^2}{m_b^2} \right)}^{\sim 10^{-8}} \right] \tag{2.1}$$

You could think of the bottom quark coming in through perhaps:



However, there is one subtly in that we have to decide what we mean by the coupling, α. The bottom mass, m_b, does effect the coupling in (for example) $\overline{MS}$ by contributing to its running. We have, $\alpha(M_W) = \frac{1}{127}$ and $\alpha(0) \approx 1/137$. Thus more precisely if α is a parameter of the SM which is fixed by doing perhaps Z boson physics, then α for Hydorgen depends on the bottom quark. But we can take a different attitude - a low energy approach. Simply extract $\alpha(0)$ from some low energy atomic experiments and if this is the way we define α then the value can be used in other experiments and we don't need to know anything about m_b. In this case we don't need to know anything about the high energy theory unless we are doing experiments up there. Adding in new higher order particles you could effect what you mean by your parameters.

We have,

$$\mathcal{L}(p, e^-, \gamma, b; \alpha, m_b) = \mathcal{L}(p, e^-, \gamma; \alpha') + \mathcal{O}\left(\frac{1}{m_b^2}\right) \tag{2.2}$$

We missed many other details when we first discussed Hydrogen.

1. We were insenstive to quarks in proton. The reason we didn't discuss is because typical momentum transfer in hydrogen is $|\mathbf{p}| \sim m_e\alpha$ This is much small then the proton size which is about 200MeV.

2. We could further ignore the proton mass, $m_e\alpha \ll m_p \sim 1$GeV and the proton acts like a static charge.

3. We also used the fact that $m_e\alpha \ll m_e$ and used a non-relativistic Lagrangian.

These conclusions hold despite of UV divergences. For example, prior to regulation we an infinite bottom quark correction.

In general EFT's are used in two distinct ways, "top-down" and "bottom-up".

2.2.1 Top-down

"Top-down": In this situation we know what the high energy theory is and it is well understood but we find it useful to have a simpler theory to do some low energy physics.

$$\begin{tikzcd} \boxed{\text{High energy: Theory 1}} \\ \boxed{\text{Low energy: Theory 2}} \arrow[u] \end{tikzcd}$$

So what we can do is just start calculating things in theory 1 and "integrate out" (remove) heavier particles and in doing that we can do what's called matching onto a low energy theory. We can use this ability to do calculations in the high energy theory to find what the operators are of the low energy theory just by direct calculation and also if there are new low energy constants that show up we can calculate the value of those constants.

Schematically we have,

$$\mathcal{L}_{high} \to \sum_n \mathcal{L}_{low}^{(n)} \tag{2.3}$$

where the sum in n is an expanion in decreasing relevance. $\mathcal{L}_{high}$ and $\mathcal{L}_{low}$ need to agree in the infrared (IR). The place where they differ is in the ultraviolet (UV). The desired precision tells us when to stop (what n).

Some examples of top-down EFT's are when we integrate out the heavy particles, W, Z, top or using Heavy Quark Effective Field Theory (HQFT) for charm and bottom quarks.

2.2.2 Bottom-up

The underlying theory is unknown or matching is too difficult to carry out (e.g. non-perturbative). You may need to know something about the high energy theory such as that it is Lorentz invariant, a gauge theory, etc. but not the full theory.

You construct,

$$\sum_n \mathcal{L}_{low}^{(n)} \tag{2.4}$$

by writing down the most general possible operators that you can think of consistent with what ever degrees of freedom we have and the symmetries that we are imposing.

The couplings are unknowns however they can be fit to experiment so the effective theory may still be powerful since you can make more predictions then the number of parameters you can have (like for Hydrogen). The desired precision again tells us at what order to stop expanding.

The classic example of this example is chiral perturbation theory where the low energy theory is composed of π, K. The SM itself is another example.

2.2.3

Comment: So far when discussing this sum over n we were really thinking about expansions in powers, $m^2/\Lambda^2 \ll 1$. However there are also logs and renormalization of $\mathcal{L}_{low}^{(n)}$. This allows us (t) sum seriese of $\log \frac{m_1}{m_2}$ for $m_2 \ll m_1$.

Chapter 3

Standard Model as EFT

For an EFT we have,

$$\sum_n \mathcal{L}^{(n)}_{low} = \mathcal{L}^{(0)} + \mathcal{L}^{(1)} + ... \tag{3.1}$$

$\mathcal{L}^{(0)}$ is the SM that we normally talk about. We now discuss the SM degrees of freedom. The SM is a gauge theory:

$$\underbrace{SU(3)_C}_{G^\mu_A} \times \overbrace{SU(2)_L}^{W^a_\mu} \times \underbrace{U(1)_Y}_{B^\mu} \tag{3.2}$$

where $A = 1, ..., 8$ and $a = 1, 2, 3$. The boson masses are

$$m_\gamma = 0, m_{gluons} = 0, m_W = 80\text{GeV}, m_Z = 91\text{GeV}, m_H = 126\text{GeV}$$

The fermion masses have a broad spectrum:

Type	Name	Masses
Quarks	u_L, u_R	$1.5 - 3.3$MeV
	d_L, d_R	$3.5 - 6$MeV
	s_L, s_R	100 ± 30MeV
	c_L, c_R	1.27 ± 0.01GeV
	b_L, b_R	4.20 ± 0.12GeV
	t_L, t_R	171.2 ± 1GeV
Leptons	e_L, e_R	0.511MeV
	μ_L, μ_R	105MeV
	τ_L, τ_R	1777MeV
	ν_{e_L}	?
	ν_{μ_L}	?
	ν_{τ_L}	?
	Sterile neutrinos?	

where for neutrinos we only know that $\Delta m_0^2 \approx 8 \times 10^{-5}\text{eV}^2$ from colliders and $\Delta m_{atm}^2 \approx 2 \times 10^{-3}\text{eV}^2$ from atmospheric neutrinos.

From a “top-down” effective field theory point of view you can integrate out the heaviest particles step by step. This is not the sense in which we are thinking about it now. We’re looking at it from a “bottom-up” approach and trying to figure what is beyond the scale of the top quark. The lowest order Lagrangian is:

$$\mathcal{L}^{(0)} = \mathcal{L}_{gauge} + \mathcal{L}_{fermion} + \mathcal{L}_{Higgs} + \mathcal{L}_{\nu_R} \tag{3.3}$$

where

$$\mathcal{L}_{gauge} = -\frac{1}{4}B^{\mu\nu}B_{\mu\nu} - \frac{1}{4}W^a_{\mu\nu}W^{\mu\nu}_a - \frac{1}{4}G^a_{\mu\nu}G^{\mu\nu}_A \tag{3.4}$$

$$\mathcal{L}_{fermion} = \sum_L \bar{\psi}_L i \not{D} \psi_L + \sum_R \bar{\psi}_R i \not{D} \psi_R \tag{3.5}$$

where $iD_\mu = i\partial_\mu + g_1 B_\mu Y + g_2 W^a_\mu T^a + g A^A_\mu T^A$.

The next step is the power counting which is related to what we left out. We define

$$\epsilon = \frac{M_{SM}}{\Lambda} \tag{3.6}$$

where $M_{SM} \sim m_t, m_W, m_Z, m_H ...$ and $\Lambda \sim M_{GUT}, M_{\cancel{SUSY}}, M_{Planck}$, the mass scale of high energy. The new physics can be described by higher dimension operators ($dim > 4$) built from SM fields.

3.1 What does “Renormalizable” mean?

A theory is renormalizable if at any order in perturbation theory the UV divergences from loop integrals can be absorbed can be into a finite number of parameters.

The effective field theory definition of renormalizabile is more general. For a theory to be renormalizable, the divergences must be absorbalbe into a finite number of parameters order by order in its expansion parameter. This allows for an infinite number of parameters but only a finite number at some fixed order of ϵ.

If $\mathcal{L}^{(0)}$ is renormalizable in the traditional sense. This means that you don’t see the highest energy scales in your lowest order Lagrangian; we do not know directly Λ from just looking at $\mathcal{L}^{(0)}$.

3.2 Marginal,Irrelevant, and Relevant Operators

We consider the case where mass dimension determines the power counting (PC). Suppose we have the of ϕ^4 theory as an effective theory:

$$S[\phi] = \int d^dx \left(\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4 - \frac{\tau}{6!}\phi^6 + ... \right) \tag{3.7}$$

We can look at the dimensions of the various objects:

$$[\phi] = \frac{d-2}{2}$$
$$\left[d^4x\right] = d$$
$$\left[m^2\right] = 2$$
$$[\lambda] = 4 - d$$
$$[\tau] = 6 - 2d$$

Suppose we want to study a correlation function, $\langle\phi(x_1)...\phi(x_n)\rangle$ at long distance (small momenta). To enforce this condition we write, $x^\mu = sx'^\mu$ and demand $s \to \infty$ and x' to be fixed[1]

We define

$$\phi'(x') = s^{(d-2)/2}\phi(x) \tag{3.8}$$

in order to normalize the kinetic term:

$$S'[\phi'] = \int d^dx' \left[\frac{1}{2}\partial'^\mu\phi'\partial'_\mu\phi' - \frac{1}{2}m^2s^2\phi'^2 - \frac{\lambda}{4!}s^{4-d}\phi'^4 - \frac{\tau}{6!}s^{6-2d}\phi'^6 + ...\right] \tag{3.9}$$

We now have,

$$\langle\phi(sx'_1)...\phi(sx'_n)\rangle = s^{n(2-d)/2}\langle\phi'(x'_1)...\phi'(x'_n)\rangle \tag{3.10}$$

Lets now take $d = 4$ and consider the case where s gets large. In this case the m^2 term becomes more and more important. It is called *relevant*. The τ term is less important. It is called *irrelevant*. The λ term is equally important and is called *marginal*. We can see that this is directly related to the dimension of the operator (as the higher the dimension of the operator, the greater the number of ϕ's in it). In summary:

- Relevant: $dim < d$ $\quad$ $([m^2] > 0)$
- Marginal: $dim = d$ $\quad$ $([\lambda] = 0)$
- Irrelevant: $dim > d$ $\quad$ $([\tau] < 0)$

Lets take s to be finite but large. The dimension of parameters (or operators) tells us the importance of the various terms. Use mass scale of parameters. If we associate a new scale of new physics, Λ, then we have,

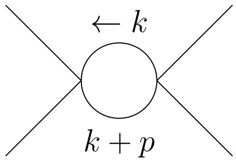
$$(m^2) \sim (\Lambda^2) \quad , \quad \lambda \sim (\Lambda^0) \quad , \quad \tau \sim (\Lambda)^{-2} \tag{3.11}$$

We can do a power counting in this Λ. At large distances, $s \to \infty$ means small momemnta $p \ll \Lambda$.

Note: relevant operators can upset PC since they are set by the kinetic term. To get rid of this issue take $m = 0$ or fine tune it to be small, $m^2s^2 \sim p^2$.

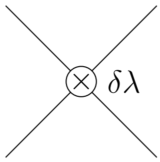
We can now start thinking about divergences. We can have two four-point interactions:

[1] This is fundamentally a bit flawed since we are scaling time differently then space and its not exactly clear what s is doing. We will elaborate on this point later.

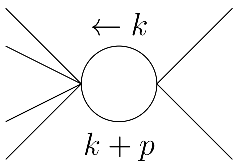


$$\sim \lambda^2 \int \frac{d^dk}{(k^2-m^2)((k+p)^2-m^2)}$$

The superficial degree of divergence is Λ^{d-4}. So if $d = 4$ then we have a logarithmic divergence (which corresponds to a $\frac{1}{\epsilon}$ in dim-reg). This renormalizes the $\lambda\phi^4$ operator and you add the counterterm:

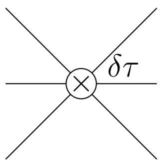


We now continue and think about other diagram. Lets consider the τ renormalization:

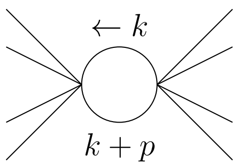


$$\sim \lambda\tau \int \frac{d^dk}{(k^2-m^2)((k+p)^2-m^2)}$$

The diagram has the same divergence as before but now the diagram its renormalizing is one with 6 points on the outside:

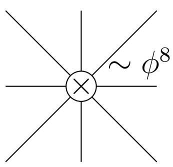


We could also include two τ diagrams:



$$\sim \tau^2 \int \frac{d^dk}{(k^2-m^2)((k+p)^2-m^2)}$$

This again has the same divergence but renormalizes a new interaction that we haven't included yet:



In order to renormalize the diagram we need to include a ϕ^8 operator. Since ϕ^8 is not included in $S[\phi]$ (without the dots) the theory is non-renormalizable in the traditional sense.

But if $\tau \sim \frac{1}{\Lambda^2}$ is small then $p^2\tau^2 \ll 1$ then the theory can be renormalized order by order in $\frac{1}{\Lambda}$. We need to add ϕ^8 at order $\sim \frac{1}{\Lambda^4}$.

To include all corrections up to $\sim \frac{1}{\Lambda^r}$ or $\frac{1}{s^r}$ we need to include all operators with dimensions up to the level, $[\mathcal{O}] \le d + r$. Here we see how power counting is connected to dimensions.

Note that we made an assumption we said that $x^\mu = s x'^\mu$, i.e., we scaled each component in the same way. You don't need to do this, but works well for the SM. For the Standard Model, $\mathcal{L}^{(0)}$ then we know as part of the way of constructing it we write down all the operators with $[\mathcal{O}] \le 4$ and it is renormalizable in the traditional sense.

The first order SM correction is $\mathcal{L}^{(1)}$ and we can write it of the form,

$$\mathcal{L}^{(1)} = \frac{C}{\Lambda}\mathcal{O}_5 \tag{3.12}$$

where $\mathcal{O}_5$ is a dimension 5 operator and the dimension of $C = 0$ which we take to be approximately 1. We made Λ explicit and we are allowed to do this due to our earlier arguements using s. Since nothing in $\mathcal{L}^{(0)}$ really tells us about Λ, we're free to take it as big as we want, $\Lambda \gg m_t, m_W$.

3.3 Corrections to $\mathcal{L}^{(0)}$

We have,

$$\mathcal{L} = \underbrace{\mathcal{L}^{(0)}}_{\sim\Lambda^0} + \underbrace{\mathcal{L}^{(1)}}_{\sim\Lambda^{-1}} + \underbrace{\mathcal{L}^{(2)}}_{\sim\Lambda^{-2}} + ... \tag{3.13}$$

for $p^2 \sim m_t^2$ we have $p^2 \sim m_t^2$ which we assume is much smaller then Λ^2. We now try to construct $\mathcal{L}^{(1)}$ and $\mathcal{L}^{(2)}$.

We assume there won't be any Lorentz or gauge symmetry breaking terms. i.e., each $\mathcal{L}^{(i)}$ is $SU(3) \times SU(2) \times U(1)$ invariant and Lorentz invariant.

We then construct $\mathcal{L}^{(i)}$ from the same degrees of freedom of $\mathcal{L}^{(0)}$ and assume Higgs VEV remains the same as the value in $\mathcal{L}^{(0)}$ (for $E \gg v$ we see the full gauge symmetry). Thus we assume that there no new particles produced at p, only at Λ.

Gauge theory is very restricted for dimension 5 and there is only one operator(here we ignore flavor indices but in reality there should be a sum over all flavors):

$$\mathcal{L}^{(1)} = \frac{C}{\Lambda}\epsilon_{ij}\bar{L}_L^{c,i} H^j \epsilon_{k\ell} L_L^k H^\ell \tag{3.14}$$

where $\bar{L}_L{}^{c,i} = (L_L^i)^T i\gamma_2\gamma_0$ and

$$H = \begin{pmatrix} h^+ \\ h^0 \end{pmatrix}, \quad L_L = \begin{pmatrix} \nu_L \\ e_L \end{pmatrix} \tag{3.15}$$

This operator is interesting from a phenomelogical point of view since if we replace $H \to \begin{pmatrix} 0 \\ v \end{pmatrix}$ we get a Majorana mass term for the left-handed neutrinos:

$$\frac{1}{2} m_\nu \nu_L^a \nu_L^b \epsilon_{ab} \tag{3.16}$$

Since we know that $m_\nu \leq 0.5\text{eV}$ we have an important constraint on the scale of new physics

$$\Lambda \geq 6 \times 10^{14}\text{GeV} \tag{3.17}$$

(for $C_5 \sim 1$). This term violates lepton number. Similarly, you can also write down dimension 6 operators that violate varyon number. If you impose conservation of lepton number and baryon number there are 80 dimension-6 operators:

$$\mathcal{L}^{(2)} = \sum_{i=1}^{80} \frac{C_i}{\Lambda^2} \mathcal{O}_i^{(6)} \tag{3.18}$$

80 seems like a large number but for any observable only a manageable number contribute to the process. For any new theory at Λ a particular pattern of C_i's are expected. We list a few of these operators to get a feel for them:

$$\mathcal{O}_G = f_{ABC} G^{A,\mu\nu} G_\nu^{B\lambda} G_{\lambda\mu}^C \tag{3.19}$$

$$\mathcal{O}_{LQ} = (\bar{L}_L \gamma^\mu \sigma^a L_L)(Q_L \gamma_\mu \sigma_a Q_L) \tag{3.20}$$

$$\mathcal{O}_W = \bar{L}_L \sigma^{\mu\nu} \sigma^a e_R H W_{\mu\nu}^a \tag{3.21}$$

$$\mathcal{O}_B = \bar{L}_L \sigma^{\mu\nu} e_R H B_{\mu\nu} \tag{3.22}$$

where $Q_L \equiv \begin{pmatrix} u_L \\ d_L \end{pmatrix}$. $\mathcal{O}_W$ and $\mathcal{O}_B$ contribute to the muon anomalous magnetic moment. We have,

$$(g-2)_\mu = \binom{\text{SM contributions}}{\text{from } \mathcal{L}^{(0)}} + C \frac{4 m_\mu v}{\Lambda^2} \tag{3.23}$$

If you take into account how well we have measured $g-2$ we have,

$$\Lambda \gtrsim 100\text{TeV} \tag{3.24}$$

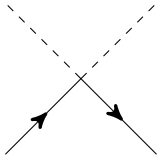
For the remaining 76 see Buchuller and Wyler (1985). They weren't the first to get the operators however they were the first to get 80 operators. Their analysis used the tree level equations of motion derived from $\mathcal{L}^{(0)}$ to reduce the number of operators. For example the equation of motion:

$$i \not{D} e_R^i = g_e^{ij} H^\dagger L_L^j \tag{3.25}$$

(the analogue of $\not{p} u(p) = m u(p)$ for a Dirac spinor). This is obviously fine at lowest order since external lines are put on-shell in Feynman rules. For example an operator that's not listed in the 80 is

$$(H^\dagger H)(\bar{e}_R i \not{D} e_R) \tag{3.26}$$

corresponds to the Feynman rule:



$$= \bar{u}\not{p}u = m\bar{u}u$$

which connects this operator to the corresponding left handed operator. This makes sence if the particles are on-shell however what about loops or propagators?

3.4 Representation Independent Theorem

Let $\phi = \chi F(\chi)$ where $F(0) = 1$ and ϕ and χ are scalars. Then calculations of observables done with ϕ, $\mathcal{L}(\phi)_{\text{quantized}_\phi}$ will give the same results as those with $\mathcal{L}(\phi) = \mathcal{L}(\chi F(\chi))_{\text{quantized}_\chi}$.

For example consider

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}m^2\phi^2 - \lambda\phi^4 + \eta g_1 \phi^6 + \eta g_2 \phi^3 \partial^2 \phi \tag{3.27}$$

with $\eta \ll 1$. The statement that we want to explore is that we can use the equations of motion (EOM),

$$\partial^2\phi = -m^2\phi - 4\lambda\phi^3 \tag{3.28}$$

to drop the last term. This is equivalent to making the field redefinition $\phi \to \phi + \eta g_2 \phi^3$[2]. We have,

$$\frac{1}{2}(\partial_\mu \phi)^2 \to \frac{1}{2}(\partial_\mu \phi)^2 - \eta g_2 \phi^3 \partial^2 \phi + \mathcal{O}(\eta^2) \tag{3.29}$$

$$m^2\phi^2 \to m^2\phi^2 + 2\eta g_2 m^2 \phi^4 + \mathcal{O}(\eta^2) \tag{3.30}$$

$$\lambda\phi^4 \to \lambda\phi^4 + 4\eta g_2 \lambda \phi^6 + \mathcal{O}(\eta^2) \tag{3.31}$$

where have integrated by parts to get Eq. 3.29. Inserting these into the Lagrangian we have,

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}m^2\phi^2 - \lambda'\phi^4 + \eta g_1' \phi^6 + \mathcal{O}(\eta^2) \tag{3.32}$$

where the primed parameters are functions of the unprimed ones. Notice that we have successfully eliminated one of the terms in the Lagrangian and all the other terms have the same form.

Genearlized Theorem: Field redefinitions that preserve symmetries and have same 1 particle states before and after the field redefinition allow classical equations of motion to simplify a local $\mathcal{L}_{EFT}$ without changing observables.

[2]Initially this may seem a bit weird as we are defining one field to be a product of fields. However, since both objects transform the same way this doesn't violate Lorentz symmetry.

We prove this in general below. For the details see (C.Arzt, hep-ph/9304230) or (H. Georgi, "On-shell Effective Field Theory").

$$\mathcal{L}_{EFT} = \sum_{n=0}^{\infty} \eta^n \mathcal{L}^{(n)} \tag{3.33}$$

Consider removing the operator $\eta T[\phi] D^2\phi$ in $\mathcal{L}^{(1)}$, where ϕ is complex scalar and T is any local functional that meets the symmetries of the problem. The generating function is given by

$$Z[J_k] = \int \prod_i \mathcal{D}\phi_i \exp\left\{ i \int d^dx \left[\mathcal{L}^{(0)} + \eta \overbrace{\left(\mathcal{L}^{(1)} - TD^2\phi\right)}^{\text{what we want}} + \eta T D^2\phi \right] + \sum_k J_k\phi_k + \mathcal{O}(\eta^2) \right\} \tag{3.34}$$

where Green's functions are obtained by functional derivatives with respect to the J_k's. We now make a change of variables in the path integral. When we make a change of variable in the Lagrangian we also need to change the integration measure and possibly the source term.

We let,

$$\phi^\dagger = \phi'^\dagger + \eta T \tag{3.35}$$

then

$$\begin{aligned} Z = \int \prod_i \mathcal{D}\phi'_i \left[\frac{\delta\phi^\dagger}{\delta\phi'^\dagger}\right] \exp\Bigg\{ i \int d^dx \Bigg[\mathcal{L}^{(0)} + \eta T \overbrace{\left[\frac{\delta\mathcal{L}^{(0)}}{\delta\phi^\dagger} - \partial_\mu \frac{\delta\mathcal{L}^{(0)}}{\delta\partial_\mu\phi^\dagger}\right]}^{\text{EOM for } \phi^\dagger} + \eta\left(\mathcal{L}^{(1)} - TD^2\phi\right) \\ + \eta T D^2\phi + \sum_k J_k\phi_k + J_{\phi^\dagger}\eta T + \mathcal{O}(\eta^2) \Bigg] \Bigg\} \end{aligned} \tag{3.36}$$

where we have integrated by parts in $\delta\mathcal{L}$. The claim is that without changing the S matrix we can remove the Jacobian and source term factors so only need change of variable in $\delta\mathcal{L}$ to do what we want.

$\delta\mathcal{L}$ needs $\phi'^\dagger + \eta T$ to transform as $\phi^\dagger$ does (to respect symmetries).

$$\mathcal{L}^{(0)} = (D^\mu\phi)^\dagger (D_\mu\phi)^\dagger - m^2\phi^\dagger\phi + ... \tag{3.37}$$

$$= (D_\mu\phi')^\dagger (D_\mu\phi') - m^2\phi'^\dagger\phi' + \eta T\left[-D^2\phi' - m^2\phi'\right] + (...)' \tag{3.38}$$

where we have integrated the covariant derivative term by parts[3]. The $-\eta TD^2\phi'$ term conveniently cancels the desired term in the action. Now you may be worried about all

[3] As long as you are dealing with a gauge invariant quantity you can integrate by parts for example for the invariant, $\Phi^\dagger\Phi$:

$$(D_\mu\Phi)^\dagger\Phi + \Phi^\dagger(D_\mu\Phi) = \partial_\mu\Phi^\dagger\Phi - ig(\Phi^\dagger T^a\Phi)A^a_\mu + \Phi^\dagger\partial_\mu\Phi + igA^a_\mu\Phi^\dagger T^a\Phi = \partial_\mu(\Phi^\dagger\Phi)$$

the other terms that get induced. However, the point of the effective theory is that you already wrote down all the possible terms allowed by the symmetries. So you can just shift your old terms. The terms in $(...)'$ are already operators present in $(...)$ so we can just shift couplings.

We now move onto the Jacobian and do something similar to the Fadeev-Poppov procedure. Recall that

$$\det(\partial^\mu D_\mu) = \int \mathcal{D}\bar{c}\mathcal{D}c \exp\left[i \int d^4x \bar{c}\left[-\partial^\mu D_\mu\right] c\right] \tag{3.39}$$

where c and $\bar{c}$ are the ghost fields. Here we write $\frac{\delta\phi^\dagger}{\delta\phi'^\dagger} = 1 + \eta\frac{\delta T}{\delta\phi'^\dagger}$ as

$$-\mathcal{L}_{ghost} = \bar{c}c + \eta\bar{c}\frac{\delta T}{\delta\phi'^\dagger}c \tag{3.40}$$

term in the Lagrangian. It turns out that these ghosts will have a mass $\sim \frac{1}{\sqrt{\eta}} = \Lambda$. The ghosts then decouple from the theory just like other particles at this mass scale that we left out.

To see how this works lets pick a particular example:

$$T = -\partial^2\phi^\dagger + \lambda\phi^\dagger(\phi^\dagger\phi) \tag{3.41}$$

With this we have,

$$-\mathcal{L}_{ghost} = \bar{c}(1 - \eta\partial^2 + 2\eta\lambda\phi^\dagger\phi)c \tag{3.42}$$

Here our kinetic term has the wrong dimensions. This is a consquence of not being careful enough when applying the Faddeev-Poppov results. We need to rescale $c \to c/\sqrt{\eta}$:

$$-\mathcal{L}_{ghost} \to \bar{c}(\Lambda^2 - \partial^2 + 2\lambda\phi^\dagger\phi)c \tag{3.43}$$

We now see that we have mass term that is proportional to Λ^2. The ghost masses arose from the $\bar{c}c$ term in the Lagrangian. This was a consequence of having a linear trasnformation,

$$\phi^\dagger = \phi'^\dagger + \eta T\left[\phi\right] \tag{3.44}$$

As when quantizing a gauge theory, ghosts always appear in loops. Due to their heavy mass they can be integrated out just like other heavy particles.

We now go back to the source shift. The source, $j_{\phi^\dagger}$ is used to take derivatives and get correlation functions. Consider,

$$G^{(n)} = \langle 0|\, \mathcal{T}\phi(x_1)...\phi(x_n)...\,|0\rangle \tag{3.45}$$

(we write only real ϕ here for notational simplicity).

When we make a field redefinition we have,

$$G^{(n)} = \langle 0|\, \mathcal{T}(\phi(x_1) + \eta T_{x_1})...(\phi(x_n) + \eta T_{x_n})...\,|0\rangle \tag{3.46}$$

where we have transformed each field $\phi(x_i)$ by ηT_{x_i}.

The extra terms will certainly modify the Green's function. However, to see whether they have any impact on the physics we should consider observables, not just Green's functions. Recall that the LSZ formula says

$$\int d^4x_i e^{+iP_ix_i} \langle 0| \mathcal{T}\phi(x_1)...\phi(x_n)... |0\rangle \xrightarrow{p_i^0 \to \sqrt{\mathbf{p}_i^2+m_i}} \left(\prod_i \frac{\sqrt{Z}i}{(p_i^2 - m_i^2 + i\epsilon)}\right) \langle p_1p_2...| S |p_jp_{j+1}...\rangle \tag{3.47}$$

The claim is that the change to the source will drop out for $\langle S\rangle$. Suppose that

$$\phi \to \phi + \eta\phi = (1+\eta)\phi \tag{3.48}$$

then

$$T\partial^2\phi = \eta\phi\partial^2\phi \tag{3.49}$$

The matrix element for four fields for example,

$$(1+\eta)^4 \langle 0| \mathcal{T}\phi(x_1)...\phi(x_4) |0\rangle \tag{3.50}$$

However, when we calculate the $\sqrt{Z}$ factor we get an extra $1+\eta$ for each field because of the scaling in 3.48, which cancels out the $(1+\eta)^4$, leaving the S matrix unchanged.

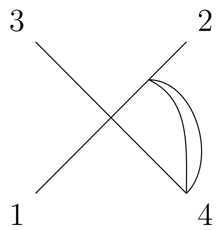
Lets now consider a less trivial example:

$$\phi = \phi' + \eta\, \overbrace{g_2\phi'^3}^{T} \tag{3.51}$$

We get several extra terms in the S matrix we write down one of them:

$$\eta g_2 \langle 0| \mathcal{T}\phi(x_1)\phi(x_2)\phi(x_3)\phi^3(x_4) |0\rangle + ... \tag{3.52}$$

This won't effect the leading term since having a ϕ^3 means you don't have a 1 particle state. In Feynman diagram language (in position space):



the $\phi^3(x_4)$ is less singular, has no single particle pole, and hence gives no contribution to $\langle S\rangle$.

As a final example consider

$$\phi = \phi' + \partial^2\phi' = \phi' + \eta\underbrace{(\partial^2 + m^2)\phi'}_{\text{no pole}} - \overbrace{\eta m^2\phi'}^{\text{Same as first example}} \tag{3.53}$$

We have shown the three different ways you can shift the field and none of them gave a Jacobian. This can be shown more generally. Therefore, we conclude that as long as you have the linear term in your field redefinition you don't need to worry about changes to the Jacobian or the source term.

Chapter 4

Loops, Renormalization, and Matching

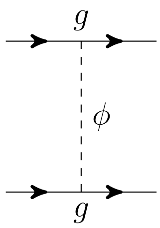
Lets take a theory with a heavy scalar ϕ of mass M and light fermion of mass m. The UV theory is

$$\mathcal{L}_{UV} = \bar{\psi}(i\not{\partial} - m)\psi + \frac{1}{2}\left[(\partial_\mu\phi)^2 - M^2\phi^2\right] + g\phi\bar{\psi}\psi \tag{4.1}$$

and we are in a situation where $m^2 \ll M^2$. To descirbe ψ's at low energies, $p^2 \ll M^2$ we can remove ("integrate out"), ϕ. The low energy effective theory is going to look like,

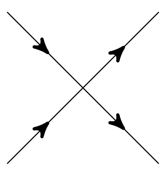
$$\mathcal{L} = \bar{\psi}(i\not{\partial} - m)\psi + \frac{a}{M^2}(\bar{\psi}\psi)^2 + \ldots \tag{4.2}$$

In the UV theory at tree level we have,



$$= \frac{(ig)^2 i}{q^2 - M^2} = \frac{ig^2}{M^2} + \mathcal{O}\left(\frac{q^2}{M^4}\right)$$

By comparing with the effective theory tree level result we can fix the coefficient of a,



$$= \frac{ia}{M^2}$$

which sets $a = g^2$.

This gets a bit more involved when discussing loops. Loop integrals often diverge and you have to regulate. We need to cutoff (UV) divergences to obtain finite results and introduce cutoff parameters. Some examples include,

$$p^2_{Euclidean} \leq \Lambda^2, d = 4 - 2\epsilon, \text{lattice spacing}, \ldots \tag{4.3}$$

Renormalization: pick a scheme that gives definite meaning to the parameters to each coefficient/operator in the EFT. This may also introduce parameters. Such parameters include, μ in $\overline{MS}$, $p^2 = -\mu_R^2$ for off-shell momentum subtraction, Λ for Wilsonian renormalization (i.e. renormalization with a cutoff).

For Wilsonian renormalization the bare, renormalized, and counterterm coefficients are related by,

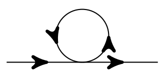
$$a^{bare}(\Lambda_{UV}) = a^{ren}(\Lambda) + \delta a(\Lambda_{UV}, \Lambda) \tag{4.4}$$

in dim-reg:

$$a^{bare}(\epsilon) = a^{ren}(\mu) + \delta a(\epsilon, \mu) \tag{4.5}$$

One of the things that makes this tricky is when you start thinking about your power counting.

Consider the four-fermion operator we considered above but with contracting two lines,



This corrects the ψ mass, m by

$$\Delta m \sim \frac{ia}{M^2} \int \frac{d^4k}{(2\pi)^4} \frac{(\overbrace{\not{k}}^{0} + m)}{k^2 - m^2} = \frac{am}{M} \int \frac{d^4k_E}{(2\pi)^4} \frac{1}{k_E^2 + m^2} \tag{4.6}$$

where we did a Wick rotation to get a Euclidean integral. If the integral is dominated by $k_E \sim m$. Then this integral scales like $\sim m^2$ which gives,

$$\Delta m \sim a \frac{m^3}{M^2} \tag{4.7}$$

which is what we would like since we can do power counting in $\frac{m}{M}$.

We need to be more careful and introduce a regulator. We do this in two ways. One way is to introduce a cutoff, $\Lambda_{UV} \sim M$. This gives,

$$\begin{aligned} a\frac{m}{M^2} \int \frac{d^4k_E}{(2\pi)^4} \frac{1}{k_E^2 + m^2} &= a\frac{m}{M^2} \frac{2}{(4\pi)^2} \int_0^{\Lambda_{UV}} \frac{dk_E k_E^3}{k_E^2 + m^2} && (4.8) \\ &= a\frac{m}{M^2 (4\pi)^2} \left[\Lambda_{UV}^2 - m^2 \log\left(1 + \frac{\Lambda_{UV}^2}{M^2}\right)\right] && (4.9) \\ &= a\frac{m}{(4\pi)^2} \left[\frac{\Lambda_{UV}^2}{M^2} + \frac{m^2}{M^2} \log \frac{m^2}{\Lambda_{UV}^2} - \frac{m^4}{M^2 \Lambda_{UV}^2} + ... \right] && (4.10) \end{aligned}$$

Thus our earlier discussion was incomplete. We have a $\sim m^3/M^2$ term but we also have an order 1 correction, Λ_{UV}^2/M^2. We cannot expand in $\frac{m^2}{M^2}$ so power counting fails. We can get around this if we absorb a piece of the integral:

$$\int_{\Lambda}^{\Lambda_{UV}} \frac{d^4k_E}{(2\pi)^4} \frac{1}{k_E^2 + m^2} = \frac{1}{(4\pi)^2} \left\{ \left(\Lambda^2 + m^2 \log \frac{m^2}{\Lambda^2 + m^2} + ... \right) + \left(\Lambda_{UV}^2 - \Lambda + m^2 \log \frac{\Lambda^2 + m^2}{\Lambda_{UV}^2} \right) \right\} \tag{4.11}$$

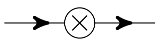
into $\delta m(\Lambda_{UV}, \Lambda)$. This improves things, leaving $\frac{\Lambda^2}{M^2}$ and $\log \frac{m^2}{\Lambda^2}$ (instead of Λ_{UV} terms). [Q 1: How exactly does this work?]

Lets now repeat this procedure with a new regulator using the $\overline{MS}$ scheme:

$$a\frac{m}{M^2}\int \frac{d^d k_E}{(2\pi)^d}\frac{1}{k_E^2+m^2} = a\frac{m}{M^2}\frac{2(\mu^2 e^{\gamma_E}/4\pi)^\epsilon}{(4\pi)^d\Gamma(d/2)}\int_0^\infty \frac{dk_E k_E^{d-1}}{k_E^2+m^2} \tag{4.12}$$

$$= a\frac{m}{(4\pi)^2}\left[\frac{m^2}{M^2}\left(-\frac{1}{\epsilon}+\log\frac{m^2}{\mu^2}-1+\gamma-\log 4\pi\right)+\mathcal{O}(\epsilon)\right] \tag{4.13}$$

From this result we are getting something that is the size we expected for $k^\mu \sim m$ from our power counting arguement. The $\overline{MS}$ counterterm is



$$= a\frac{m^2}{(4\pi)^2 M^2}\left(\frac{1}{\epsilon}+1-\gamma+\log 4\pi ...\right)$$

The two regulators give us similar results but not identical results. Using dim-reg we still preserve the power counting in the sense that we still have a $\frac{m^2}{M^2}$ factor in front. There is no annoying quadratic divergence. What we say is that in dim-reg the regularization does not "break" the power counting, we can power count regularized graphs. When using a cutoff we can say power counting only applies to renormalized couplings and operators order by order. You are adding counterterms to restore the power counting that you want.

In principal any regulator is fine, but we make computations easier if our regulator preserves symmetries (gauge invariance, Lorentz, chiral, ...) and power counting by not yielding a mixing of terms of different order in the expansion. For dimensional power counting this corresponds to using a "mass independent" regulator (like dim-reg)[1]. In general operators will always mix with other operators of the same dimension and same quantum numbers (i.e.,$\mathcal{O}^{bare} = Z_{ij}\mathcal{O}_j^{ren}$).

4.0.1 Dimensional Regularization

Axioms (we define $đ^d k \equiv \frac{d^d k}{(2\pi)^4}$):

- Linearity: $\int đ^d k\,[af(k)+bg(p)] = a\int đ^d k f(k) + b\int đ^d k g(k)$
- Translation:$đ^d k f(k+p) = \int đ^d k f(k)$
- Lorentz Invariance
- Scaling: $\int đ^d k f(sk) = s^d \int đ^d k f(k)$

[1]While it does introduce a mass scale, μ, its put in softly

These conditions give a unique regulator up to normalization, dim-reg (see Colling's pg 65 for a proof).

Useful formula:

$$d^d p = dpp^{d-1} d\Omega_d = \underbrace{dpp^{d-1}}_{\text{UV div occurs in 1D integration}} d(\cos\theta)(\sin\theta)^{d-3} d\Omega_{d-1} \tag{4.14}$$

We use $d = 4 - 2\epsilon$.

- $\epsilon > 0$ tames UV
- $\epsilon < 0$ tames IR

Facts:

1. $\int \text{đ}^d p (p^2)^\alpha = 0$ (Collins, pg 71). While this is true for any α its simple to prove in the case of $k > 0, k \in \mathbb{Z}$:

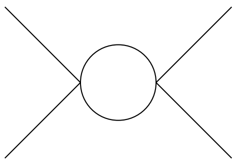
$$\int \text{đ}^d p (p+q)^{2k} = \int \text{đ}^d p \left[p^{2k} + (...)\, p^{2k-2} q^2 + (...)\, p^{2k-4} q^4 + ... \right] \tag{4.15}$$

but we could have also shifted the original relation so we have,

$$\int \text{đ}^d p (p)^{2k} = \int \text{đ}^d p \left[p^{2k} + (...)\, p^{2k-2} q^2 + (...)\, p^{2k-4} q^4 + ... \right] \tag{4.16}$$

Thus all the terms except the first term must be zero. But this is true for any q and any k. Thus $\int \text{đ}^d p (p^2)^\alpha = 0$ for $\alpha = k$.

This can sometimes be dangerous. Consider for example the diagram:



with zero momentum and zero mass. This is given by

$$\int \frac{\text{đ}^d p}{p^4} \to \frac{-i}{16\pi^2} (4\pi) \Gamma(\epsilon) \overset{\substack{\text{IR regulator} \\ \uparrow}}{\mu^\epsilon} \tag{4.17}$$

This integral is zero as mentioned above.

If we set $d = 4 - 2\epsilon$ then we regulate the UV divergence:

$$\int \frac{\text{đ}^{4-2\epsilon} p}{p^4} = -\frac{i}{16\pi^2} \left(\frac{1}{\epsilon} - \gamma + \log 4\pi - \log \mu^2 \right) \tag{4.18}$$

$$\Rightarrow \frac{1}{\epsilon} = -\gamma + \log 4\pi - \log \mu^2 \tag{4.19}$$

However, we also have an IR divergence which could regulate by working in $d+2\epsilon$ dimensions:

$$\int \frac{\bar{d}^{4+2\epsilon}p}{p^4} = -\frac{i}{16\pi^2}\left(-\frac{1}{\epsilon_{IR}} - \gamma + \log 4\pi - \log \mu^2\right) \tag{4.20}$$

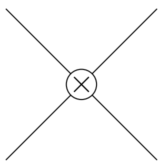
$$\frac{1}{\epsilon} = -\underbrace{\left(-\gamma + \log 4\pi - \log \mu^2\right)}_{\frac{1}{\epsilon_{IR}}} \tag{4.21}$$

We get the same value for ϵ in the UV and IR. Thus we can write the integral as,

$$\int \frac{\bar{d}^d p}{p^4} = \frac{i}{16\pi^2}\left(-\frac{1}{\epsilon_{UV}} - \frac{1}{\epsilon_{IR}}\right) \tag{4.22}$$

$$= 0 \tag{4.23}$$

However, even though the UV divergence cancels the IR diveregence you still need to add a counterterm, since counterterms need to cancel UV divergences.



$$= \frac{-i}{16\pi^2}\frac{1}{\epsilon_{UV}}$$

2. Dim-reg is well defined even with both UV and IR divergences. You can use analytic continuation. Suppose you have some integral,

$$\int \bar{d}^d p f(p^2) = \frac{2}{(4\pi)^{d/2}\Gamma(d/2)}\int_0^\infty p^{d-1} f(p^2) dp \tag{4.24}$$

which is well defined for $0 < d < d_{max}$.

The problem with negative d is that you get some infrared divergences. To obtain range $-2 < d < d_{max}$ you can do the following:

$$\int \bar{d}^d p f(p^2) = \frac{2}{(4\pi)^{d/2}\Gamma(d/2)}\left\{\int_c^\infty dp p^{d-1} f(p^2) + \int_0^c dp p^{d-1}\left[f(p^2) - f(0)\right] + \frac{f(0)c^d}{d}\right\} \tag{4.25}$$

Then regulate the IR and UV peices with different values of d. So for $-2 < d < 0$ take $c \to \infty$. gives

$$\int \bar{d}^d p f(p^2) \frac{2}{(4\pi)^{d/2}\Gamma(d/2)}\int_0^\infty dp p^{d-1}\left[f(p^2) - f(0)\right] \tag{4.26}$$

[Q 2: What is going on here...?]

In MS scheme you introduce μ to keep dimensionless couplings dimensionless under renormalization. If you have an interaction term, $g_{bare}\phi\bar{\psi}\psi$ then the bare coupling has dimensions,

$$[g_{bare}] = 1 + 3 - d = \epsilon \tag{4.27}$$

To "remove" the dimensions we set:

$$g_{bare} = Z_g \mu^{\epsilon} \underbrace{g(\mu)}_{\text{dim. less}} \tag{4.28}$$

For the term

$$\frac{a_{bare}}{M^2}\left(\bar{\psi}\psi\right)^2 \tag{4.29}$$

we have,

$$[a_{bare}] = 4 - d \tag{4.30}$$

and so,

$$a_{bare} = Z_a \mu^{2\epsilon} a(\mu) \tag{4.31}$$

The $\overline{MS}$ is simply a rescaling:

$$\mu_{MS}^2 \to \mu_{\overline{MS}}^2 e^{\gamma_E/4\pi} \tag{4.32}$$

$\overline{MS}$ is a useful scheme because

- it preserves symmetries
- it's easy to calculate in
- It often gives "manifest PC"

However it also has its downsides:

- Phyiscal picture is less clear, for example you can lose positive definiteness for can loose for renormalized quantities.
- You can introduce "renormalons", i.e., poor convergence at large orders in perturbation theory (we'll talk more about this later)
- $\overline{MS}$ does not satisfy the "decoupling theorem" that says the following: If the remaining low energy theory is renormalizable and we use a physical renormalization scheme, then all the effects of the heavy particles due to heavy particles appear as changes to couplings or are suppressed as $1/M$.

$\overline{MS}$ is not physical. It is mass independent which is something that we like however its also whats causing this problem. It does not see mass thresholds. You must instead implement this decoupling "by hand" by removing particles of mass M for $\mu \leq M$.

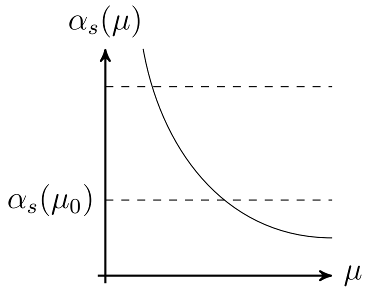
As an example of how this works we go over it in the context of QCD. In QCD we have the β function:

$$\beta(g) = \mu \frac{d}{d\mu} g(\mu) = -\frac{g^3}{16\pi^2} \underbrace{\left(11 - \frac{2}{3} n_f\right)}_{b_0} + \mathcal{O}(g^5) < 0 \tag{4.33}$$

Solving this equation for $\alpha_s(\mu) \equiv \frac{g^2}{4\pi}$ gives,

$$\alpha_s(\mu) = \frac{\alpha_s(\mu_0)}{1 + \alpha_s(\mu_0) \frac{b_0}{2\pi} \log \frac{\mu}{\mu_0}} \tag{4.34}$$

which gives,



You can also associate to this solution an intrinsic mass scale (dimensional transformation):

$$\Lambda_{QCD}^{\overline{MS}} \equiv \mu \exp\left[\frac{-2\pi}{b_0 \alpha_s(\mu)}\right] \tag{4.35}$$

This scale is independent of whether its evaluated at μ or μ_0 since,

$$\begin{aligned} \Lambda_{QCD}(\mu_0) &= \mu_0 \exp\left[-\frac{2\pi}{b_0 \alpha_s(\mu_0)}\right] && (4.36) \\ &= \mu_0 \exp\left[-\frac{2\pi}{b_0}\left(-\frac{b_0}{2\pi}\log\frac{\mu}{\mu_0} + \frac{1}{\alpha_s(\mu)}\right)\right] && (4.37) \\ &= \mu \exp\left[-\frac{2\pi}{b_0 \alpha_s(\mu)}\right] && (4.38) \\ &= \Lambda_{QCD}(\mu) && (4.39) \end{aligned}$$

You can then use this scale to write,

$$\alpha_s(\mu) = \frac{2\pi}{b_0 \log \frac{\mu}{\Lambda_{QCD}}} \tag{4.40}$$

Λ_{QCD} is the scale is where $\alpha_s \sim 1$ and QCD becomes non-perturbative. This scale is dependent on

i) order of the loop expansion $\beta(g)$.

ii) number of light fermions, n_f

iii) renormalization scheme (when you go beyond 2 loops)

The issue with this is that apriori top and up quarks which have very different masses contribute to b_0 and it seems like they do so for any μ. The decoupling theorem gaurentees that the top quark would not appear in the running for low momenta in a physical scheme.

The solution to this is to implement the decoupling by hand by integrating out the heavy fermion at $\mu \sim m$. This is an example of matching. We define different b_0 depending on the scale that we are at.

$$b_0 = \begin{cases} 11 - \frac{2}{3} \cdot 6 & \text{for } \mu \geq m_t \\ 11 - \frac{2}{3} \cdot 5 & \text{for } m_b \leq \mu \leq m_t \\ \vdots & \qquad \vdots \end{cases} \tag{4.41}$$

4.0.2 Matching Conditions

To do this we use what are called *matching conditions.* At a scale $\mu = \mu_m \sim m$ we demand that S matrix elements with light external particles agree between the high energy (theory 1) and low energy (theory 2) theories.

If we do this for QCD and we do it at lowest order then the picture is,

Lowest Order Condition

$$\begin{array}{ccc} & \text{run} \downarrow \alpha_s^{(6)} & \\ m_t & \text{- - - - - - - - - -} & \text{match} \\ & \text{run} \downarrow \alpha_s^{(5)} & \\ m_b & \text{- - - - - - - - - -} & \text{match} \\ & \text{run} \downarrow \alpha_s^{(4)} & \\ m_c & \text{- - - - - - - - - -} & \text{match} \\ & \text{run} \downarrow \alpha_s^{(3)} & \\ & \vdots & \end{array}$$

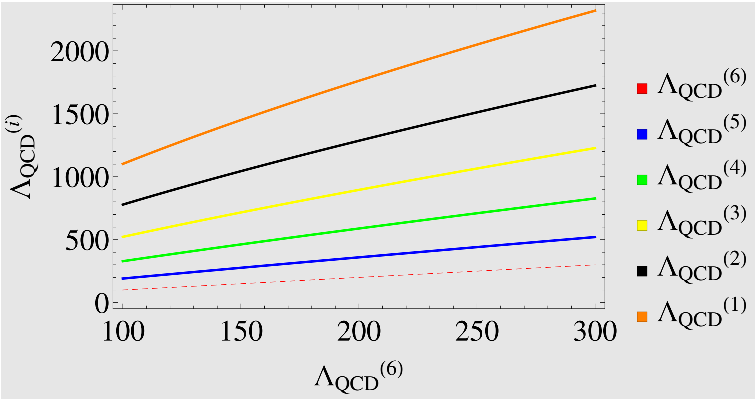
where we have denoted the coupling with n_f flavors as $\alpha_s^{(n_f)}$. This has some interesting implications. For example the value of Λ_{QCD} changes with scale. Using

$$\alpha_s = \frac{12\pi}{(33 - 2N_q)\log \mu^2/\Lambda_{QCD}^2} \tag{4.42}$$

and setting $\alpha_s^{(5)}$ equal to $\alpha_s^{(6)}$ at $\mu = m_t$ gives,

$$\left(\frac{m_t}{\Lambda_{QCD}^{(6)}} \right)^{2/23} \Lambda_{QCD}^{(6)} = \Lambda_{QCD}^{(5)} \tag{4.43}$$

So the scale of non-perturbativity changes with the scale. This effect is shown below:



We see that when working at low energies that non-perturbative scale is as large as a couple GeV, while at high energies, it is as low as 100 MeV.

For example for crossing between μ_b we have,

$$\alpha_s^{(5)}(\mu_b) = \alpha_s^{(4)}(\mu_b) \quad \text{(at leading order)} \tag{4.44}$$

where $\mu_b \sim m_b$. [Q 3: How would this look like in a physical scheme?]

The coupling is actually not continuous in $\overline{MS}$ at higher orders in the coupling. An explicit calculation gives,

$$\alpha_s^{(4)} = \alpha_s^{(5)} \left[1 + \frac{\alpha_s^{(5)}}{\pi} \left(-\frac{1}{6} \log \frac{\mu_b^2}{m_b^2} \right) + \left(\frac{\alpha_s^{(5)}}{\pi} \right)^2 \left(\frac{11}{72} - \frac{11}{24} \log \frac{\mu_b}{m_b} + \frac{1}{36} \log^2 \frac{\mu_b}{m_b} \right) + ... \right] \tag{4.45}$$

which is discontinuous at all scales, μ_b. This comes from calculating the running at 4 flavors and 5 flavors and then setting them equal at the renormalization scale. There are no large logs as long as we match for $\mu = \mu_b \sim m_b$.

The general procedure for massive particles (any operators/couplings),

$$\begin{gathered} m_1 \gg m_2 \gg m_3 \gg ... \gg m_n \\ \mathcal{L}_1 \to \mathcal{L}_2 \to \mathcal{L}_3 \to ... \to \mathcal{L}_n \end{gathered}$$

(1) Match $\mathcal{L}_1$ at $\mu_1 \sim m_1$ onto $\mathcal{L}_2$.

(2) Compute the β-functions and anomalous dimensions in $\mathcal{L}_2$ and evolve/run the couplings down

(3) Match $\mathcal{L}_2$ at $\mu \sim m_2$ onto $\mathcal{L}_3$

(4) $\vdots$

(5) Say we're interested in dynamics at scale $\mu \sim m_n$, then we compute final matrix elements in $\mathcal{L}_n$.

4.0.3 Massive SM Particles

Suppose we work in the SM but we integrate out the top, Higgs, W, Z. Other possibilities are $m_t \gg M_W, M_Z$. However, integrating out just the top is problematic for two reasons. One issue is it breaks $SU(2) \times U(1)$ gauge invariance since we remove t from the doublet, $\begin{pmatrix} t_L \\ b_L \end{pmatrix}$. The bigger issue is that

$$\frac{m_Z}{m_t} \sim \frac{1}{2} \tag{4.46}$$

and this is a large expansion parameter.

By removing t, H, W, Z simultaneously we "miss" the running, $m_t \to m_W$. In other words we treat

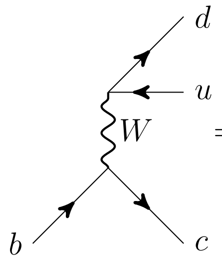
$$\underbrace{\alpha_s(m_W) \log \frac{m_W^2}{m_Z^2}}_{\text{counted as } \mathcal{O}(1)} \tag{4.47}$$

perturbatively.

As an example consider $b \to c\bar{u}d$. In the SM,

$$\mathcal{L}_{SM} = \frac{g_2}{\sqrt{2}} W_\mu^+ \bar{u}_L \gamma^\mu V_{CKM} d_L \tag{4.48}$$

We first do treelevel matching,



$$= \left(\frac{ig_2}{\sqrt{2}}\right)^2 (-i) V_{cb} V_{ud}^* \left(g^{\mu\nu} k^2 - \frac{k^\mu k^\nu}{m_W^2}\right) \frac{1}{k^2 - m_W^2} \left[\bar{u}^c \gamma_\mu P_L u^b\right] \left[\bar{u}^d \gamma_\nu P_L v^u\right]$$

The momenta transfer is

$$k^\mu = p_b^\mu - p_c^\mu = p_u^\mu + p_d^\mu \tag{4.49}$$

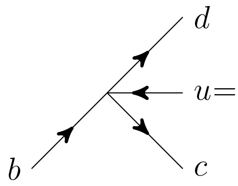
with the momenta of order the masses which we take $\sim m_b$. Furthermore,

$$\not{p}_b u_b = m_b u_b, \text{etc.} \tag{4.50}$$

and the leading term in the propagator is

$$\frac{i g^{\mu\nu}}{m_W^2} + \mathcal{O}\left(\frac{m_b^2}{m_W^4}\right) \tag{4.51}$$

The Feynman rule in the effective theory is,



$$= -\frac{4iG_F}{\sqrt{2}} V_{cb}V_{ud}^* \left[\bar{u}^c \gamma_\mu P_L u_b\right] \left[\bar{u}_d \gamma^\mu P_L v^\mu\right]$$

so we have,

$$G_F = \frac{\sqrt{2}g_2^2}{8m_W^2} \tag{4.52}$$

In the EFT after removing t, W, Z, H, the Hamiltonian is called the "Electroweak Hamiltonian". At tree level we have,

$$H_W = -\mathcal{L}_W = \frac{4G_F}{\sqrt{2}} V_{cb}V_{ub}^* \left[\bar{c}\gamma_\mu P_L b\right] \left[\bar{d}\gamma^\mu P_L u\right] \tag{4.53}$$

4.0.4 Most General Basis of Operators

- Instead of going through the calculation lets think what kind of terms could show up in the effective theory. At $\mu = m_W$ we can treat b, c, d, u as if they are massless to get coefficients, "C". Then their propagators will be proportional to a gamma matrix. Furthermore, recall that QCD does not change chirality. This means that we can't have an even number of gamma matrices in the bilinears since,

  $$\bar{\psi}_1 P_R \gamma_\mu \gamma_\nu P_L \psi_2 = 0 \tag{4.54}$$

  and similary for any number of bilinears.

  Lastly, any higher order odd number of gamma matrix can be reduced to one gamma matrix. Thus we only have terms of the form,

  $$\bar{c}\gamma_\mu P_L b \tag{4.55}$$

- Spin Fierz identities,

  $$\left(\bar{\psi}_1 \gamma_\mu P_L \psi_2\right)\left(\bar{\psi}_3 \gamma^\mu P_L \psi_4\right) = (-1)^2 \left(\bar{\psi}_1 \gamma_\mu P_L \psi_4\right)\left(\bar{\psi}_3 \gamma^\mu P_L \psi_2\right) \tag{4.56}$$

  where the two minus signs arise from the Fierz identity and from anticommuting two fields. Using this relation we can always write the high order operators as,

  $$\left(\bar{c}\Gamma b\right)\left(\bar{d}\Gamma u\right) \tag{4.57}$$

- Color Fierz can be used to simplify the results as well as the completeness relation,

  $$\left(T_a\right)_{\alpha\beta}\left(T_a\right)_{\gamma\delta} = \frac{1}{2}\left(\delta_{\alpha\delta}\delta_{\beta\gamma} - \frac{1}{N}\delta_{\alpha\beta}\delta_{\gamma\delta}\right) \tag{4.58}$$

When we take this into account we end up with two possible operators,

$$H_W = \frac{4G_F}{\sqrt{2}} V_{cb} V_{ud}^* \left[C_1(\mu)\mathcal{O}_1(\mu) + C_2(\mu)\mathcal{O}_2(\mu)\right] \tag{4.59}$$

where

$$\mathcal{O}_1 \equiv \left(\bar{c}^\alpha \gamma_\mu P_L b_\alpha\right)\left(\bar{d}_\beta \gamma^\mu P_L u^\beta\right) \tag{4.60}$$

$$\mathcal{O}_2 \equiv \left(\bar{c}^\beta \gamma_\mu P_L b^\alpha\right)\left(\bar{d}_\alpha \gamma^\mu P_L u_\beta\right) \tag{4.61}$$

$$C_i(\mu) = C_i\left(\frac{\mu}{M_W}, \alpha_s(\mu)\right)^2 \tag{4.62}$$

and the α and β are color indices.

Matching at $\mu = m_W$ gives,

$$C_1(1, \alpha_s(m_W)) = 1 + \mathcal{O}\left(\alpha_s(m_W)\right) \tag{4.63}$$

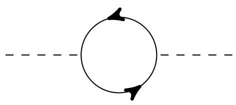
$$C_2(1, \alpha_s(m_W)) = 0 + \mathcal{O}\left(\alpha_s(m_W)\right) \tag{4.64}$$

where $C_2 = 0$ at tree level since the W boson doesn't carry color.

You may wonder how this would work for mesons as we only did the matching for quarks. However, we state without proof that the matching is independent of choice of states and IR regulator as long as same choice is made in both theories in 1 and 2. Result is valid for B, D, π states even though we use quark states for matching.

4.0.5 Renormalize EFT to 1-loop in $\overline{MS}$

Loop calculations are straightforward in four-fermion theories with two related subtleties. Recall that when performing a loop calculating we usually trace over the spinor indices and get a negative sign. Neither of these things hold in four-fermion theory. To see why recall why they are necessary in Yukawa theory for example. In that case we have diagrams like this,



$$= (-ig)^2 \int d^4w\, d^4z \,\langle 0|\, \phi_x \bar{\psi}_w \psi_w \phi_w \bar{\psi}_z \psi_z \phi_z \phi_y \,|0\rangle$$

But the fermion contraction is

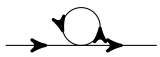
$$\psi_x \bar{\psi}_y = S_F(x-y)^{\alpha\beta} \tag{4.65}$$

where $S_F^{\alpha\beta}$ is the position space fermion propagator. Notice that the order of the fermions is $\psi\bar{\psi}$. To fix this we take the trace and move the fermions over,

$$(-ig)^2 \int d^4w\, d^4z \,\langle 0|\, \phi_x \mathrm{Tr}\left[\psi_w \bar{\psi}_z \psi_z \bar{\psi}_w\right] \phi_w \phi_z \phi_y \,|0\rangle\, (-1) \tag{4.66}$$

where we get a negative sign due to the anticommuting nature of fermions.

With this in mind we now consider the four-fermion interaction:



$$= \frac{ia}{M^2}\int d^4z \, \langle 0|\, \psi_x \bar{\psi}_z \psi_z \bar{\psi}_z \psi_z \bar{\psi}_y \,|0\rangle$$

This is already in the form that can be simplified as a product of propagators. Thus we don't need to take the trace and we also avoid the negative sign.

With this in mind we can move on to renormalization. The renormalized field is proportional to the bare field,

$$\psi = Z_\psi^{-1/2}\psi_0 \tag{4.67}$$

In appendix 4.A we show that the wavefunction renormalization is given by,

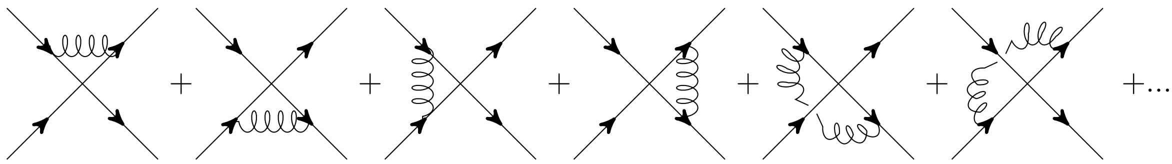
$$Z_\psi = 1 - \frac{\alpha_s C_F}{4\pi\epsilon}, \quad C_F = \frac{4}{3} \tag{4.68}$$

We will use Feynman gauge and let,

$$\langle \bar{u}dc|\mathcal{O}_1|b\rangle = s_1 \tag{4.69}$$

$$\langle \bar{u}dc|\mathcal{O}_2|b\rangle = s_2 \tag{4.70}$$

The QCD 1 loop corrections to the couplings are,



We regulate the IR with off-shell momenta, p (masses to zero). We have (the $^{(0)}$ denotes bare.), [Q 4: Show this!]

$$\begin{aligned}\langle \mathcal{O}_1\rangle^{(0)} &= \left[1 + 2C_F\frac{\alpha_s}{4\pi}\overbrace{\left(\frac{1}{\epsilon} + \log\frac{\mu^2}{-p^2}\right)}^{\text{cancelled by } Z_\psi}\right]s_1 + \overbrace{\frac{3}{N_c}\frac{\alpha_s}{4\pi}\left(\frac{1}{\epsilon} + \log\frac{\mu^2}{-p^2}\right)}^{\text{CT for } \mathcal{O}_1}s_1 \\ &\quad - 3\frac{\alpha_s}{4\pi}\underbrace{\left(\frac{1}{\epsilon} + \log\frac{\mu^2}{-p^2}\right)}_{\text{CT for } \mathcal{O}_2}s_2 + \underbrace{\cdots}_{\text{const terms}} \end{aligned} \tag{4.71}$$

$$\langle \mathcal{O}_2\rangle^{(0)} = \text{same with } s_1 \leftrightarrow s_2 \tag{4.72}$$

We say that $\mathcal{O}_1$ has mixed into $\mathcal{O}_2$.

There are two different methods we could use to carry out the renormalization. The first method is called composite operator renormalization.

$$\mathcal{O}_i^{(0)} = Z_{ij}\mathcal{O}_j(\psi^{(0)}) \tag{4.73}$$

You can think of anomalous dimensions as running the coefficient. These are not related to wavefunction renormalization, they are just the operators renormalization. When you take the matrix element you need the wavefunction renormalization as well,

$$\langle \mathcal{O}_i \rangle^{(0)} = Z_\psi^{-2} Z_{ij} \langle \mathcal{O}_j \rangle \tag{4.74}$$

Isolating for the renormalized operators,

$$\langle \mathcal{O}_j \rangle = Z_\psi^2 (Z^{-1})_{ji} \langle \mathcal{O}_i \rangle^{(0)} \tag{4.75}$$

The second method is renormalize coefficients instead. This is closer to the way we are used to doing calculations in renormalization. We write the Hamiltonian first in terms of bare coefficients and then switch to renormalized coefficients.

$$\begin{aligned}
\mathcal{H} &= C_i^{(0)} \mathcal{O}_i(\psi^{(0)}) && (4.76)\\
&= (Z_{ij}^c C_j) Z_\psi^2 \mathcal{O}_i && (4.77)\\
&= C_i \mathcal{O}_i + \underbrace{(Z_\psi^2 Z_{ij}^c - \delta_{ij}) C_j \mathcal{O}_i}_{CT} && (4.78)\\
&= C_j^{ren} \mathcal{O}_j^{ren} && (4.79)
\end{aligned}$$

Here $C_i \mathcal{O}_i$ still divergences but its divergences are cancelled off the by counterterms as usual.

These two ways of thinking about things are equivalent and we can prove it. Consider the Hamiltonian,

$$Z_\psi^2 Z_{ij}^c C_j \langle \mathcal{O}_i \rangle^{(0)} = C_j \langle \mathcal{O}_j \rangle = C_j Z_{ji}^{-1} Z_\psi^2 \langle \mathcal{O}_i \rangle^{(0)} \tag{4.80}$$

which implies that

$$Z_{ij}^c = Z_{ji}^{-1} \tag{4.81}$$

and hence we have proven the equivalence. In our example,

$$Z_{ij} = \mathbb{1}_{ij} - \frac{\alpha_s}{4\pi} \frac{1}{\epsilon} \begin{pmatrix} 3/N_c & -3 \\ -3 & 3/N_c \end{pmatrix} \tag{4.82}$$

4.0.6 Anomalous Dimensions for Operators

We can now construct the anomalous dimension of the operators. Consider, We must have,

$$0 = \mu \frac{d}{d\mu} \mathcal{O}_i^{(0)} = \left(\mu \frac{d}{d\mu} Z_{ij'} \right) \mathcal{O}_{j'} + Z_{ij} \left(\mu \frac{d}{d\mu} \mathcal{O}_j \right) \tag{4.83}$$

where we have used the notations from method 1. Rearranging gives,

$$\mu \frac{d}{d\mu} \mathcal{O}_j \equiv -\gamma_{ji} \mathcal{O}_i \tag{4.84}$$

where

$$\gamma_{ji} = Z_{jk}\left(\mu\frac{d}{d\mu}Z_{ki}\right) \tag{4.85}$$

The anomalous dimension is determined by the Z factors. To find the anomalous dimension you need,

$$\mu\frac{d}{d\mu}\alpha_s(\mu) = \underbrace{-2\epsilon\alpha_s}_{\text{Matters at 1 loop}} + \overbrace{\beta\left[\alpha_s\right]}^{\text{Drop}} \tag{4.86}$$

The anomalous dimension doesn't depend on ϵ and is given by,

$$\gamma_{ji} = -\frac{\alpha_s}{2\pi}\begin{pmatrix} 3/N_c & -3 \\ -3 & 3/N_c \end{pmatrix} \tag{4.87}$$

Diagonalizing this operator gives two new operators,

$$\mathcal{O}_{\pm} = \mathcal{O}_1 \pm \mathcal{O}_2 \tag{4.88}$$

which coefficients $C_{\pm}$. Now no mixing between the $+$ and $-$ basis and we have a simple differential equation from the anomalous dimension,

$$\mu\frac{d}{d\mu}\mathcal{O}_{\pm} = \gamma_{\pm}\mathcal{O}_{\pm} \tag{4.89}$$

and

$$\gamma_{+} = -\frac{\alpha_s}{2\pi}\left(\frac{3}{N_c} - 3\right) \tag{4.90}$$

$$\gamma_{-} = -\frac{\alpha_s}{2\pi}\left(\frac{3}{N_c} + 3\right) \tag{4.91}$$

The Hamiltonian is then,

$$H_W = C_1\mathcal{O}_1 + C_2\mathcal{O}_2 = C_+\mathcal{O}_+ + C_-\mathcal{O}_- \tag{4.92}$$

which leads to,

$$C_{\pm} = \frac{1}{2}\left(C_1 \pm C_2\right) \tag{4.93}$$

Tree level matching gives,

$$C_{\pm}(\mu = m_W) = \frac{1}{2} \tag{4.94}$$

In summary we have,

$$H_W = \left(\frac{4G_F}{\sqrt{2}}V_{cb}^*V_{ud}^*\right)\sum_{i=1,2} C_i^{(0)}\mathcal{O}_i(\psi^{(0)}) \tag{4.95}$$

$$= \left(\frac{4G_F}{\sqrt{2}}V_{cb}^*V_{ud}^*\right)\sum_{i=1,2} C_i(\mu)\mathcal{O}_i(\mu) \tag{4.96}$$

$$= \left(\frac{4G_F}{\sqrt{2}}V_{cb}^*V_{ud}^*\right)\sum_{i=\pm} C_i(\mu)\mathcal{O}_i(\mu) \tag{4.97}$$

Notice that H_W is independent of the renormalization scale. We can use this observation to calculate the anomalous dimension of the coefficient. You can think of anomalous dimensions as either running the operators or running the coefficients. We have (in a general basis),

$$0 = \mu \frac{d}{d\mu} H_W = \left(\mu \frac{d}{d\mu} C_i \right) \mathcal{O}_i - C_j \left(\gamma_{ji} \mathcal{O}_i \right) \tag{4.98}$$

where we have used the definition of the anomalous dimensions, $\mu \frac{d}{d\mu} \mathcal{O}_i \equiv \gamma_{ij} \mathcal{O}_j$. The equation should hold for all operators. Thus we must have,

$$\mu \frac{d}{d\mu} C_i = C_j \gamma_{ji} = \left(\gamma^T \right)_{ij} C_j \tag{4.99}$$

We can solve this equation by moving to the $\pm$ basis. This gives,

$$\mu \frac{d}{d\mu} \log C_\pm(\mu) = \gamma_\pm \left(\alpha_s(\mu) \right) \tag{4.100}$$

where we write the α_s dependence of γ to emphasize that it is a coupled equation and we can't just simply integrate over μ. Performing a change of variables on the $\beta(g)$ function we can write,

$$\mu \frac{d}{d\mu} \alpha_s(\mu) = -\frac{b_0}{2\pi} \alpha_s^2 \tag{4.101}$$

We can then solve the differential equation above by a change of variable:

$$\mu \rightarrow \alpha_s : \frac{d\mu}{\mu} = -\frac{2\pi}{b_0} \frac{d\alpha_s}{\alpha_s^2} \tag{4.102}$$

Substitutin back into the first equation and integrating from μ_W to μ gives(note that $\mu_W < \mu$ so we are "integrating backwords"),

$$\log \left(\frac{C_\pm(\mu)}{C_\pm(\mu_W)} \right) = -\frac{2\pi}{b_0} \int d\alpha_s \frac{\gamma_\pm}{\alpha_s^2} \tag{4.103}$$

Doing the integral gives,

$$\log \left(\frac{C_\pm(\mu)}{C_\pm(\mu_W)} \right) = -\frac{1}{b_0} \left(\frac{3}{N} \mp 3 \right) \log \left(\frac{\alpha_s(\mu)}{\alpha_s(\mu_W)} \right) \tag{4.104}$$

$$= a_\pm \log \left(\frac{\alpha_s(\mu)}{\alpha_s(\mu_W)} \right) \tag{4.105}$$

Taking $N_c = 3$ we have, where

$$a_\pm = \begin{cases} \frac{2}{b_0} \\ -\frac{4}{b_0} \end{cases} \tag{4.106}$$

Here the boundary condition is $C_{\pm}(\mu_W)$, where $\mu_W \sim m_W$. Typically, $\mu_W = m_W, 2m_W, m_W/2$. You should think of $C_{\pm}(\mu_W)$ as a fixed order series in $\alpha_s(\mu_W)$. Isolating for the coefficient we have,

$$C_{\pm}(\mu) = C_{\pm}(\mu_W) e^{a_{\pm} \log \frac{\alpha_s(\mu)}{\alpha(\mu_W)}} = C_{\pm}(\mu_W) \left(\frac{\alpha_s(\mu_W)}{\alpha_s(\mu)} \right)^{a_{\pm}} \tag{4.107}$$

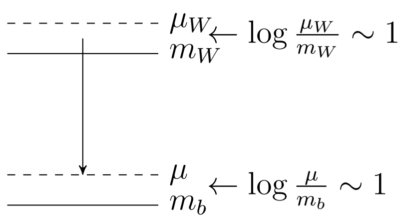
For $b \to c\bar{u}d$ to avoid large logs you want to take the renormalization scale around the b quark mass, $\mu \sim m_b \ll m_W$. Result for $C_{\pm}(\mu)$ sums the leading logarithms (LL),

$$\sim \frac{1}{2} + \alpha_s(\mu_W) \log \frac{m_W}{m_b} + (\alpha_s^2 \log^2 \frac{m_W}{m_b}) + (\alpha_s^3 \log^3 \frac{m_W}{m_b}) + ... \tag{4.108}$$

Counting here is

$$\alpha_s(\mu_W) \log \frac{m_W}{m_b} \sim 1 \tag{4.109}$$

The physical picture is,



Since we diagonalized our anomalous dimension matrix we got decoupled C_i's. More generally we have, General form,

$$C^i(\mu) = C^j(\mu_W) \underbrace{U^{ji}(\mu_W, \mu)}_{\text{evolution}} \tag{4.110}$$

This gives,

$$H_W = \left(\frac{4G_F}{\sqrt{2}} V_{ud}^* V_{cb} \right) \sum_{i,j} \underbrace{C_j(\mu_W) U_{ji}(\mu_W, \mu_b)}_{C_i(\mu_b)} \mathcal{O}_i(\mu_b) \tag{4.111}$$

$C_j(\mu_W)$ is a fixed order calculation, U_{ji} is the anomalous dimension and $\mathcal{O}_i(\mu_b)$ are the matrix elements at $\mu_b \sim m_b$.

When we go to higher orders we compute,

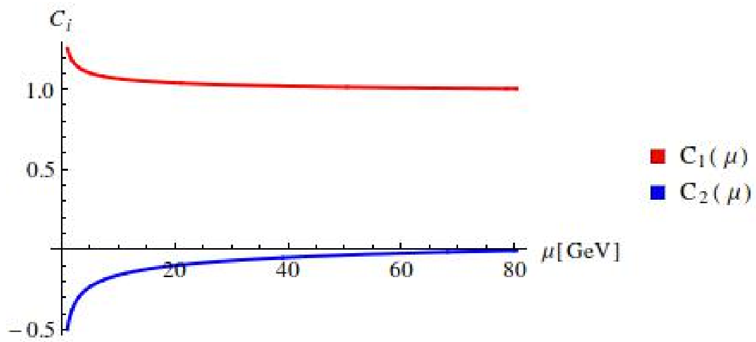
$$C_i(\mu) \sim \underbrace{\sum_k [\alpha_s \log]^k}_{LL} + \underbrace{\alpha_s \sum_k [\alpha_s \log]^k}_{NLL} + \underbrace{\alpha_s^2 \sum_k [\alpha_s \log]^k}_{NNLL} + ... \tag{4.112}$$

through the RG equation. This called RG improved perturbation theory.

Notice that we needed to go to one loop to get the running of $C_{\pm}$ from the tree-level result. If we go one order higher then we need to use the 1 loop result to do matching and the two-loop to find the anomalous dimensions. This trend continues,

	Matching (μ_W)	Running γ
LL	tree-level	1-loop
NLL	1-loop	2-loop
NNLL	2-loop	3-loop
$\vdots$	$\vdots$	$\vdots$

Recall that $\mathcal{O}_2$ has $C_2 = 0$ at tree-level, but it is non-zero at leading order if we shift scales. To see this we calculate $C_\pm(m_b)$ using Eq. 4.107 and use them to compute $C_{1,2}$. This gives,



At LL we have,

$$C_1(m_b) \approx 1.11, C_2(m_b) \approx -0.26 \tag{4.113}$$

We can now consider a physical application of $b \to c\bar{u}d$ through

$$\bar{B} \to D\pi \tag{4.114}$$

Without renormalization group improvement we have,

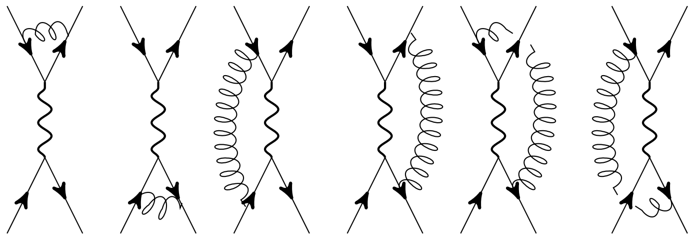
$$\langle D\pi|H_W|B\rangle = C_i(m_W)\,\langle D\pi|\mathcal{O}_i(m_W)|B\rangle \tag{4.115}$$

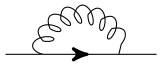
The matrix is element is highly non-perturbative since the operator is a weak scale operator evaluated at the b mass, which leads to large logs, $\log\frac{m_W}{m_b}$. So what we do is instead is we work in the RG improved version,

$$\langle D\pi|H_W|B\rangle = \underbrace{C_i(m_b)}_{\text{LL summed here}}\,\langle D\pi|\mathcal{O}_i(m_b)|B\rangle \tag{4.116}$$

where the matrix element no longer has large logs and can be calculated on a lattice for example.

Physically, $C_i(m_b)$ are the right couplings to use. We now do a comparison of the results in the full theory with the effective theory. We already renormalized the EFT in $\overline{MS}$ ("theory 2"). First consider the full theory ("theory 1"). The calculation involves conserved currents so UV divergences in vertex and wavefunction cancel. The result for the full theory will be independent of UV divergences unlike the effective theory. The full theory diagrams are,





This is a much harder calculation. It yields,

$$i\mathcal{M}^{1-loop} = \left[1 + 2C_F \frac{\alpha_s}{4\pi} \log \frac{\mu^2}{-p^2}\right] s_1 + \frac{3}{N_c} \frac{\alpha_s}{4\pi} \log \frac{m_W^2}{-p^2} s_1 + ... \tag{4.117}$$

where p^2 is the IR regulator and the ellipses represent s_2 terms as well as non-logarithmic terms.

We now compare this result to what we got in the EFT,

$$\left\langle \mathcal{O}_1^{1-loop} \right\rangle = \left[1 + 2C_F \frac{\alpha_s}{4\pi} \log \frac{\mu^2}{-p^2}\right] s_1 + \frac{3}{N_c} \frac{\alpha_s}{4\pi} \log \frac{\mu^2}{-p^2} s_1 + ... \tag{4.118}$$

(the only difference in s_1 is that instead of m_W in the second logarithm we get a μ)

Comments:

(1) In EFT computation is much easier and only involves triangle loops. The reason computations are easier is because you are dealing with one scale at a time which makes computations easier and $1/\epsilon$ term is all that's required to compute anomalous dimensions and its even easier.

(2) In EFT $m_W \to \infty$ so m_W doesn't exist in the theory and must be replaced by μ's.

(3) $\log(-p^2)$ terms match. In other words the infrared structure of the two theories agrees. This is an important check that our EFT has the right degrees of freedom. This is trivial in this example, but in more complicated examples its not always easy to identify the correct degrees of freedom.

(4) Differences of $\mathcal{O}(\alpha_s)$ of renormalized calculations gives one-loop matching.

$$\begin{aligned} 0 &= i\mathcal{M}^{1-loop} - [C_1 \langle \mathcal{O}_1 \rangle + C_2 \langle \mathcal{O}_2 \rangle] && (4.119) \\ &= s_1 - (1)s_1 + i\mathcal{M}^{1-loop} - C_1^{(1)} s_1 - (1) \langle \mathcal{O}_1 \rangle^{O(\alpha_s)} + ... && (4.120) \end{aligned}$$

which allows you to calculate C_1.

For $s_1 \times \log$ terms,

$$i\mathcal{M}^{\mathcal{O}(\alpha_s s_1)} - \langle \mathcal{O}_1 \rangle^{\mathcal{O}(\alpha_s s_1)} = C_1^{(1)} s_1 \tag{4.121}$$

$$\frac{3}{N_c} \frac{\alpha_s C_F}{4\pi} \log \frac{m_W^2}{-p^2} - \frac{3}{N_c} \frac{\alpha_s C_F}{4\pi} \log \frac{\mu^2}{-p^2} = C_1^{(1)} \tag{4.122}$$

which implies that,

$$C_1^{(1)} = -\frac{3}{N_c}\frac{\alpha_s C_F}{4\pi}\log\frac{\mu^2}{m_W^2} \tag{4.123}$$

So what the matching is doing is

$$\text{full theory} = \text{large momenta} \times \text{small momenta} \tag{4.124}$$

or schematically,

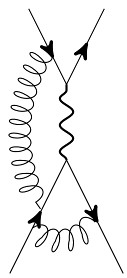
$$\log\frac{m_W^2}{-p^2} = \log\frac{m_W^2}{\mu^2} + \log\frac{\mu^2}{-p^2} \tag{4.125}$$

In other words μ^2 separates large m_W^2 from small $-p^2$, or in product form,

$$\left(1+\alpha_s\log\frac{m_W^2}{-p^2}\right) = \left(1+\alpha_s\log\frac{m_W^2}{\mu^2}\right)\left(1+\alpha_s\log\frac{\mu^2}{-p^2}\right) \tag{4.126}$$

(5) Order by order in α_s the $\log\mu$'s in $C(\mu)\mathcal{O}(\mu)$ cancel and the result is μ-independent at each order in α_s.

(6) Note that at the level of log's the full theory has less information. To compute 2-loop, $\alpha_s^2\log^2\frac{m_W^2}{-p^2}$, in the full theory you would need to calculate diagrams such as,



In EFT we just needed the 1-loop anomalous dimension of the operators. The shows the power of taking a parameter, m_W, and turning it into a scale since then you can use the whole power the renormalization group. [Q 5: How do we have the α_s^2 terms?]

(7) The one-loop anomalous dimension, γ_{ij} are scheme indepedendent for all mass-independent schemes.

4.0.7 Loss and Constants

We will now study the full 1-loop matching at NLL. The coefficients, matrix elements, anomalous dimensions at NLO are all scheme dependent but when we put it all together we get a scheme independent result. In other words,

$$C(\mu)\,\langle\mathcal{O}(\mu)\rangle \tag{4.127}$$

is a physical observable and independent of scheme (much like its independence of μ). For simplicity we ignore mixing and write,

$$\mathcal{M}^{EFT} = C(\mu)\,\langle\mathcal{O}(\mu)\rangle \tag{4.128}$$

$$\mathcal{M}^{full} = 1 + \frac{\alpha_s(\mu)}{4\pi}\left[-\frac{\gamma^{(0)}}{2}\log\frac{m_W^2}{-p^2} + \mathcal{M}^{(1)}\right] \tag{4.129}$$

$$\mathcal{M}^{EFT} = C_\mu\left[1 + \frac{\alpha_s}{4\pi}\left(-\frac{\gamma^{(0)}}{2}\right)\log\frac{\mu^2}{-p^2} + B(1)\right] \tag{4.130}$$

where $A^{(1)}$ and $B^{(1)}$ are just natural numbers which are scheme dependent. We have,

$$C(\mu_W) = 1 + \frac{\alpha_s(\mu_W)}{4\pi}\left[\frac{\gamma^{(0)}}{2}\log\frac{\mu_W^2}{m_W^2} + \underbrace{A^{(1)} - B^{(1)}}_{\text{To get correct constant}}\right] \tag{4.131}$$

$$= 1 + \frac{\alpha_s(m_W)}{4\pi}\left[A^{(1)} - B^{(1)}\right] \tag{4.132}$$

We outline the NLL computation,

$$\mu\frac{d}{d\mu}C(\mu) = \gamma\left[\alpha_s\right]C(\mu) \tag{4.133}$$

$$\mu\frac{d}{d\mu}\log C(\mu) = \gamma\left[\alpha_s\right] = \gamma^{(0)}\frac{\alpha_s}{4\pi} + \gamma^{(1)}\left(\frac{\alpha_s}{4\pi}\right)^2 + ... \tag{4.134}$$

where here we need $\gamma^{(1)}$, the two-loop coefficient. We can solve this differential equation as we did before,

$$\frac{d\mu}{\mu} = \frac{d\alpha_s}{\beta} \tag{4.135}$$

where

$$\beta\left[\alpha_s\right] = \beta_0 + \beta_1 + ... \tag{4.136}$$

Using renormalization group evolution we can write the all order solution as,

$$\log\frac{C(\mu)}{C(\mu_W)} = \int_{\alpha_s(\mu_W)}^{\alpha_s(\mu)} d\alpha_s\frac{\gamma(\alpha_s)}{\beta(\alpha_s)} \tag{4.137}$$

where we keep the second order term in $\gamma\left(\alpha_s\right)$ (we write $C\pm = \frac{1}{2} + \alpha_s(...)$). We can write,

$$C(\mu) = C(\mu_W)U(\mu_W,\mu) \tag{4.138}$$

where

$$U(\mu_W,\mu) = \exp\int d\alpha_s\frac{\gamma}{\beta} \tag{4.139}$$

We then take $\mu_W = m_W$ and get,

$$U^{NLL}(\mu_W, \mu) = \left[1 + \frac{\alpha_s(\mu) \cdot J}{4\pi}\right] \left(\frac{\alpha_s(m_W)}{\alpha_s(\mu)}\right)^{\gamma^{(0)}/2b_0} \left[1 - \frac{\alpha_s(m_W)}{4\pi} J\right] \tag{4.140}$$

where [Q 6: These equations may have a different convention for $\beta_0, \beta_1, \ldots.$]

$$J = \frac{\gamma^{(0)}\beta_1}{2\beta_0^2} - \frac{\gamma^{(1)}}{2\beta_0} \tag{4.141}$$

We can combine this equation with our equation for $C(\mu) = 1 + \frac{\alpha_s(m_W)}{4\pi}\left[A^{(1)} - B^{(1)}\right]$,

$$C(\mu) = \left[1 + \frac{\alpha_s(\mu)}{4\pi} J\right] \left(\frac{\alpha_s(m_W)}{\alpha_s(\mu)}\right)^{\gamma^{(0)}/2\beta_0} \left[1 + \frac{\alpha_s(m_W)}{4\pi}\left(A^{(1)} - B^{(1)} - J\right)\right] \tag{4.142}$$

which has NLO matching and NLL running.

The claim is that $B^{(1)}, \gamma^{(1)}, J, C, \langle \mathcal{O} \rangle$ are all scheme dependent but $\beta_0, \beta_1, \gamma^{(0)}, A^{(1)}, B^{(1)} + J, C\langle \mathcal{O} \rangle$ are scheme inpedendent. We do not prove this here.

We can make the following conclusions

(1) $A^{(1)} - B^{(1)} - J$ is scheme independent. A cancellation occurs between $\gamma^{(1)}$ and $B^{(1)}$(matching and anomalous dimension's cheme dependencies cancel)

(2) LL result, $\left(\frac{\alpha_s(m_W)}{\alpha_s}\right)^{\gamma^{(0)}/2\beta_0}$ is scheme independent

(3) Scheme dependence of $\left(1 + \frac{\alpha_s(\mu)}{4\pi} T\right)$ in $C(\mu)$ is cancelled by scheme dependence of $\langle \mathcal{O}(\mu) \rangle$ at lower end of RG integration.

The lesson is if you take numbers at NLL from the literature you need to make sure you are working in the same scheme as in the literature or you will get wrong results.

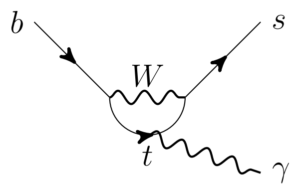
Remarks (subtleties) in full NLL analysis:

(1) $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ is inherently four dimensional and must be treated carefully in dim.reg.

(2) Evanescent Operators: $\{1, \gamma_5, \gamma_\mu, \gamma_\mu\gamma_5, \sigma_{\mu\nu}\}$ is not a complete basis in d dimensions. Additional operatore are called evanescent operators which involve Dirac structures that vanish as $\epsilon \to 0$.

4.0.8 Phenomenology from H_W

Consider $b \to s\gamma$. This is a flavor changing neutral current process and so can't occur at tree level. In the SM we have,



If we integrate out the W and the top then we get a local operator. We can now enumerate all the possible operators,

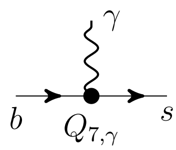
$$Q_{7\gamma} = \frac{e}{8\pi^2} m_b \bar{s}\sigma^{\mu\nu}(1+\gamma_5) b F_{\mu\nu} \tag{4.143}$$

$$Q_{8G} = \frac{g}{8\pi^2} m_b \bar{s} T^a \sigma^{\mu\nu}\left(1+\gamma_5\right) b G^a_{\mu\nu} \tag{4.144}$$

$$Q_1 = \left[\bar{s}\gamma^\mu(1-\gamma_5)u\right]\left[\bar{u}\gamma_\mu(1-\gamma_5)d\right] \tag{4.145}$$

$$Q_2, ..., Q_{10} \tag{4.146}$$

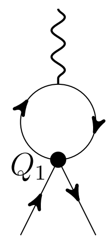
where $Q_2, .., Q_{10}$ are more four-quark operators. In the SM we just have $Q_{7\gamma}$, and the rest can be used to constrain new physics. To leading order we have,



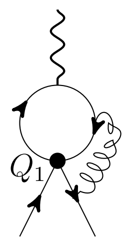
The coefficient at lowest order is found by the loop diagram above,

$$C^{LO}_{7\gamma} = C^{LO}_{7\gamma}\left(\frac{m_W}{m_t}\right) \approx -0.195 \tag{4.147}$$

The next thing you can think of doing is,



This diagram is actually 0 in a good scheme for γ_5. The diagram,



Doing the calculation gives the LL solution,

$$C_{7\gamma}(\mu) = \eta^{16/23} C_{7\gamma}^{LO} + \frac{8}{3}\left(\eta^{14/23} - \eta^{16/23}\right) C_{8G}^{LO} + \sum_{i=1}^{8} h_i \eta^{a_i} C_i^{LO} \tag{4.148}$$

where

$$\eta \equiv \frac{\alpha_s(m_W)}{\alpha_s(\mu)} \tag{4.149}$$

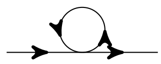
and the right scale to use is $\mu \approx m_b$. If we plug in numbers for the various terms we have

$$C_{7\gamma}(\mu) \approx -0.300 \tag{4.150}$$

which is a pretty large change to $C_{7\gamma}$ (50% larger then the non-evolved result). So you need to make sure you properly understand the SM result when looking for new physics. If you didn't take this into account you would think that there is new physics here.

4.A QCD Wavefunction Renormalization

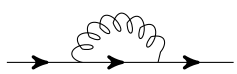
In this section we calculate the wavefunction renormalization of the quark fields below the weak scale. Here we ignore the effects of QED which is valid as long as we work in a regime that $\alpha_s \gg \alpha$. Furthermore, when calculating wavefunction renormalization we can ignore the weak interaction since the lowest order diagram,



is independent of the incoming momenta. The Lagrangian in terms of renormalized variables is given by,

$$\mathcal{L} = \mathcal{L}_{kin} - m\bar{\psi}\psi - \frac{1}{4} G^a_{\mu\nu} G^{a,\mu\nu} + (Z_\psi - 1)\bar{\psi} i \not{\partial} \psi + ... \tag{4.151}$$

The only diagram that contributes to the quark self-energy at lowest order in α_s is,



(note that ghosts only couple to gluons so they don't conribute at this order). This diagram is given by,

$$i\mathcal{M} = \int \not{d}^4\ell \gamma_\mu \frac{\not{\ell} + m}{\ell^2 - m^2 + i\epsilon} \gamma^\mu \frac{1}{(\ell - p)^2 + i\epsilon} (-ig)^2 (i)(-i) (T^a)^i{}_k (T^a)^k{}_j \tag{4.152}$$

Simplifying gives,

$$2g^2 \int dx \not{p} x \left[\frac{i}{16\pi^2} \left(\frac{1}{\epsilon} + \log 4\pi - \gamma - \log \Delta \right) \right] C_F \delta_{ij} \tag{4.153}$$

where $\Delta \equiv -p^2x(1-x) + m^2(1-x)$ and $C_F = 4/3$. The counterterm is given by

$$i(Z_\psi - 1)\not{p}\delta_{ij} \tag{4.154}$$

Thus,

$$\frac{d}{d\not{p}}\left(\mathcal{M} + (Z_\psi - 1)\right) = \frac{2g^2}{16\pi^2}\int dx\left(\frac{1}{\epsilon} + \log 4\pi - \gamma - \log\Delta\right) + \not{p}x\frac{2\not{p}x(1-x)}{\Delta} + (Z_\psi - 1) \tag{4.155}$$

So in $\overline{MS}$ we have,

$$Z_\psi = 1 - \frac{2g^2C_F}{16\pi^2}\int dx\left(\frac{1}{\epsilon} + \log 4\pi - \gamma\right) \tag{4.156}$$

$$= 1 - \frac{2\alpha_s C_F}{4\pi}\left(\frac{1}{\epsilon} + \log 4\pi - \gamma\right) \tag{4.157}$$

In the notes for simplicity the professor just drops irrelevant γ and $\log 4\pi$ terms.

4.B Operator Renormalization of Gauge Interaction

Peskin and Schroeder mention a few times that the anomalous dimension of a gauge interaction operator is zero. The justification for this is that the charge operator shouldn't get modified under anomalous dimensions. Explicitly if you have a conserved current then you also have an algebra,

$$\left[Q^a, Q^b\right] = if^{abc}Q_c \tag{4.158}$$

This equation can only hold if the charges are dimensionless. But the charges are given by,

$$Q^a = \int d^3x j^0 \tag{4.159}$$

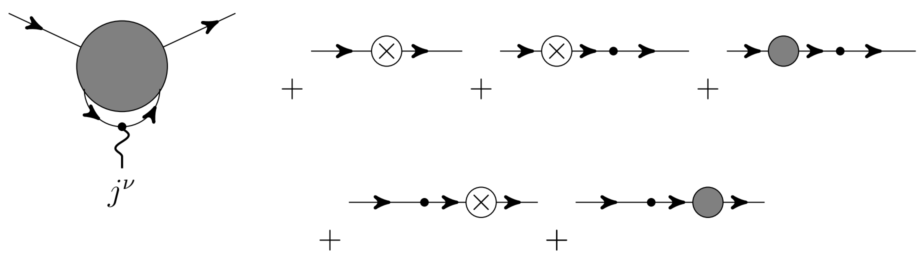
and so if the current operator gets an anomalous dimension, Q becomes dimensionful, spoiling the algebra.

This can also be shown explicitly. Lets consider for simplicity the QED Lagrangian.

The two point Green's function with the current is,

$$\tilde{G}^{(2;1)} = \langle 0|\, T\bar{\psi}_1(p_1)\psi_2(p_2)j^\nu(k)\, |0\rangle \tag{4.160}$$

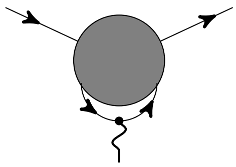
which diagramatically takes the form,



The wavefunction renormalization terms are accounted for by introducing a wavefunction renormalization for the fermion. The final five diagrams can be summarized by a current counterterm:

$$i\left(\frac{Z_\psi}{Z_{j^\nu}}-1\right) \tag{4.161}$$

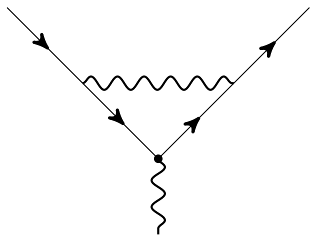
We use as our renormalization condition:



j^ν

$$=\frac{i\not{p}}{p^2}ie\gamma_\nu\frac{i(\not{p}+\not{k})}{(p-k)^2}$$

We need to calculate the diagram,



We work in $\overline{MS}$ with massless fermions and only keep the divergent pieces. The part without the external lines is

$$iG=\int d^4\ell\frac{\gamma_\mu\not{\ell}\not{\ell}\gamma^\mu}{\ell^6}(i)^2(-i)(-ie)^2 \tag{4.162}$$

$$=-\frac{2ie^2}{16\pi^2\epsilon} \tag{4.163}$$

Due to our renormalization condition this must be cancelled by the CT.

The wavefunction renormalization is

$$Z_\psi=1-2\frac{e^2}{16\pi^2\epsilon} \tag{4.164}$$

So we have,

$$Z_j=Z_\psi\left(1+\frac{2e^2}{16\pi^2\epsilon}\right)=1 \tag{4.165}$$

as desired.

Chapter 5

Chiral Lagrangians

This example has the following properties

- It is an example of a bottom-up EFT
- It has a non-linear symmetry representation and their connection to field redefinition
- In this example power counting instead done by loops but instead by powers of momenta - This gives a non-trivial power counting.
- It obeys what's known as a power-counting theorem which tells you which diagrams you need to calculate

Consider the QCD Lagrangian for massless quarks,

$$\mathcal{L}_{QCD} = \bar{\psi} i \not{D} \psi = \bar{\psi}_L i \not{D} \psi_L + \bar{\psi}_R i \not{D} \psi_R \tag{5.1}$$

which has the symmetry,

$$\psi_L \to L\psi_R \tag{5.2}$$
$$\psi_R \to R\psi_R \tag{5.3}$$

where L and R are independent transformations. We have a group $G(L, R)$ which will be broken spontaneously by the Higgs. We can either take the light quarks to be up, down, and strange or omit the strange:

$G = (L, R) \to H$	ψ	Goldstones	Expansion
$\underbrace{SU(3)_L}_{8 \text{ gen}} \times \underbrace{SU(3)_R}_{8 \text{ gen}} \to \underbrace{SU(3)_V}_{8 \text{ gen}}$	$\begin{pmatrix} u \\ d \\ s \end{pmatrix}$	$\pi^\pm, \pi^0, K^\pm, K^0, \bar{K}^0, \eta$	$\frac{m_{u,d,s}}{\Lambda_{QCD}} \sim \frac{1}{3}$
$\underbrace{SU(2)_L}_{3 \text{ gen}} \times \underbrace{SU(2)_R}_{8 \text{ gen}} \to \underbrace{SU(2)_V}_{8 \text{ gen}}$	$\begin{pmatrix} u \\ d \end{pmatrix}$	$\pi^\pm, \pi^0$	$\frac{m_{u,d}}{\Lambda_{QCD}} \sim \frac{1}{50}$

You get a much better expansion parameter if you work with $SU(2)$ but then you get less information.

Matching at "Λ_{QCD}" is non-perturbative so instead you construct Λ_{EFT} for Goldstones based just on symmetry breaking pattern. You will again get $C_i\mathcal{O}_i(\pi, K, \eta)$, but in this case we can calculate the matrix elements since they are perturbative, but the C_i's are non-perturbative and need to be derived from lattice QCD or experiment.

Note: Any other theroy with same symmetry breaking will give the same chiral Lagrangian, $\mathcal{L}_\chi$, just different C_i's. This is just one example of chiral Lagrangian's and we will use it as an example to illustrate our bullets above.

Consider for example the linear σ-model. We define π as a matrix:

$$\pi = \sigma + i\boldsymbol{\tau}\cdot\boldsymbol{\pi} \tag{5.4}$$

where $\boldsymbol{\tau}$ are the Pauli matrices. The full theory is[1],

$$\begin{aligned}\mathcal{L}_\sigma = \frac{1}{4}\mathrm{Tr}\left(\partial_\mu\pi^\dagger\partial^\mu\pi\right) + \frac{\mu^2}{4}\mathrm{Tr}\left(\pi^\dagger\pi\right) - \frac{\lambda}{4}\left(\mathrm{Tr}\left(\pi^\dagger\pi\right)\right)^2 \\ + \bar{\psi}_L i\partial\!\!\!/\psi_L + \bar{\psi}_R i\partial\!\!\!/\psi_R - g\left(\bar{\psi}_L\pi\psi_R + \psi_R\pi^\dagger\psi_L\right)\end{aligned} \tag{5.5}$$

The theory has an $SU(2)_L\times SU(2)_R$ symmetry:

$$\psi_L \to L\psi_L \quad , \quad \psi_R \to R\psi_R \quad , \quad \pi \to L\pi R^\dagger \tag{5.6}$$

The π transformation is a linear infinestimal transformation rule for $\boldsymbol{\pi}, \sigma$.

We have,

$$L = e^{i\boldsymbol{\alpha}_L\cdot\boldsymbol{\tau}} \tag{5.7}$$

$$R = e^{i\boldsymbol{\alpha}_R\cdot\boldsymbol{\tau}} \tag{5.8}$$

The scalar potential is,

$$V = -\frac{\mu^2}{2}\left(\sigma^2 + \boldsymbol{\pi}^2\right) + \frac{\lambda}{4}\left(\sigma^2+\boldsymbol{\pi}^2\right)^2 \tag{5.9}$$

$$= \frac{\lambda}{4}\left(\sigma^2 + \boldsymbol{\pi}^2 - \frac{\mu^2}{\lambda}\right)^2 + \mathrm{const} \tag{5.10}$$

[1] It may not be obvious that such a Lagrangian represents a $SU(2)_L\times SU(2)_R$ symmetry. To see this we need to define π as part of a bigger (Hermitian) matrix,

$$\Pi \equiv \begin{pmatrix} 0 & \pi \\ \pi^\dagger & 0 \end{pmatrix}$$

which leads to the transformation matrix:

$$U \equiv \begin{pmatrix} L & 0 \\ 0 & R \end{pmatrix}$$

and we have under a $SU(2)_L\times SU(2)_R$ transformation,

$$\Pi \to U\Pi U^\dagger$$

Finding the minima, its easy to see that if $\mu^2 > 0$ we have a minima at μ^2/λ.

We give σ a VEV:

$$\langle 0|\sigma|0\rangle \equiv v = \frac{\mu^2}{\lambda} \quad , \quad \langle \boldsymbol{\pi} \rangle = 0 \quad , \quad \tilde{\sigma} = \sigma - v \tag{5.11}$$

The new Lagrangian is,

$$\begin{aligned}\mathcal{L} = &\frac{1}{2}\left[\partial_\mu\tilde{\sigma}\partial^\mu\tilde{\sigma} - 2\mu^2\tilde{\sigma}^2\right] + \frac{1}{2}\left(\partial_\mu\boldsymbol{\pi}\cdot\partial^\mu\boldsymbol{\pi}\right) + \lambda v\tilde{\sigma}\left(\tilde{\sigma}^2 + \boldsymbol{\pi}^2\right) - \frac{\lambda}{4}\left[\tilde{\sigma}^2 + \boldsymbol{\pi}^2\right]^2 \\ &+ \bar{\psi}_L i\partial\!\!\!/\psi_L + \bar{\psi}_R i\partial\!\!\!/\psi_R - gv\left(\bar{\psi}_L\psi_L + \bar{\psi}_R\psi_R\right) \\ &- g\left(\bar{\psi}_L(\tilde{\sigma} - i\boldsymbol{\tau}\cdot\boldsymbol{\pi})\psi_L + \bar{\psi}_R(\tilde{\sigma} - i\boldsymbol{\tau}\cdot\boldsymbol{\pi})\psi_R\right)\end{aligned} \tag{5.12}$$

We have a remanent $SU(2)$ vector symmetry:

$$\tilde{\sigma} \to \tilde{\sigma} \quad , \quad \boldsymbol{\pi} \to V\pi V^\dagger \tag{5.13}$$

The $\tilde{\sigma}$ has a mass but the pions are massless:

$$m_{\tilde{\sigma}}^2 = 2\mu^2 = 2\lambda\sigma^2 \quad , \quad m_\psi = gv \quad , \quad m_{\boldsymbol{\pi}} = 0 \tag{5.14}$$

We now try a few field redefinitions which we use as an organizational tool. We can make the square root redefinition:

$$S = \sqrt{(\tilde{\sigma} + v)^2 + \boldsymbol{\pi}^2} - v = \tilde{\sigma} + ... \tag{5.15}$$

$$\boldsymbol{\phi} = \frac{v\boldsymbol{\pi}}{\sqrt{(\tilde{\sigma} + v)^2 + \boldsymbol{\pi}^2}} = \boldsymbol{\pi} + ... \tag{5.16}$$

or equivalently,

$$\sigma = (S + v)\sqrt{1 - \frac{\boldsymbol{\phi}^2}{v^2}} \tag{5.17}$$

$$\boldsymbol{\pi} = (S + v)\,\frac{\boldsymbol{\phi}}{v} \tag{5.18}$$

We have,

$$\begin{aligned}\mathcal{L} = &\frac{1}{2}\left[(\partial^\mu S)^2 - 2\mu^2 S^2\right] + \frac{1}{2}\left(\frac{v + S}{v}\right)^2\left[(\partial_\mu\boldsymbol{\phi})^2 + \frac{(\boldsymbol{\phi}\cdot\partial_\mu\boldsymbol{\phi})^2}{v^2 - \boldsymbol{\phi}^2}\right] \\ &- \lambda v S^2 - \frac{\lambda}{4}S^4 + \bar{\psi}i\partial\!\!\!/\psi - g\frac{v + S}{v}\bar{\psi}\left[(v^2 - \boldsymbol{\phi}^2)^{1/2} - i\boldsymbol{\phi}\cdot\boldsymbol{\tau}\right]\psi\end{aligned} \tag{5.19}$$

In this expression we have omitted most of the non-renormalizable terms.

There is an alternative representation that's commonly found in the literature where we take (we write a prime on the $\boldsymbol{\pi}$ field since it is slightly different then the ways we defined it above),

$$\pi = \sigma + i\boldsymbol{\tau}\cdot\boldsymbol{\pi} = (v + S)\Sigma \quad , \quad \Sigma = \exp\left(\frac{i\boldsymbol{\tau}\cdot\boldsymbol{\pi}'}{v}\right) \tag{5.20}$$

which gives,

$$\mathcal{L} = \frac{1}{2}\left[(\partial^\mu S)^2 - 2\mu^2 S^2\right] + \frac{(v+S)^2}{4}\mathrm{Tr}\left(\partial_\mu \Sigma \partial^\mu \Sigma^\dagger\right) - \lambda v S^3 - \frac{\lambda}{4}S^4$$
$$+ \bar{\psi} i \partial\!\!\!/ \psi - g(v+S)\left(\bar{\psi}_L \Sigma \psi_R + \bar{\psi}_R \Sigma \psi_L\right) \tag{5.21}$$

The final representation will be different and is known as the non-linear chiral Lagrangian. To get there we just drop S, ψ (the massive fields). In other words, we are integrating out the heavier degrees of freedom. This gives,

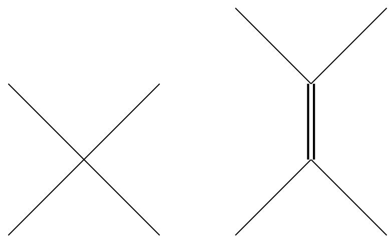
$$\mathcal{L}_\chi = \frac{v^2}{4}\mathrm{Tr}\left[\partial_\mu \Sigma \partial^\mu \Sigma^\dagger\right] \tag{5.22}$$

The first three actions we wrote down were identical but this final one is only equivalent for low energy phenomenology to $\mathcal{L}_\chi$.

For example consider Goldstone boson scattering:

$$\pi^+\pi^0 \to \pi^+\pi^0 \tag{5.23}$$

where $q \equiv p_+' - p_+ = p_0 - p_0'$. We can have,



The amplitude in the different representation is[2],

$$\text{Linear} = -i2\lambda + (-2i\lambda v)^2 \frac{i}{q^2 - m^2} = (-2i\lambda)\left(1 + \frac{2\lambda v^2}{q^2 - 2\lambda v^2}\right) = iq^2/v^2 + ... \tag{5.24}$$

$$\text{Square Root} = iq^2/v^2 + \mathcal{O}(q^4) \tag{5.25}$$

$$\text{Exponential} = iq^2/v^2 + \mathcal{O}(q^4) \tag{5.26}$$

$$\text{Non-linear} = iq^2/v^2 + 0 \tag{5.27}$$

Thus as expected to leading order all the amplitudes are the same (they are the same to all orders in the first three representations). However, the change is in the way we think about each diagram. The linear representation doesn't contain any derivative interactions. That's why a propagator was needed to get all diagrams of over q^2/v^2. From this point of view the linear representation is the least convenient. It hides what we think of as being leading order. Every other representation can work well.

[2]To calculate the amplitudes you needs to be careful with symmetry factors and use $m = 2\mu^2 = 2\lambda v^2$

The non-linear Lagrangian is the most convenient since it only has a low energy, Σ field with derivative couplings. for $SU(2)_L \times SU(2)_R$ we have,

$$S \to S \quad , \quad \Sigma \to L\Sigma R^\dagger \tag{5.28}$$

with

$$\Sigma = \exp\left(\frac{i\boldsymbol{\tau}\cdot\boldsymbol{\pi}}{v}\right) \tag{5.29}$$

and $\boldsymbol{\pi}$ transforms non-linearly. In an infinestimal transformation we have,

$$\Sigma \to \Sigma' = L\Sigma R^\dagger \tag{5.30}$$

$$1 + i\tau^a\frac{\pi'_a}{v} = (1 - i\tau^a\alpha_{L,a})\left(1 + i\tau^a\frac{\pi_a}{v}\right)(1 + i\tau^a\alpha_{R,a}) + \mathcal{O}(\pi^2) \tag{5.31}$$

$$i\tau^a\frac{\pi'_a}{v} = i\tau^a\frac{\pi_a}{v} - i\frac{\tau^a}{2}(\alpha_{L,a} - \alpha_{R,a}) + \mathcal{O}(\pi^2) \tag{5.32}$$

$$\Rightarrow \pi'_a = \pi_a - \frac{v}{2}(\alpha_{L,a} - \alpha_{R,a}) + \mathcal{O}(\pi^2) \tag{5.33}$$

Thus to lowest order the transformed π field is shifted. This tells you that the theory must be derivatively coupled. The π^2 and higher terms mix between the different pions.

To get to our final convenient Lagrangian we went through the linear sigma model. This was an unecessary detour and we would like to write down $\mathcal{L}_\chi$ from the start. To do that we go back to study the symmetry breaking pattern.

Consider a system invariant under a group G with elements g. The vacuum state ($|0\rangle$) is not invariant under G but only under a subgroup of G, H, with elements, h. Explicitly:

$$U(g)\,|0\rangle \neq |0\rangle \tag{5.34}$$

$$U(h)\,|0\rangle = |0\rangle \tag{5.35}$$

The number of massive bosons are equal to the dimension of H, while the number of Goldstone bosons are equal to the dimension of the coset, G/H (which is also equal to the number of broken generators). We say the the Goldstone bosons live in the coset space, G/H.

The symmetry breaking pattern is $G \to H$ coset is parametrized by Σ. We have a generator,

$$g \in (L, R) \to h \in (V, V) \tag{5.36}$$

We have,

$$g = (g_L, g_R) = \Xi(x)h \tag{5.37}$$

where $\Xi(x)$ is some field to parameterize fluctuations where we pull out the subgroup, h. The notation is such that,

$$(g_1, g_2)\,(g_3, g_4) \equiv (g_1g_3, g_2g_4) \tag{5.38}$$

We can use this to find a convenient choice of Ξ:

$$\overbrace{(g_L, g_R)}^{\Xi(x)}\overbrace{(g_V, g_V)}^{\in H} = (g_L g_V, g_R g_V) = \left(g_L g_R^\dagger g_R g_V, g_R g_V\right) \tag{5.39}$$

$$= (\underbrace{g_L g_R^\dagger}_{\Sigma = g_L g_R^\dagger}, 1)\underbrace{(g_R g_V, g_R g_V)}_{\in (V,V)} \tag{5.40}$$

$\Sigma \equiv g_L g_R^\dagger$ parametrizes the coset and transforms as,

$$\Sigma \to L\Sigma R^\dagger \tag{5.41}$$

In terms of broken generators, X, then a very general definition of what Ξ is given by,

$$\Xi(x) = \exp\left(iX^a \frac{\pi^a(x)}{v}\right) \tag{5.42}$$

This is found using what's known as the CCWZ perscription.

You can pick different choices for the broken generators, X^a. If we take

$$X^a = \tau_L^a \tag{5.43}$$

then we get what we got above:

$$\Xi(x) = (\underbrace{e^{i\boldsymbol{\tau}_L\cdot\boldsymbol{\pi}/v}}_{\Sigma(x)}, \underbrace{e^0}_{1}) \tag{5.44}$$

and we can also derive, $\Sigma \to L\Sigma R^\dagger$.

Another possible choice is given by,

$$X^a = \tau_L^a - \tau_R^a \tag{5.45}$$

which gives a different $\Xi(x)$ and is usually denoted by ξ ([Q 7: If I understand correctly, $\xi(x) = e^{i(\boldsymbol{\tau}_L+\boldsymbol{\tau}_R)\cdot\boldsymbol{\pi}/v}$, where the plus sign came from compex conjugationg in $g_L g_R^\dagger$])

We avoid further discussion, but the key point is that there is a way of thinking about chiral Lagrangian's even from starting from the low energy point of view.

For QCD, common convension is, $v = f/\sqrt{2}$. We have[3],

$$\Sigma = \exp\left(\frac{2iM}{f}\right) \quad , \quad M = \frac{\pi^a\tau^a}{\sqrt{2}} = \begin{pmatrix} \pi^0/\sqrt{2} & \pi^+ \\ \pi^- & -\pi^0/\sqrt{2} \end{pmatrix} \tag{5.46}$$

which gives,

$$\mathcal{L}_\chi = \frac{f^2}{8}\mathrm{Tr}\left(\partial_\mu\Sigma^\dagger\partial^\mu\Sigma\right) \tag{5.47}$$

$$= \frac{f^2}{8}\mathrm{Tr}\left\{\left(-\frac{2i}{f}\partial_\mu M - \frac{1}{2f^2}\left[M, \partial_\mu M\right]^\dagger + ...\right)\left(\frac{2i}{f}\partial^\mu M - \frac{1}{2f^2}\left[M, \partial_\mu M\right] + ...\right)\right\} \tag{5.48}$$

$$= \frac{1}{2}\mathrm{Tr}\left(\partial_\mu M\partial^\mu M\right) + \frac{1}{6f^2}\mathrm{Tr}\left(\left[M, \partial_\mu M\right]^2\right) + ... \tag{5.49}$$

[3]In order to derive the form of M we need to put the Pauli matrices in the adjoint representation These are the states that are charge eigenstates and hence the interaction basis of EM.

where we have used that the trace of 3 M's is zero since $\mathrm{Tr}\left[\tau_i\tau_j\tau_k\right]=0$.

In the SM the symmetry is also broken explicitly (technically its broken spontaneously through the Higgs but this breaking happens at a much higher scale and so its effectively a explicit breaking at this low scale). To add this effect we do what's known as a spurion analysis.

A spurion analysis lets you find the correct low energy physics terms to put in your Lagrangian. They are unique in the sense that any UV physics will integrate out to this form. The idea is to find the term that you know breaks your symmetry. Then let the coupling of that term transform as if it were a field (which we call a "spurion") such that the term is now invariant under that symmetry. Now write down all terms that are invariant assuming this new transformation rule. Once this is done you can return the spurion to its constant value.

Consider the QCD Lagrangian,

$$\mathcal{L}_{QCD}=\bar{\psi}\left(i\not{D}-m\right)\psi \tag{5.50}$$

The mass term breaks the chiral symmetry as it is given by,

$$\mathcal{L}_m=-\bar{\psi}_L m_q\psi_R-\bar{\psi}_R m_q\psi_L \quad , \quad m_q=\begin{pmatrix} m_u & 0 \\ 0 & m_d \end{pmatrix} \tag{5.51}$$

The Lagrangian is invariant under the chiral symmetry,

$$\psi\rightarrow\begin{pmatrix} L & 0 \\ 0 & R \end{pmatrix}\psi \tag{5.52}$$

if m_q were to transform as,

$$m_q\xrightarrow{G}Lm_qR^\dagger \tag{5.53}$$

We now find all terms invariant under this transformation law. In this case we get one extra term, $\mathcal{L}_\chi^{mass}=v_0\mathrm{Tr}\left(m_q^\dagger\Sigma+m_q\Sigma^\dagger\right)$, with v_0 being some constant with dimensions of $(\mathrm{mass})^3$. Now we can return to treating m_q as a constant. To find the masses of the pion we expand Σ:

$$m_q\Sigma^\dagger=m_q\left(1-\frac{2iM}{f}-\frac{4M^2}{2!f^2}-...\right) \tag{5.54}$$

which gives,

$$\begin{aligned}
\mathcal{L}_m &= v_0\mathrm{Tr}\left[m_q\Sigma^\dagger+h.c.\right] && (5.55)\\
&= -4\frac{v_0}{f^2}\mathrm{Tr}\left[m_qM^2\right] && (5.56)\\
&= -4\frac{v_0}{f^2}\mathrm{Tr}\left[\begin{pmatrix} m_u & 0 \\ 0 & m_d \end{pmatrix}\begin{pmatrix} \pi^0/\sqrt{2} & \pi^+ \\ \pi^- & -\pi^0/\sqrt{2} \end{pmatrix}\begin{pmatrix} \pi^0/\sqrt{2} & \pi^+ \\ \pi^- & -\pi^0/\sqrt{2} \end{pmatrix}\right] && (5.57)\\
&= -\underbrace{4\frac{v_0}{f^2}\left(m_u+m_d\right)}_{m_\pi^2}\left(\frac{1}{2}(\pi^0)^2+\pi^+\pi^-\right) && (5.58)
\end{aligned}$$

Note that π^0 is a real scalar so the pions are all degenerate.

We can now couple currents to our fields as well. We can have a left handed fermion current (could be coupled to the $W^\pm$ for example),

$$J^a_{L,\mu} = \bar{\psi}\gamma_\mu P_L \tau^a \psi \tag{5.59}$$

To find out how to add such a term to the Lagrangian we can think of getting the current through,

$$J^a_{L,\mu} = -\frac{\delta \mathcal{L}}{\delta \ell^a_\mu(x)} \tag{5.60}$$

for some ℓ^μ which will be our spurion,

$$\ell^\mu = \ell^\mu_a \tau^a \tag{5.61}$$

(this is closely related to the W_μ boson).

We take it to transform like a left handed gauge field [4] ,

$$\ell^\mu \rightarrow L(x)\ell^\mu L^\dagger(x) + \left(\partial_\mu L(x)\right) L^\dagger(x) \tag{5.62}$$

This gives you an invariant in the original theory through the covariant derivative,

$$D^\mu \Sigma = \partial^\mu \Sigma + i\ell^\mu \Sigma \tag{5.63}$$

The spurion is letting us track the symmetry breaking.

5.1 Feynman Rules, Power Counting, and Loops

The Lagrangian is,

$$\mathcal{L}^{(0)} = \frac{f^2}{8}\mathrm{Tr}\left(\partial^\mu \Sigma^\dagger \partial_\mu \Sigma\right) + v_0 \mathrm{Tr}\left(m_q^\dagger \Sigma + m_q \Sigma^\dagger\right) \tag{5.64}$$

Note that f and v_0 are not small so there is no small coupling constant to expand around as done in typical perturbation theory. Instead we will do our power counting in powers of momenta or pion mass. We have,

$$\partial^2 \sim p^2 \sim m_\pi^2 \sim \frac{v_0 m_q}{f^2} \tag{5.65}$$

[4] To see why this is the correct transformation law consider the gauge covariant derivative of the weak interaction, $D_\mu\psi = \partial_\mu\psi - igW^a_\mu t_a \psi$. Under a gauge transformation we have,

$$\begin{aligned} igW_\mu\psi &= \partial_\mu\psi - D_\mu\psi \\ &\rightarrow \partial_\mu\left(L(x)\psi\right) - L(x)D_\mu\psi \\ &= \left((\partial_\mu L)L^\dagger + LW_\mu L^\dagger\right)\psi' \end{aligned}$$

which implies that a gauge field transforms as

$$igW_\mu \rightarrow (\partial_\mu L)L^\dagger + LW_\mu L^\dagger$$

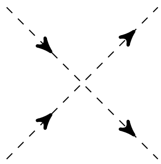
We expand

$$\frac{p^2}{\Lambda_\chi^2}, \frac{m_\pi^2}{\Lambda_X^2} \ll 1 \tag{5.66}$$

where Λ_χ is some cutoff scale. This is both a derivative and m_π expansion.

If we look at the original Lagrangian we have, [Q 8: Calculate these Feynman rules]

$$\to \frac{i}{p^2 - m_\pi^2}$$

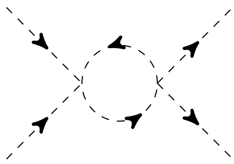


$$\sim \frac{p^2}{f^2} = \frac{m_\pi^2}{f^2}$$

from the lowest order expansion of the Lagrangian,

$$\mathcal{L} = \frac{1}{6f^2}\mathrm{Tr}\left([M, \partial_\mu M]\,[M, \partial^\mu M]\right) + \frac{4v_0}{f^2}\mathrm{Tr}\left(m_q M^2\right) \tag{5.67}$$

5.1.1 Loops and Λ_χ



$$\sim \int d\!\!{}^-{}^d\ell \frac{(p'_- - \ell)^2}{f^2}\frac{1}{(\ell^2 - m_\pi^2)}\frac{1}{[(\ell - p_+ - p_-)^2 - m_\pi^2]}\frac{(\ell - p'_+)^2}{f^2}$$

Here we use dim-reg since dim-reg preserves chiral symmetry. The finite part of the loop is roughly given by,

$$\frac{(p^4, p^2 m_\pi^2, m_\pi^4)}{(4\pi)^2 f^4} \sim \underbrace{\frac{(p^2, m_\pi^2)}{f^2}}_{\text{tree level 4-pt}} \frac{p^2 \text{ or } m_\pi^2}{(4\pi f)^2} \tag{5.68}$$

Loops are suppressed by $\frac{p^2}{\Lambda_\chi^2}$ where $\Lambda_\chi = 4\pi f \sim$ GeV. In practice we know that the ρ meson appears at about ~ 770MeV, so our cutoff is roughly of the correct order.

We now introduce use a convenient rescaling:

$$\frac{f^2}{8}\chi \equiv m_q v_0 \tag{5.69}$$

now the chiral Lagrangian at tree level is given by,

$$\mathcal{L}_\chi^{(0)} = \frac{f^2}{8}\mathrm{Tr}\left(\partial_\mu \Sigma^\dagger \partial^\mu \Sigma\right) + \frac{f^2}{8}\mathrm{Tr}\left(\chi^\dagger \Sigma + \chi \Sigma^\dagger\right) \tag{5.70}$$

We want to use dim-reg and so we need to look at dimensions of objects:

$$[M] = 1 - \epsilon \tag{5.71}$$
$$[f] = 1 - \epsilon \tag{5.72}$$

Therefore[5],

$$f^{bare} = \mu^{-\epsilon} f \tag{5.73}$$
$$v_0^{bare} = \mu^{-2\epsilon} v_0 \tag{5.74}$$

There are no μ's in physical quantities such as in, $m_\pi, m_q, ...$. Thus we must have,

$$\frac{v_0^{bare}}{f_{bare}^2} = \frac{v_0}{f^2} \tag{5.75}$$

Notice that there is an important difference between a chiral pertrubation theory and a gauge theory. In a gauge theory the couplings get a renormalization factors, Z_i. But in chiral pertrubation theory the loops don't renormalize the leading order Lagrangian, they renormalize something else. The two point functions never get renormalized since loops are suppressed by either p^2 or m_π^2.

When you do the loop calculation you do get a UV divergence,

$$\text{loop value } \sim \frac{1}{\epsilon} + \log\frac{\mu^2}{p^2}, \frac{1}{\epsilon} + \log\frac{\mu^2}{m_\pi^2} \tag{5.76}$$

Generically both the momentum and pion mass will show up. The loops enter at

$$\mathcal{O}(p^4) \sim \mathcal{O}(p^2 m_\pi^2) \sim \mathcal{O}(m_\pi^4) \tag{5.77}$$

To cancel the $1/\epsilon$ we need a counterterm, but not in the leading order Lagrangian but instead at the loop Lagrangian. At this order we also have new operators that are the same order in our power counting.

For $SU(2)$:[Q 9: Write down all possible terms explicitly.]

$$\mathcal{L} = L_1 \left(\mathrm{Tr}\left(\partial_\mu \Sigma \partial^\mu \Sigma^\dagger\right)\right)^2 + L_2 \mathrm{Tr}\left(\partial_\mu \Sigma \partial_\nu \Sigma^\dagger\right) \mathrm{Tr}\left(\partial^\mu \Sigma \partial^\nu \Sigma^\dagger\right) + ... \tag{5.78}$$

where the ellipses stand for terms involving one m_q (or equivalently one χ) and two ∂'s or 2 m_q's. Its δL_i counterterms that cancel the $1/\epsilon$ UV divergences. The theory is renormalizable order by order.

One can show that the equation of motion is [Q 10: Calculate]

$$(\partial^2\Sigma)\Sigma^\dagger - \Sigma(\partial^2\Sigma^\dagger) - \chi\Sigma^\dagger + \Sigma\chi^\dagger + \frac{1}{2}\mathrm{Tr}\left(\chi\Sigma^\dagger - \Sigma\chi^\dagger\right) = 0 \tag{5.79}$$

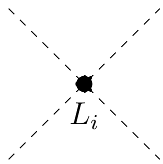
[5] The 2ϵ instead of ϵ can be seen from eq. 5.69 and the fact that m_q and χ are physical quantities.

This can be used to get rid of the $\partial^2\Sigma$ contributions. Furthermore, we also have some $SU(2)$ identities, [Q 11: prove]

$$\mathrm{Tr}\left(\partial^\mu\Sigma\partial_\mu\Sigma^\dagger\partial_\nu\Sigma\partial^\nu\Sigma^\dagger\right) = \frac{1}{2}\left[\mathrm{Tr}\left(\partial_\mu\Sigma\partial^\mu\Sigma^\dagger\right)\right]^2 \tag{5.80}$$

$$\mathrm{Tr}\left(\partial^\mu\Sigma\partial^\nu\Sigma^\dagger\partial_\mu\Sigma\partial_\nu\Sigma^\dagger\right) = ... \tag{5.81}$$

At $\mathcal{O}(p^4)$ we include both loops $\sim p^4 \log\frac{\mu^2}{p^2}$ and we include terms like $\sim p^4 L_{1,2}(\mu)$ terms from



The μ dependence cancels between the two contributions. One can think of μ as a cutoff between the low energy physics in loops and the high energy coefficient, $L_i(\mu)$. The difference between this and integrating out the massive particle is not the physics about where things go because the low energy physics always goes in the matrix elements and the high energy physics is in the couplings. The difference is that we can calculate the matrix elements explicitly because our theory is in terms of the right degrees of freedoms and the coefficients are unknowns. This is the difference between the bottom up (which we work in here) and top down approach which we take here.

We expect,

$$\frac{L_i(\mu)}{f^2} = \frac{1}{(4\pi f)^2}\left[a_i \log\frac{\mu}{\Lambda_\chi} + b_i\right] \tag{5.82}$$

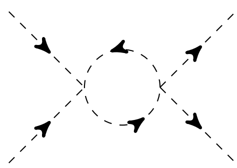
and

$$a_i \sim b_i \sim 1 \tag{5.83}$$

This is "Naive Dimensional Analysis". Changing μ moves pieces back and forth from low energy to high energy. The sum is μ independent but each indiviaul part is not, thus we expect them to be the same order of magnitude, hence $a_i \sim b_i \sim 1$.

Typically we pick $\mu \approx m_\rho$, m_χ, or something in between $\mu \approx 1\text{GeV}$. The idea is to put the large logs in the matrix elements not in the coefficients. Here there is not an infinite series of large logarithms that you need to resum which is related to the fact that the kinetic term never gets renormalized. We simply have one log in the renormalization group. We calculate matrix elements and then fit the coefficients L_i to data.

Now lets consider what would happen if we used a hard cutoff, Λ. The advantage is that we would be able to explicitly see the difference between low and high energy physics. The (large) price to pay is,



$$\sim \frac{\Lambda^4}{\Lambda_{\chi^2}}, \frac{\Lambda^2 p^2}{\Lambda_\chi^2}, \frac{p^4\Lambda}{\Lambda_\chi^4}$$

Λ^4/Λ_χ^4 breaks chiral symmetry since it has no counter term. $\Lambda^2 p^2/\Lambda_\chi^2$ breaks power counting. This can be absorbed into your leading order counting. The $p^4 \log \Lambda/\Lambda_\chi^4$ can also be absorbed in $L_i p^4$ as in dim-reg. We won't use a cutoff since we don't want to think of terms that break power-counting or chiral symmetry.

5.1.2 Infrared Divergences

∂^μ couplings make IR nicer and we since usually have good $p^2, m_\pi^2 \to 0$ limit of your results.

5.1.3 Phenomenology

As a phenomenological example consider $\pi\pi \to \pi\pi$ scattering. Below inelastic thresholds (no $\pi\pi \to 4\pi$), the scattering is particularly simple. The S matrix is [Q 12: How did we get this?]

$$S_{\ell,I} = e^{2i\delta_{\ell,I}} \tag{5.84}$$

where I is the isospin and ℓ is the angular momentum. So in other words the S matrix is just a phase as in simple quantum mechanical scattering.

One can then use an effective range expansion,

$$p^{2\ell+1} \cot \delta_{\ell,I} = -\frac{1}{a_{\ell,I}} + \frac{r_0^{\ell,I}}{2} p^2 + ... \tag{5.85}$$

This is a general result from QM. If you do an explicit calculation using chiral perturbation theory then you can find the $a_{\ell,I}$ coefficients.

5.1.4 General Power Counting

Lets consider an arbitrary diagram in this theory with N_v vertices, N_I internal lines, N_E external lines, and N_L number of loops. We don't want to restrict ourselves to leading order vertices so we write,

$$N_v = \sum_n N_n \tag{5.86}$$

where N_n is the number of vertices that go as p^n, m_π^n.

We assume we use dim-reg so we can just count mass dimension. We count the Λ_χ factors for matrix elements, $\mathcal{M}$, of N_E external pions.

$$(\Lambda_\chi)^{\sum_n N_n(4-n)} \tag{5.87}$$

For lowest order, $n = 2 \to f^2$. For $n_4 \to L_i$ which were dimensionless. There are alos f's that come with the pions,

$$(\Lambda_\chi)^{-2N_I - N_E} \tag{5.88}$$

The Euler identity tells us that

$$N_I = N_L + N_v - 1 \tag{5.89}$$

which we can use to eliminate N_I.

$$\mathcal{M} \sim (\Lambda_\chi)^{\sum_n N_n(4-n) - N_E - 2N_L - 2\sum_n N_n + 2} E^D f\left(\frac{E}{\mu}\right) \tag{5.90}$$

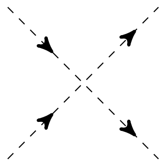
$$\sim (\text{mass})^{4-N_E} \tag{5.91}$$

where E is m_π or p and D is some parameter which initially we know nothing about. We have different objects giving mass dimensions, f's and E's. But because we know the mass dimension of the final answer we get,

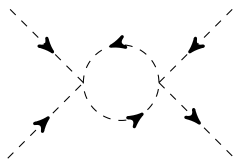
$$D = 2 + \sum_n N_n (n-2) + 2N_L \tag{5.92}$$

which implies that $D \geq 2$ and when we add vertices or loops we get more E's.

People often refer to this process as p-counting. We just need to count loops and higher order vertices. For example, $\mathcal{L}^{(0)}$:



$D = 2$



$D = 4$

5.1.5 $SU(3)$

In $SU(3)$ we have,

$$m_q = \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_d & 0 \\ 0 & 0 & m_s \end{pmatrix} \tag{5.93}$$

we have,

$$M \equiv \frac{\pi^a \lambda^a}{\sqrt{2}} = \begin{pmatrix} (\pi^0 + \eta^0)/\sqrt{2} & \pi^+ & K^+ \\ \pi^- & -(\pi^0 - \eta)/\sqrt{2} & K^0 \\ K^- & \bar{K}^0 & -\sqrt{2}\eta \end{pmatrix} \tag{5.94}$$

where we have written out the mesons in the charged basis. There is a freedom in what basis to pick and in different situations different bases are more convenient. If we expand,

$$\text{Tr}\left(\Sigma m_q^\dagger + m_q \Sigma^\dagger\right) \tag{5.95}$$

then you get masses for the mesons. Since the group is larger the theory is more predictive and you also get,

$$m_{K^0}^2 = m_{\bar{K}^0}^2 = \frac{4v_0}{f^2}(m_d + m_s) \tag{5.96}$$

and there is a mixing between η and π^0 since they have the same quantum numbers:

$$m^2 = \frac{4v}{f^2}\begin{pmatrix} m_u & (m_u - m_d)/\sqrt{3} \\ (m_u - m_d)/\sqrt{3} & (4m_s + m_u + m_d)/3 \end{pmatrix} \tag{5.97}$$

where the mixing is isospin violating since it takes the form, $m_u - m_d$. Often when you do calulations keeping m_u, m_d, m_s as all separate parameters is a little much. We often ignore isospin violation and take

$$m_{u,d} \equiv \hat{m} \equiv \frac{1}{2}(m_u + m_d), \quad m_s \gg \hat{m} \tag{5.98}$$

For higher orders we have,

$$\mathcal{L}^{(0)} = \frac{f^2}{8}\mathrm{Tr}\left[D_\mu\Sigma^\dagger D^\mu\Sigma + \chi^\dagger\Sigma + \chi\Sigma^\dagger\right] \tag{5.99}$$

where

$$D_\mu\Sigma \equiv \partial_\mu\Sigma + i\ell_\mu\Sigma \tag{5.100}$$

For power counting purposes:

$$\Sigma \sim 1, \quad D_\mu\Sigma \sim p, \quad \ell_\mu \sim p, \quad \Sigma \sim m_q \sim p^2 \tag{5.101}$$

and we can then enumerate our higher order terms as[6],

$$\begin{aligned}
\mathcal{L}^{(1)} = {} & L_i\left[\mathrm{Tr}\left(D_\mu\Sigma D^\mu\Sigma^\dagger\right)\right]^2 + L_2\mathrm{Tr}\left(D_\mu\Sigma D^\nu\Sigma^\dagger\right)\mathrm{Tr}\left(D^\mu\Sigma D^\nu\Sigma^\dagger\right) \\
& + L_3\mathrm{Tr}\left[D_\mu\Sigma D^\mu\Sigma^\dagger D_\nu\Sigma D^\nu\Sigma^\dagger\right] + L_4\mathrm{Tr}\left(D_\mu\Sigma D^\mu\Sigma^\dagger\right)\mathrm{Tr}\left(\chi\Sigma^\dagger + \Sigma\chi^\dagger\right) \\
& + L_5\mathrm{Tr}\left(D_\mu\Sigma D^\mu\Sigma^\dagger(\chi\Sigma^\dagger + \Sigma\chi^\dagger)\right) + L_6\left[\mathrm{Tr}\left(\chi\Sigma^\dagger + \Sigma\chi^\dagger\right)\right]^2 \\
& + L_7\left[\mathrm{Tr}\left(\chi^\dagger\Sigma - \Sigma^\dagger\chi\right)\right]^2 + L_8\mathrm{Tr}\left(\chi\Sigma^\dagger\chi\Sigma^\dagger + \Sigma\chi^\dagger\Sigma\chi^\dagger\right) \\
& + L_9\mathrm{Tr}\left[L_{\mu\nu}D^\mu\Sigma D^\nu\Sigma^\dagger\right]
\end{aligned} \tag{5.102}$$

where

$$L_{\mu\nu} \equiv [D_\mu, D_\nu] = \partial_\mu\ell_\nu - \partial_\nu\ell_\mu + i\left[\ell_\mu, \ell_\nu\right] \tag{5.103}$$

We have 9 operators when we couple ℓ^μ and χ. In the above we have used both the equations of motion and $SU(3)$ relations.

[6] In the following we also impose parity as we know its a good symmetry in QCD.

We can make a correspondence between $SU(2)$ and $SU(3)$. In the $SU(2)$ theory the Kaon is in the coefficients. A proper correspondence gives,

$$2L_1^{(2)} + L_3^{(2)} = 2L_1 - L_3 - \frac{1}{96(4\pi)^2}\left(1 + \log\frac{\mu^2}{m_K^2}\right) \tag{5.104}$$

so what you think of the coefficients of your theory depend on the matter that you put in.

Renormalization of L_i gives,

$$L_i = \bar{L}_i + \delta L_i \tag{5.105}$$

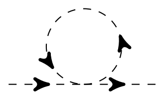
where

$$\delta L_i = \frac{\gamma_i}{32\pi^2}\Big(\frac{1}{\epsilon} - \log 4\pi + \gamma_E \underbrace{-1}_{\text{convention}}\Big) \tag{5.106}$$

The pion mass is,

$$m_0^2 = \frac{4v_0}{f^2}\left(m_u + m_d\right) \equiv 2B_0\hat{m} \tag{5.107}$$

A mass shift is given by,



In $SU(2)$ we have,

$$m_\pi^2 = m_0^2\left[1 - \frac{16m_0^2}{f^2}\left(2\bar{L}_4^{(2)} + \bar{L}_5^{(2)} - 4\bar{L}_6^{(2)} - \bar{L}_8^{(2)}\right) + \frac{m_0^2}{(4\pi f)^2}\log\frac{m_0^2}{\mu^2}\right] \tag{5.108}$$

In $SU(3)$ we have,

$$f_\pi = f\left(1 - 2\mu_\pi - \mu_K + \frac{16\hat{m}B_0}{f^2}\bar{L}_5 + \left(m_s + 2\hat{m}_0\right)\frac{16B_0}{f^2}\bar{L}_4\right) \tag{5.109}$$

where,

$$\mu_i \equiv \frac{m_i^2}{(4\pi f)^2}\log\frac{m_i^2}{\mu^2} \tag{5.110}$$

and f is the parameter in $\mathcal{L}^{(0)}$.

Chapter 6

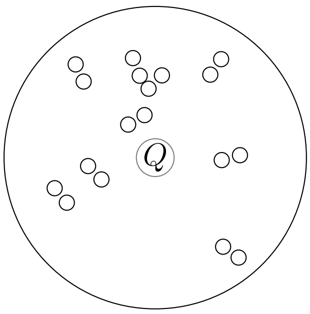
Heavy Quark Effective Theory

With this effective theory our goals are

- Build *Lagrangian* with labelled fields, H_v.
- Heavy Quark Symmetry with covariant representations
- Anomalous dimensions that are functions
- Reparameterization Invariance
- Limitations of $\overline{MS}$ which arise from power-like scale separation and renormalons

When we work with Heavy Quark Effective Theory (HQET) we don't want to remove the heavy quarks from the theory, but we want to see the effects of the light degrees of freedom on the heavy quark. In other words, we just want to "tickle" it with light particles. In other words its an EFT for sources that can wiggle.

A typical picture would be,



One heavy quark and lots of light junk. In the quark model the degrees of freedom are $Q\bar{q}$ mesons. There are two scales in the problem. The size and the mass,

$$r^{-1} \sim \Lambda_{QCD} \ll m_Q \tag{6.1}$$

We want to describe fluctuations of Q due to lighter degrees of freedom. We take,

$$\lim_{m_Q \to \infty} \mathcal{L} = \lim_{m_Q \to \infty} \bar{Q} \left(i\not{D} - m_Q \right) Q \tag{6.2}$$

where $D_\mu \equiv \partial_\mu - igA_\mu^a T^a$. Notice that the mass is upstairs so its not obvious as it was in chiral perturbation theory how to proceed.

Instead lets start with the Feynman rules. Consider the propagator for a heavy quark. We want it to be on-shell up to corrections of order Λ_{QCD}. We furthermore, give it four velocity v^μ with $v^2 = 1$. With this in mind we can write,

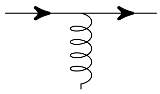
$$p^\mu = m_Q v^\mu + k^\mu \tag{6.3}$$

where $k^\mu \sim \Lambda_{QCD}$ descirbes how off-shell the heavy quark is. We can now write,

$$i\frac{(\not{p} + m_Q)}{p^2 - m_Q^2 + i\epsilon} = \frac{i\,(m_Q\not{v} + m_Q + \not{k})}{2m_Q v\cdot k + k^2 + i\epsilon} \tag{6.4}$$

$$= i\left(\frac{1+\not{v}}{2}\right)\frac{1}{v\cdot k + i\epsilon} + \mathcal{O}\left(\frac{1}{m_Q}\right) \tag{6.5}$$

We can also consider the vertices in this theory,



$$= -ig\gamma^\mu T^A$$

If we are dealing with n-point functions then we will always have projectors around the vertex due to the propagator,

$$\frac{1+\not{v}}{2}\gamma^\mu\frac{1+\not{v}}{2} = \frac{1}{4}\left[\gamma^\mu + \{\not{v},\gamma^\mu\} + \not{v}\gamma^\mu\not{v}\right] \tag{6.6}$$

$$= \frac{1}{4}\left[2v^\mu + \not{v}2v^\mu\right] \tag{6.7}$$

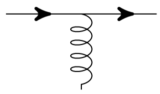
$$= v^\mu\frac{1+\not{v}}{2} \tag{6.8}$$

$$= \frac{1+\not{v}}{2}v^\mu\frac{1+\not{v}}{2} \tag{6.9}$$

where we have used the fact that

$$\frac{1+\not{v}}{2}\frac{1+\not{v}}{2} = \frac{1+\not{v}}{2} \tag{6.10}$$

So we can make the replacement $\gamma^\mu \to v^\mu$. The gauge vertex becomes,



$$= -igv^\mu T^A$$

We see that the interaction is going to be independent of the spin structure of the heavy quark!

With this propagator and interaction we can write down the HQET Lagrangian,

$$\mathcal{L}_{HQET} = \bar{Q}_v iv\cdot DQ_v \tag{6.11}$$

where the Q_v field satisfies

$$\frac{1+\not{v}}{2}Q_v = Q_v \tag{6.12}$$

and

$$iv\cdot D = iv\cdot\partial - gv\cdot A \tag{6.13}$$

This field describes the heavy quark.

Our derivation of the HQET Lagrangian was in an indirect derivation. We now rederive this result directly instead.

We start with a convenient field redefintion,

$$Q(x) \equiv e^{-im_Q v\cdot x}\left[Q_v(x) + B_v(x)\right] \tag{6.14}$$

$$\Leftrightarrow Q_v = e^{im_Q v\cdot x}\frac{1+\not{v}}{2}Q(x), \quad B_v = e^{im_Q v\cdot x}\frac{1-\not{v}}{2}Q(x) \tag{6.15}$$

where

$$\frac{1+\not{v}}{2}Q_v = Q_v, \quad \frac{1-\not{v}}{2}B_v = B_v \tag{6.16}$$

Rearranging gives,

$$\not{v}Q_v = Q_v, \quad \not{v}B_v = -B_v \tag{6.17}$$

To understand the motivation for extracting the phase consider acting on Q_v with the momentum operator,

$$\mathcal{P}_\mu Q_v(x) = e^{im_Q v\cdot x}\frac{1+\not{v}}{2}\left(m_Q v_\mu + \mathcal{P}_\mu\right)Q(x) \tag{6.18}$$

Therefore, the momentum of Q_v (and B_v) is the same as Q except shifted by $m_Q v_\mu$ (the momenta of the heavy quark).

We can break the $\not{D}$ into two pieces:

$$i\not{D} = \not{v}iv\cdot D + i\not{D}_T \tag{6.19}$$

where

$$D_T^\mu \equiv D^\mu - v^\mu v\cdot D \tag{6.20}$$

such that $v\cdot D_T = 0$.

So,

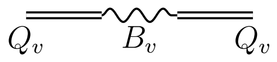
$$\begin{aligned}
\mathcal{L}_{QCD} &= \left[\bar{Q}_v + \bar{B}_v\right]e^{im_Q v\cdot x}\left\{\not{v}iv\cdot D + i\not{D}_T - m_Q\right\}e^{-im_Q v\cdot x}\left[Q_v + B_v\right] && (6.21)\\
&= \left[\bar{Q}_v + \bar{B}_v\right]e^{i(\ldots)}e^{-i(\ldots)}\left((\not{v}-1)m_Q + \not{v}iv\cdot D + i\not{D}_T\right)(Q_v + B_v) && (6.22)\\
&= \bar{Q}_v iv\cdot DQ_v - \bar{B}_v\left(iv\cdot D + 2m_Q\right)B_v + \bar{Q}_v i\not{D}_T B_v + \bar{B}_v i\not{D}_T Q_v && (6.23)
\end{aligned}$$

where we have used[1],

$$\not{D}_T\frac{1-\not{v}}{2} = \frac{1+\not{v}}{2}\not{D}_T \tag{6.24}$$

Notice that the Q_v field is massless while the B_v field is very heavy! In the case with only external Q_v fields with $m_Q\to\infty$, B_v decouples. Diagramatically,

[1]We purposely express the final result in terms of both D_μ and $D_{T,\mu}$ for reasons that will become clear later.



$$\sim \frac{1}{m_Q} \to 0$$

So if we drop the B_v field we get the result above,

$$\mathcal{L}_{HQET} = \tilde{Q}_v iv \cdot D Q_v \tag{6.25}$$

Physically what is happening is Q_v corresponds to the heavy particles and B_v corresponds to the heavy antiparticles. By making the phase redefinition that we did then we chose to expand about the particles. If we chose the opposite phase we would have expanded around the antiparticles instead.

We summarize some important points:

1. The above field redefinition was at tree level and is valid to leading order in $1/m_Q$ and $\alpha_s(m_Q)$. It correctly describe the coupling to $k^\mu \ll m_Q$ gluons at leading orders.

2. The antiparticles are integrated out. Its easiest to see that if we go to the rest frame,

$$v_r^\mu = (1,0,0,0)^T \tag{6.26}$$

which gives,

$$\frac{1+\not{v}}{2} = \frac{1+\gamma_0}{2} \tag{6.27}$$

Thus in the Dirac representation for the gamma matrices,

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \tag{6.28}$$

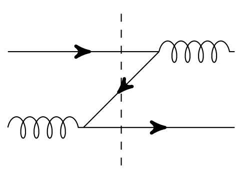
we have,

$$\frac{1+\gamma_0}{2} u_{Dirac} = \begin{pmatrix} \psi_v \\ 0 \end{pmatrix} \begin{array}{l} \leftarrow \text{particles} \\ \leftarrow \text{antiparticles} \end{array} \tag{6.29}$$

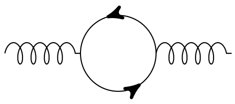
This will be true independently of which representation you use the for gamma matrices, but the Dirac representations makes the differentiation between particles and antiparticles manifest.

The way to think of this physically is we are studying heavy particles close to their mass shell. We measure fluctations near the heavy mass, m_Q. If you are perturbing about m_Q then the antiparticles are "very far away" since the splitting is $2m_Q$. Thus there is no pair creation.

This can be seen from drawing a time-ordered perturbation theory diagram (this is not a Feynman diagram!) for pair creation,



The intermediate state has 3 heavy particles and its off-shell by $2m_Q$. This can alternatively be seen from a vacuum polarization Feynman diagram,



Thus the number of heavy quarks in preserved. We have an extra $U(1)$ symmetry for HQET which is not existent in QCD.

3. The extra $U(1)$ is actually part of a bigger symmetry called, Heavy Quark Symmetry. It is part of a flavor symmetry,

$$U(N_h) \tag{6.30}$$

for N_h heavy quarks. The reason this symmetry exists is that $\mathcal{L}_{HQET}$ is independent of m_Q and hence doesn't see the flavor of that quark.

We also have an new spin symmetry, $SU(2)$, since our Lagrangian is independent of the two remaining spin components,

$$\bar{Q}_v i \underbrace{v \cdot D}_{\text{no spin matrices (such as } \gamma_\mu)} Q_v \tag{6.31}$$

In the rest frame the heavy quark spin transformation is,

$$S_Q^i = \frac{1}{2}\begin{pmatrix} \sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix} = \frac{1}{2}\gamma_5 \gamma^0 \gamma^i \tag{6.32}$$

and the infinestismal version of the transformation takes the form,

$$Q_v' = (1 + i\boldsymbol{\theta} \cdot \boldsymbol{S}_Q)\, Q_v \tag{6.33}$$

which gives,

$$\delta \mathcal{L} = \bar{Q}_v \left[i v \cdot D, i\boldsymbol{\theta} \cdot \mathbf{S}_Q \right] Q_v = 0 \tag{6.34}$$

since $v \cdot D$ doesn't carry any matrix structure.

Furthermore in the rest frame we have,

$$\not{v}_r Q_{v_r}' = \gamma_0 \left(1 + i\boldsymbol{\theta} \cdot \mathbf{S}_Q\right) Q_{v_r} \tag{6.35}$$

$$= Q_v' \tag{6.36}$$

where we denote the rest frame variables with a subscript r.

But boosting both sides of the equation we see that

$$U(\Lambda)\not{v}_r U(\Lambda^{-1}) U(\Lambda) Q_{v_r}' = U(\Lambda) Q_{v_r}' \tag{6.37}$$

$$\not{v} Q_v' = Q_v' \tag{6.38}$$

so the rotation acts within the two component subspace.

All together we have $U(2N_h)$ symmetry where Q_v field is a fundamental in the group. For example,

$$\begin{pmatrix} 1 \\ 0 \\ \vdots \end{pmatrix} \leftarrow b\text{-quark with spin-up} \tag{6.39}$$

and some other component may be a charm quark in a spin up, b quark with spin down, etc.

4. The velocity, v, appears in the phase and on the field, Q_v. This operator is useful because its a conserved quantity by low energy QCD interactions. In our definition,

$$p = m_Q v + k \tag{6.40}$$

k^μ can change the momenta, but not v^μ.

5. The power counting is done in $1/m_Q$. The mode expansion is

$$Q(x) = \int \frac{đ^3 p}{\sqrt{2E_p}} \sum_s \left(a_p^s u^s e^{-ip\cdot x} + b_p^{s\,\dagger} v^s e^{ip\cdot x} \right) \tag{6.41}$$

This implies that

$$Q_v = \frac{1+\not{v}}{2} \int \frac{đ^3 p}{\sqrt{2E_p}} \sum_s \left(a_p^s u^s e^{-ik\cdot x} + b_p^{s\dagger} v^s e^{ik\cdot x + 2im_Q v\cdot x} \right) \tag{6.42}$$

What we did with the $Q(x)$ field is that we pulled out the phase, $e^{-im_Q v\cdot x}$. That leaves in Q_v just $e^{-ik\cdot x}$. So derivatives acting on Q_v give,

$$i\partial^\mu Q_v \sim k^\mu Q_v \tag{6.43}$$

and so there are no m_Q's in the derivative. That's the magic of making this field redefinition. Since we want to count m_Q we want to make these m_Q's explicit. So coordinate, x corresponds in Q_v corresponds to the low energy variations, i.e. over scales, $\ll m_Q$.

In subleading Lagrangians and external operators, all the powers of m_Q are going to be explicit which makes doing the power counting very simple.

6.0.1 States

So far we have ignored states. These have one last hiding m_Q. The usual relativistic normalization for relativistic particles is

$$\langle H(p')|H(p)\rangle = 2E_p (2\pi)^3 \delta^3(\mathbf{p} - \mathbf{p}') \tag{6.44}$$

but $E_p = \sqrt{m_H^2 + \mathbf{p}^2}$ has a heavy quark mass hiding in it ($m_H \approx m_Q$). States defined from $\lim_{m_Q\to\infty} \mathcal{L}_{HQET}$ are different by their norm and $1/m_Q$ corrections. Instead we define,

$$|H(\mathbf{p})\rangle = \sqrt{m_H}\left[|H(v)\rangle + \mathcal{O}\left(\frac{1}{m_Q}\right)\right] \tag{6.45}$$

which gives,

$$\langle H(v', k')|H(v, k)\rangle = 2v^0 \delta_{v,v'}(2\pi)^3 \delta^3(\mathbf{k} - \mathbf{k}') \tag{6.46}$$

This says that hadrons with differenties velocity don't interact.

6.0.2 Spectroscopy

Light quarks and gluons are still described by $\mathcal{L}_{QCD}$. As $m_Q \to \infty$, $Q\bar{q}$ has quantum numbers of $Q, \bar{q}$, and any number of quarks and gluons (the light degrees of freedom (dof)). Total angular momentum, $\mathbf{J}$, is a good quantum number. The heavy quark spin, $\mathbf{S}_Q$, is also conserved.

Therefore the angular momentum of the light dof,

$$\mathbf{S}_\ell = \mathbf{J} - \mathbf{S}_Q \tag{6.47}$$

must also be conserved. Even though its composed of a messy combination of quark and gluons popping in and out of the vacuum, it is still a good quantum number (QN). As usual we have,

$$\mathbf{J}^2 \to J(J+1) \tag{6.48}$$
$$\mathbf{S}_\ell^2 \to S_\ell\,(S_\ell + 1) \tag{6.49}$$

Since $S_Q = 1/2$ we expect to get symmetry doublets for our mesons. The lowest angular momentum states will form a pseudoscalar and a vector. Other combinations can arise from higher ℓ states.

	S_ℓ^P	Mesons	
Lightest	$\frac{1}{2}^-$	B, B^*	$j = 0, 1$
	$\frac{1}{2}^+$	B_0^*, B_1^*	$j = 0, 1$
	$\frac{3}{2}^+$	B_1, B_2^*	$j = 1, 2$
Baryons	0^+	Λ_b	$j = 1/2$
	1^+	Σ_b, Σ_b^*	$j = \frac{1}{2}, \frac{3}{2}$

6.1 Covariant Representation of Fields

We will encode Heavy Quark Symmetry (HQS) in objects with nice transformation properties. We need a way to describe the meson doublets such as, B, B^* using fields. We define $H_v^{(Q)}$ as an object that describes the ground state meson, $Q\bar{q}$.

This objects needs to annhilate both the pseudoscalar and meson fields. Vector particles have a polarization vector, ϵ_μ, with $\epsilon^2 = -1$ and $v \cdot \epsilon = 0$. The amplitude for $P_{v,\mu}^{*(Q)}$ to annhilate a vector is given by ϵ_μ [Q 13: Why?].

We would like this object to be in the $(1/2, 1/2)$ representation of the Lorentz group. Furthermore, it must transforms as,

$$H'_{v'}(x') = D(\Lambda) H_v(x) D(\Lambda)^{-1} \tag{6.50}$$

where $v' = \Lambda v$ and $x' = \Lambda x$. Lastly we require,

$$\not{v} H_v = H_v \tag{6.51}$$

such that there is no heavy antiquark. The simplest way to impose these conditions is,

$$H_v^{(Q)} = \frac{1+\not{v}}{2}\left[P_{v,\mu}^{*(Q)}\gamma^\mu + iP_v^{(Q)}\gamma_5\right] \tag{6.52}$$

With the form above we see that $H_v \not{v} = -H_v$. This holds because the polarization of spin-1 particles satisfies, $v \cdot \epsilon = 0 \Rightarrow v \cdot P_v^* = 0$.

In the rest frame, $v = (1, \mathbf{0})$, the field is given by

$$H_{v,r} = \begin{pmatrix} 0 & iP_{v,r} - \boldsymbol{\sigma} \cdot \mathbf{P}_{v,r}^* \\ 0 & 0 \end{pmatrix} \tag{6.53}$$

To derive this expression we used the fact that $P_{v,0}^* = 0$ due to gauge invariance and we work in the Dirac representation.

The $H_{v,r}$ field transforms as a $(1/2, 1/2)$ representation under, $S_Q \otimes S_\ell$. We have,

$$\left[S_Q^i, H_{v,r}\right] = \frac{1}{2}\sigma_{4\times4}^i H_{v,r}, \quad \left[S_\ell^i, H_{v,r}\right] = -\frac{1}{2}H_{v,r}\sigma_{4\times4}^i \tag{6.54}$$

where $\sigma_{4\times4}^i \equiv i\epsilon_{ijk}\left[\gamma^j, \gamma^k\right]/4$.

Under the heavy quark spin transformations we have,

$$H_v \to D(R)_Q H_v \tag{6.55}$$

which implies that

$$\delta H_v = i\left[\boldsymbol{\theta} \cdot \mathbf{S}_Q, H_v\right] \tag{6.56}$$

Plugging in the form of H_v we find,

$$\delta P_{v_r} = -\frac{1}{2}\boldsymbol{\theta} \cdot \mathbf{P}_{v_r}^* \quad , \delta P_{v_r}^* = \frac{1}{2}\boldsymbol{\theta} \times \mathbf{P}_{v_r}^* - \frac{1}{2}\boldsymbol{\theta} \cdot \mathbf{P}_{v_r} \tag{6.57}$$

So the spin symmetry transforms the scalar into the vector as expected. The power of $H_{v_r}^{(Q)}$ is that it allows you to make HQS predictions very easily.

For example consider heavy quark decay constants. For $\bar{B}$ and D decay we have,

$$\langle 0|\bar{q}\gamma^\mu\gamma_5 Q|P(p)\rangle = -if_P p^\mu \tag{6.58}$$

$$= -if_p m_p v^\mu \tag{6.59}$$

while for the vector mesons such as $D^*, \bar{B}^*$ we have,

$$\langle 0|\bar{q}\gamma^\mu Q|P^*(p,\epsilon)\rangle = f_{P^*}\epsilon^\mu \tag{6.60}$$

[Q 14: Why do we not have a $f'_{P^*}p^\mu$ contribution?] Working out the dimensions we must have that f_P is dimension one and f_{P^*} is dimension 2. We know that heavy quark symmetry must relate the two decay constants. We now work it out.

First we need to change Q in the current since we need to make the m_Q explicit. The HQET currents can be written,

$$\left(\bar{q}\Gamma Q\right)(0) = \left(\bar{q}\Gamma Q_v\right)(0) + \mathcal{O}\left(\frac{1}{m_Q}\right) \tag{6.61}$$

Under HQS we have,

$$Q_v \to D(R)Q_v \tag{6.62}$$

We want to encode in a general object that involves H_v field that transforms the same way.

We do the usual trick. We pretend that we have the transformation,

$$\Gamma \to \Gamma D(R^{-1}) \tag{6.63}$$

and then ΓH_v is invariant (only one H_v since the number of heavy quarks in conserved). Lorentz invariance requires,

$$\mathrm{Tr}(\underbrace{X}_{\text{QCD stuff}}\Gamma H_v) \tag{6.64}$$

In general X is a Lorentz bispinor and can be a complication combination of QCD interactions. The only parameter which it can depend on is v^μ.Thus it must have the form,

$$X = a_0(v^2) + \not{v}a_1(v^2) \tag{6.65}$$

But since

$$\mathrm{Tr}\left(\not{v}\Gamma H_v\right) = \mathrm{Tr}\left(\Gamma H_v \not{v}\right) = -\mathrm{Tr}\left(\Gamma H_v\right) \tag{6.66}$$

the $\not{v}$ contribution can be eliminated. Furthermore, $v^2 = 1$. Thus we can simply write, $X = a/2$ for some constant a (the $1/2$ is a convention). Evaluating the trace explicitly we have,

$$\mathrm{Tr}\left(X\Gamma H_v\right) = a\begin{cases} -iv^\mu P_v & \Gamma = \gamma^\mu\gamma_5 \\ P_v^{*,\mu} & \Gamma = \gamma^\mu \end{cases} \tag{6.67}$$

The way to think about this is that in the matrix element (and only in the matrix element!):

$$\bar{q}\Gamma Q_v = \frac{a}{2}\mathrm{Tr}\left(\Gamma H_v\right) \tag{6.68}$$

so then

$$\langle 0|\bar{q}\gamma^\mu\gamma_5 Q_v|P(v)\rangle = -iav^\mu \tag{6.69}$$

$$\langle 0|\bar{q}\gamma^\mu Q_v|P^*(v,\epsilon)\rangle = a\epsilon^{*,\mu} \tag{6.70}$$

and so $a \sim \Lambda_{QCD}^{3/2}$ by dimensions[2]. You can then relate a the the decay constants which gives,

$$f_P = \frac{a}{\sqrt{m_P}}, \quad f_{P^*} = a\sqrt{m_{P^*}} \tag{6.71}$$

You can make predictions from this:

$$f_B \sim \frac{\Lambda_{QCD}^{3/2}}{m_b^{1/2}} \sim 180\text{MeV} \tag{6.72}$$

and

$$\frac{f_B}{f_D} = \sqrt{\frac{m_D}{m_B}} \sim 0.6 \tag{6.73}$$

This gives a lot of power in semileptonic decays. For example,

$$B \to D\ell\nu \tag{6.74}$$
$$\bar{B} \to D^*\ell\nu \tag{6.75}$$

In QCD there are 6 form factors, but in HQET there is one normalized factor.

6.1.1 HQET Radiative Corrections

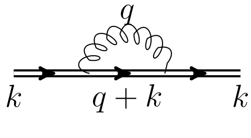
Thus far we have ignored α_s corrections. We now study what impact having labels, v^μ, have. We need to study,

- Renormalization $\quad \mathcal{L}, J^\mu$
- Matching $\quad J^\mu_{QCD} = C\left(\frac{\mu}{m_Q}\right) J^\mu_{HQET} + \mathcal{O}\left(\frac{1}{m_Q}\right)$

We begin by studying wavefunction renormalization: Just like in regular QFT our fields have a bare and a renormalized version,

$$Q_v^{(0)} = Z_h^{1/2} Q_v \tag{6.76}$$

We need to calculate,



$$= -C_F \mu^{2\epsilon} g^2 \int đ^d q \frac{v^2}{v\cdot(q+k)} \frac{e^{\epsilon\gamma_E}(4\pi)^{-\epsilon}}{q^2}$$

where we draw a heavy quark with a double line. We have one heavy quark propagator and a relativistic propagator. This diagram is IR divergent. Since we want to use counterterms to fix UV divergences we insert a small gluon mass, m, which we take to zero at the end of the calculation.

[2]Recall that the dimensions of $|P^*(v,\epsilon)\rangle$ and $|P(v)\rangle$ are $-3/2$.

We can't use Feynman trick since we don't have squares in the denomenator. Instead we use the Georgi trick,

$$2\int_0^\infty \frac{d\lambda}{(a+2b\lambda)^2} = \int_a^\infty \frac{du/b}{u^2} = \frac{1}{ab} \tag{6.77}$$

For more general propagators one can use,

$$\frac{1}{a^r b^s} = 2^s \frac{\Gamma(r+s)}{\Gamma(r)\Gamma(s)} \int_0^\infty d\lambda \frac{\lambda^{s-1}}{(a+2b\lambda)^{r+s}} \tag{6.78}$$

In our case we have, $a = q^2 - m^2, b = v\cdot(q+k)$. The denominator becomes,

$$D = (q^2 + 2\lambda v\cdot q + 2\lambda v\cdot k + i\epsilon)^2 = ((q+\lambda v)^2 - \Delta)^2 \tag{6.79}$$

where $\Delta = \lambda(\lambda - 2v\cdot k - i\epsilon)$. Thus our diagram takes the form,

$$\begin{aligned} &= -C_F \mu^{2\epsilon} g^2 (4\pi)^{-\epsilon} e^{\epsilon\gamma_E} \int d\!\!\!^-{}^d q \frac{1}{\left[q^2-\Delta\right]^2} && (6.80)\\ &= -\frac{2iC_F}{16\pi^2}\mu^{2\epsilon} g^2 e^{\epsilon\gamma_E}(4\pi)^{-\epsilon}\int_0^\infty d\lambda\Gamma(\epsilon/2)\left[\lambda^2 - 2\lambda v\cdot p + m^2\right]^{-\epsilon/2} && (6.81)\end{aligned}$$

to carry our the integral over λ we need to use a recursion relation. Consider,

$$\begin{aligned} \frac{d}{d\lambda}\left[(\lambda+b)(\lambda^2+2b\lambda+c)\right] &= (...)^a + (\lambda+b)a(\lambda^2+2b\lambda+c)^{a-1}(2\lambda+2b) && (6.82)\\ &= (...)^a + 2a(...)^{a-1}\left[(\lambda+b^2-b^2+c+b^2-c\right] && (6.83)\\ &= (...)^a + 2a\,(...)^a + 2a(b^2-c)\,(...)^{a-1} && (6.84)\\ I(a) &= \frac{1}{1+2a}\left[(\lambda+b)(\lambda^2+2b\lambda+c)^a\Big|_0^\infty - 2a(b^2-c)I(a-1)\right] && (6.85)\end{aligned}$$

but

$$\lim_{\lambda\to\infty} A^{z(\epsilon)} = 0 \tag{6.86}$$

as long as z can be analytically continued to negative values. Therefore we have,

$$I(a) = \frac{1}{1+2a}\left[-bc^a - 2a(b^2-c)I(a-1)\right] \tag{6.87}$$

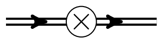
Making use of this relation we have,

$$\begin{aligned}\int_0^\infty d\lambda\left[\lambda^2 - \lambda v\cdot p + m^2\right]^{-\epsilon/2} = \frac{1}{1-\epsilon}\Big\{ & v\cdot p m^{-\epsilon} + \epsilon((v\cdot p)^2 - m^2)\\ & \times\int_0^\infty d\lambda(\lambda^2 - 2\lambda v\cdot p + m^2)^{-\epsilon/2-1}\Big\} && (6.88)\end{aligned}$$

After being multiplied by a $1/\epsilon$, the integral that's left is finite. We drop this term. We are then left with,

$$= i\frac{C_F g^2}{8\pi^2}\frac{v \cdot p}{\epsilon} \tag{6.89}$$

Therefore we need a counterterm,



$$i(Z_h - 1)v \cdot p$$

with,

$$Z_h = 1 + \frac{C_F g^2}{8\pi^2 \epsilon} \tag{6.90}$$

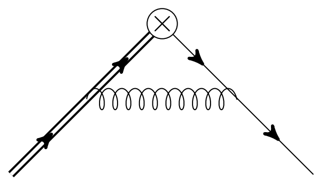
in $\overline{MS}$. Note that this is not equal to Z_q in QCD.

Now consider the renormalization of local operators. For example consider, $b \to u e^- \bar{\nu}$. We can write down the operators,

$$\mathcal{O}_r^{(0)} = \bar{q}\Gamma Q^{(0)} \tag{6.91}$$

$$\mathcal{O}_r = \frac{1}{Z_{\mathcal{O}}}\mathcal{O}_r^{(0)} = \bar{q}\Gamma Q_v + \left(\frac{\sqrt{Z_q Z_h}}{Z_{\mathcal{O}}} - 1\right)\bar{q}\Gamma Q_v \tag{6.92}$$

This amounts to the calculating,



(Combined with Z_h, Z_q)

This gives,

$$Z_{\mathcal{O}} = 1 + \frac{g^2}{8\pi^2 \epsilon} \tag{6.93}$$

So the anomalous dimension becomes,

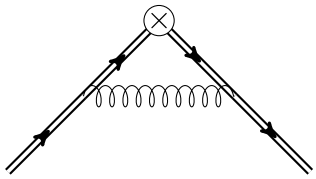
$$\gamma_{\mathcal{O}} = -\frac{g^2}{4\pi} = -\frac{\alpha_s}{\pi} \tag{6.94}$$

This gives running *below* m_Q (conserved current about m_Q had no evolution). Log's of m_Q have become UV divergences. One can show that the anomalous dimension is independent of Γ which is due to the spin symmetry of HQET.

A more interesting case is the renormalization of a heavy-heavy transition:

$$T_\Gamma = \bar{Q}_{v'}\Gamma Q_v + \left(\frac{Z_h}{Z_T} - 1\right)\bar{Q}_{v'}\Gamma Q_v \tag{6.95}$$

In example of this would be $B \to D^* e^- \bar{u}$ and we take $m_b, m_c \to \infty$. Going through the same procedure we have,



$$-iC_F g^2 v\cdot v' \int \frac{đ^d q}{q^2 (v\cdot q)(v'\cdot q)}$$

This integral has both UV and IR divergences. A careful calculation yields the UV counterterm is,

$$Z_T = 1 - \frac{g^2}{6\pi^2\epsilon}\left[w r(w) - 1\right] \tag{6.96}$$

where $w \equiv v\cdot v'$ and

$$r(w) \equiv \frac{\log(w+\sqrt{w^2-1})}{\sqrt{w^2-1}} \tag{6.97}$$

which leads to the anomalous dimension,

$$\gamma_T = \frac{g^2}{3\pi^2}\left[w r(w) - 1\right] \tag{6.98}$$

The reason we have this structure because $v^2 = v'^2 = 1$, but $v\cdot v'$ is not trivial.

Notes:

1. Answer depends on $w = v\cdot v'$. We have a current,

   $$J^{HQET,\mu}_{v,v'} = \bar{Q}_{v'}\Gamma^\mu Q_v \tag{6.99}$$

   The Wilson coefficient depends on w:

   $$C\left(\alpha_s,\mu,m_b v^\mu, m_c, v'^\mu\right) = C\left(\alpha_s,\mu,m_b^2,m_c^2,v\cdot v'\right) \tag{6.100}$$

   In $B\to D^* e^-\bar{\nu}$ we let[3],

   $$P_B^\mu = m_B v^\mu \tag{6.101}$$
   $$= m_{D^*}v'^\mu + \underbrace{q^\mu}_{\text{mom. transfer to leptons}} \tag{6.102}$$

   and

   $$q^2 = m_B^2 + m_{D^*}^2 - 2m_B m_{D^*} v\cdot v' \tag{6.103}$$

   Solving for w we find

   $$1 \le w \le 1.5 \tag{6.104}$$

   is allowed ranged for this variable. It is fixed by external kinematics.

2. Γ_T is independent of choice of spin structure, Γ, due to heavy quark symmetry.

[3] q here is a new variable unrelated to the loop momenta above.

3. What is physically happening is that we have log's of the form,
$$\log \frac{m_Q}{\Lambda_{QCD}} \tag{6.105}$$
in QCD which become,
$$\log \frac{\mu}{\Lambda_{QCD}} \tag{6.106}$$
in HQET operators and $\log \frac{m_Q}{\mu}$ hide in HQET Wilson coefficients. Anomalous dimension sums up these logs.

4. The leading log(LL) solution is to match at $\mu = m_Q$ where $C(m_Q) = 1$. Then the LL result is,
$$\begin{aligned} C_{LL}(\mu, ...) &= C(m_Q) \cdot U(m_Q, \mu) && (6.107) \\ &= \quad 1 \quad \cdot \left[\frac{\alpha_s(\mu)}{\alpha_s(m_Q)}\right]^{-\gamma/2\beta_0} && (6.108) \end{aligned}$$
where γ is a constant for $\bar{q}\Gamma Q_v$ and a function of w for $\bar{Q}_{v'}\Gamma Q_v$.

5. The HQET matrix elements also depend on μ. For example, consider the decay constant,
$$\langle 0|\bar{q}\gamma^\mu\gamma_5 Q_v|P(v)\rangle = -ia(\mu)v^\mu \tag{6.109}$$
We want, $\mu \sim 1\text{GeV} \gtrsim \Lambda_{QCD}$.

6.1.2 Matching - Perturbative corrections at $\alpha_s(m_Q)$

We use $\overline{MS}$ everywhere:

———— m_W

H_W

———— m_Q

$HQET$

———— Λ_{QCD}

In the H_W regime (well above the heavy quark mass) we have,
$$\langle q(0, s')|\bar{q}\gamma^\mu Q|Q(p, s)\rangle = \left[R^{(Q)}R^{(q)}\right]^{1/2} \bar{u}(0, s')\left[\gamma^\mu + V_1^\mu \alpha_s(\mu)\right] u(p, s) \tag{6.110}$$
where the R's are the UV finite residues in $\overline{MS}$ and V^μ is the relevant diagram.

In HQET we have,

$$\langle q(0,s')|\bar{q}\Gamma Q_v|Q(v,s)\rangle = \left[R^{(h)}R^{(q)}\right]^{1/2}\bar{u}(0,s')\left[1+V_1^{eff}\sigma_s(\mu)\right]\Gamma u(v,s) \tag{6.111}$$

An explicit calculation shows that the vector current, $\Gamma = \gamma^\mu$ in HQET has 2 currents:

$$C_1^{(v)}\bar{q}\gamma^\mu Q_v + C_2^{(v)}\bar{q}v^\mu Q_v \tag{6.112}$$

where

$$C_1^{(v)} = 1 + \frac{\alpha_s(\mu)}{\pi}\left[\log\frac{\mu_0}{\mu} - \frac{4}{3}\right] \tag{6.113}$$

$$C_2^{(v)} = \frac{2}{3}\frac{\alpha_s(\mu)}{\pi} \tag{6.114}$$

There is actually a nice trick that you can use to get these results that's more general than HQET. The idea is as follows. First pick an IR regulator to make calculation very simple. The choice that does that here is to use dim-reg for the UV in $\overline{MS}$ but also for the IR. If you do that you can convince yourself that all HQET graphs with on-shell external momenta are scaleless,

$$\propto \left(\frac{1}{\epsilon_{UV}} - \frac{1}{\epsilon_{IR}}\right) \tag{6.115}$$

The $1/\epsilon_{UV}$ get removed by counterterms in $\overline{MS}$ and you are left with $1/\epsilon_{IR}$. This makes this simple since the IR divergences in the effective theory and in the full theory have to match so the UV renormalized QCD graphs are,

$$\frac{\#}{\epsilon_{IR}} + \#\log\frac{\mu}{m_Q} + ... \tag{6.116}$$

The $1/\epsilon_{IR}$ term cancels when we substract HQET and so the matching is just the logarithmic term.

So we don't even need to calculate the HQET graphs. We can just directly get the matching!

6.2 Nonperturbative Corrections

Thus far we have considered on $\mathcal{O}(\alpha_s)$ corrections to the Lagrangian. Now we expand the extension to include $1/m_Q$ corrections. Recall that the Lagrangian to all orders in m_Q can be written,

$$\mathcal{L}_{QCD} = \bar{Q}_v iv\cdot DQ_v - \bar{B}_v\left(iv\cdot D + 2m_Q\right)B_v + \bar{Q}_v i\not{D}_T B_v + \bar{B}_v i\not{D}_T Q_v \tag{6.117}$$

where B_v is the antiparticle field that we discarded last time. We now integrate it out more carefully keeping more than just the leading order term by solving the equation of motion. This procedure is equivalent to doing the path integral for B_v.

We have,

$$\frac{\partial \mathcal{L}}{\partial B_v} = -\bar{B}_v \left(gA \cdot v - 2m_Q \right) + g\bar{Q}_v A \tag{6.118}$$
$$\frac{\partial \mathcal{L}}{\partial (\partial_\mu B_v)} = -\bar{B}_v i v_\mu + i\bar{Q}_v \gamma^\mu \tag{6.119}$$

which gives,

$$-i\partial^\mu \bar{B}_v v_\mu + i\partial_\mu \bar{Q}_v \gamma^\mu = -\bar{B}_v \left(gA \cdot v + 2m_Q \right) + g\bar{Q}_v A \tag{6.120}$$
$$\Rightarrow \left(i v^\mu \partial^\mu + gA \cdot v + 2m_Q \right) B_v = \left(\partial\!\!\!/ + gA \right) Q_v \tag{6.121}$$
$$\Rightarrow \left(i v \cdot D + 2m_Q \right) B_v = i D\!\!\!\!/_T Q_v \tag{6.122}$$

[Q 15: check the lines above]

Inserting this into the Lagrangian,

$$\begin{aligned}
\mathcal{L}_{QCD} &= \bar{Q}_v i v \cdot D Q_v - \bar{B} i D\!\!\!\!/_T Q_v + \bar{Q} i D\!\!\!\!/ B_v + \bar{B}_v i D\!\!\!\!/_T Q_v && (6.123)\\
&= \bar{Q}_v i v \cdot D Q_v + \frac{1}{2m_Q} \bar{Q} i D\!\!\!\!/_T \left(1 + \frac{i v \cdot D}{2m_Q} \right)^{-1} i D\!\!\!\!/_T Q_v && (6.124)\\
&= \bar{Q}_v i v \cdot D Q_v + \frac{1}{2m_Q} \bar{Q} i D\!\!\!\!/_T \left(1 - \frac{i v \cdot D}{2m_Q} + ... \right) i D\!\!\!\!/_T Q_v && (6.125)\\
&\approx \bar{Q}_v i v \cdot D Q_v - \frac{1}{2m_Q} \bar{Q} D\!\!\!\!/_T D\!\!\!\!/_T Q_v && (6.126)
\end{aligned}$$

where we have only kept the term lowest order in $1/m_Q$. It is convenient to split this new term into two contributions, one that breaks flavor symmetry but preserves the heavy quark spin symmetry and one that breaks both. To do this we need to rewrite the derivatives,

$$\begin{aligned}
D\!\!\!\!/_T D\!\!\!\!/_T &= \gamma_\mu \gamma_\nu D_T^\mu D_T^\nu && (6.127)\\
&= \left(g_{\mu\nu} + \frac{1}{2} \left[\gamma_\mu, \gamma_\nu \right] \right) \left(\frac{1}{2} \left\{ D_T^\mu, D_T^\nu \right\} + \frac{1}{2} \left[D_T^\mu, D_T^\nu \right] \right) && (6.128)
\end{aligned}$$

where we have split the terms into their symmetric and antisymmetric parts. Simplifying using

$$\sigma_{\mu\nu} \equiv \frac{i}{2} \left[\gamma_\mu, \gamma_\nu \right], \quad G_{\mu\nu} \equiv ig \left[D_\mu, D_\nu \right] \tag{6.129}$$

and noting that $\bar{Q}_v \left[D_{T,\mu}, D_{T,\nu} \right] Q_v = \bar{Q}_v \left[D_\mu, D_\nu \right] Q_v$ we have,

$$D\!\!\!\!/_T D\!\!\!\!/_T = D_T^2 + \frac{g}{2} \sigma_{\mu\nu} G^{\mu\nu} \tag{6.130}$$

Inserting this into our Lagrangian we have,

$$\bar{Q}_v i v \cdot D Q_v - \frac{1}{2m_Q} \bar{Q}_v \left(D_T^2 + \frac{g}{2} \sigma_{\mu\nu} G^{\mu\nu} \right) Q_v \tag{6.131}$$

Both terms spoil the heavy quark flavor symmetry as they have explicit dependence on the heavy quark mass. However, only the second term breaks the spin symmetry as its dependent on the quark spin (due to the γ matrix structure in $\sigma_{\mu\nu}$).

The first term is simply the kinetic energy of the heavy quark (of the form, $p^2/2m$) while the second is a magnetic moment interaction.

6.3 Reparameterization Invariance

You may wonder whether we missed some term since by chance it vanished at tree level. In general we must use the following procedure,

First write down all possible opertaors constrained by symmetries

1. Power counting, powers of $1/m_Q$ tell us dimension of fields needed

2. Gauge symmetry - add D^μ's.

3. Discrete symmetries - C, P, T

4. But what about Lorentz invariance? Consider the 6 generators of the Lorentz group. The rotations are,

$$M^{\mu\nu}_{TT} : M^{12}, M^{23}, M^{13} \tag{6.132}$$

Then there are boosts are:

$$v_\mu M^{\mu\nu_T} : M^{01}, M^{02}, M^{03} \tag{6.133}$$

Introducing v^μ breaks part of Lorentz invariance (since it gives a preferred frame). However, this symmetry is restored by "reparametrized invariance" (RPI).

This is an additional symmetry on v^μ and is restored order by order in m_Q. To see this consider the momenta of the heavy quark,

$$p^\mu_Q = m_Q v^\mu + k^\mu \tag{6.134}$$

The split is somewhat arbitrary since we can always move a small amount of energy from k^μ to $m_Q v^\mu$ and back. In other words we should have invariance under,

$$v^\mu \to v^\mu + \frac{\epsilon^\mu}{m_Q} \tag{6.135}$$

$$k^\mu \to k^\mu - \epsilon^\mu \tag{6.136}$$

where ϵ^μ is $\mathcal{O}(\Lambda_{QCD})$. ϵ^μ can be thought of as infinitesimal. Recall that we had $v^2 = 1$. We'd like this to be a RPI invariant so we need $\epsilon \cdot v = 0$. Thus we have 3 components of

ϵ^μ. We now change to fields. Recall that we have,

$$\not{v}Q_v(0) = Q_v(0) \tag{6.137}$$

$$\rightarrow \left(\not{v} + \frac{\not{\epsilon}}{m_Q}\right)(Q_v + \delta Q_v) = Q_v + \delta Q_v \tag{6.138}$$

$$\Rightarrow (1 - \not{v})\,\delta Q_v = \frac{\not{\epsilon}}{m_Q} Q_v \tag{6.139}$$

One such solution is (easy to check)

$$\delta Q_v \frac{\not{\epsilon}}{2m_Q} Q_v \tag{6.140}$$

Therefore RPI is

$$v \rightarrow v + \frac{\epsilon}{m_Q} \tag{6.141}$$

$$Q_v \rightarrow e^{i\epsilon\cdot x}\left(1 + \frac{\not{\epsilon}}{2m_Q}\right) Q_v \tag{6.142}$$

This restores invariance under "small boosts",

$$\epsilon \sim \Lambda_{QCD} \quad (\text{not } \epsilon \sim m_Q) \tag{6.143}$$

which are all we care about.

If we consider all the possible $1/m_Q$ operators in general consistent with the symmetries about one can show that none of the operators are missing. Including all orders in α_s we have,

$$\mathcal{L}^{(1)} = -C_K \bar{Q}_v \frac{D_T^2}{2m_Q} Q_v - C_G g \frac{\bar{Q}_v \sigma^{\mu\nu} G_{\mu\nu} Q_v}{4m_Q} \tag{6.144}$$

where C_K and C_G are the Wilson coefficients. Consider the RPI. The phase is the only LO change. $\mathcal{O}^{(0)}$ is invariance at order $\mathcal{O}(m_Q)$ since $v \cdot \epsilon = 0$.

$$\mathcal{L}^{(0)} + \delta\mathcal{L}^{(0)} = \bar{Q}_v \left(1 + \frac{\not{\epsilon}}{2m_Q}\right) e^{-i\epsilon\cdot x} i \left(v + \frac{\epsilon}{m_Q}\right) \cdot D \left[e^{i\epsilon\cdot x}\left(1 + \frac{\not{\epsilon}}{m_Q}\right) Q_v\right] \tag{6.145}$$

Using

$$\left(\frac{1+\not{v}}{2}\right)\not{\epsilon}\frac{1+\not{v}}{2} = \epsilon \cdot v = 0 \tag{6.146}$$

we get,

$$\delta\mathcal{L}^{(0)} = \bar{Q}_v \frac{i\epsilon \cdot D}{m_Q} Q_v \tag{6.147}$$

gives,

$$\delta^{(1)}_{kin} = -C_K \frac{\bar{Q}_v i\epsilon \cdot D_T Q_v}{m_Q} \tag{6.148}$$

$$\delta^{(1)}_{mag} = 0 \quad (\text{at this order}) \tag{6.149}$$

But the $\delta^{(1)}_{kin}$ to cancel the $\delta\mathcal{L}^{(0)}$ the Wilson coefficient must be one to all orders in α_s for the reparameterization invariance not to be violated (as long as your scheme and regulator don't break the symmetry). The magnetic term is not constrained and does get a contribution at higher orders which can be shown to be,

$$C_G\left(\mu\right) = \left[\frac{\alpha_s(m_Q)}{\alpha_s(\mu)}\right]^{C_A/\beta_0} \tag{6.150}$$

where $C_A = N_c = 3$.

6.4 Hadron Masses

One can use the terms derived above to construct formula for the hadron masses. A hadron mass term is

$$\Delta m = \langle H(0)| T \exp\left(-i\int d^4x\mathcal{H}\right)|H(0)\rangle \tag{6.151}$$

where T is the time ordering operator, $\mathcal{H}$ is the Hamiltonian, and $|H(0)\rangle$ is the hadron state evaluated at zero momenta. If we take H to be the quantum corrected Hamiltonian (such that it includes all loops), we can write,

$$\Delta m = -i\int d^4x\,\langle H(0)|\mathcal{H}|H(0)\rangle \tag{6.152}$$

To lowest order in m_Q we have,

$$\Delta m = -i\int d^4x\,\langle H(0)|\bar{Q}_{v_r} iv\cdot DQ_{v_r}|H(0)\rangle \tag{6.153}$$

This term respects both the heavy quark flavor and spin symmetries and is defined as $\bar{\Lambda}_H$.

To second order we need to consider the flavor symmetry-breaking terms. To do so we need to define two quantities. The first,

$$\lambda_1 \equiv -\int d^4x\,\langle H(0)|\bar{Q}_{v_r} D_T^2 Q_{v_r}|H(0)\rangle \tag{6.154}$$

doesn't depend on m_Q(since $Q_{v,r}$ and the states have the m_Q dependence removed) and so is independent of both flavor and spin of the hadron. Furthermore, we conveniotionally include a negative sign since the terrm is a kinetic energy and we expect it to be positive.

The final non-perturbative definition we need is for the spin term,

$$\langle H(0)|\bar{Q}_{v_r}\sigma_{\alpha\beta}G^{\alpha\beta}Q_{v_r}|H(0)\rangle \tag{6.155}$$

The problem is that this has dependence on the heavy quark spin and we want to extract this dependence from the non-perturbative physics. To do this it is again easiest to work in the Dirac basis. First we note that $\gamma^0 Q_{v_r} = Q_{v_r}$. In the Dirac basis this implies that Q_{v_r} only has non-zero top two components. We write,

$$Q_{v_r} = \begin{pmatrix} Q_{v_r} \\ 0 \end{pmatrix}, \quad \bar{Q}_{v_r} = \begin{pmatrix} Q_{v_r}^\dagger & 0 \end{pmatrix} \tag{6.156}$$

Furthermore,

$$\sigma^{00} = 0 \tag{6.157}$$

$$\sigma^{i0} = -\sigma^{0i} = i \begin{pmatrix} 0 & -\sigma^i \\ \sigma^i & 0 \end{pmatrix} \tag{6.158}$$

$$\sigma^{ij} = -\frac{i}{2} \begin{pmatrix} [\sigma^i, \sigma^j] & 0 \\ 0 & [\sigma^i, \sigma^j] \end{pmatrix} = \epsilon_{ijk}\sigma_k \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{6.159}$$

Due to the structure of Q_{v_r}, σ^{i0} and σ^{0i} don't contribute to the amplitude. We just have,

$$\langle H(0)|\bar{Q}_{v_r}\sigma_{\alpha\beta}G^{\alpha\beta}Q_{v_r}|H(0)\rangle = \langle S_Q|\bar{Q}_{v_r}\sigma_k Q_{v_r}|S_Q\rangle \, \langle S_q|\epsilon_{ijk}G^{ij}|S_q\rangle \tag{6.160}$$

where $|S_Q\rangle$ ($|S_q\rangle$) is the heavy (light) quark. First note that

$$Q_{v_r}\,|S_Q\rangle = u_Q \tag{6.161}$$

which gives, $\boldsymbol{\sigma} u_Q = 2\mathbf{S}_Q u_Q$. Now recall that $B_k \equiv \epsilon_{ijk}G^{ij}$. At first glance one would think that this chromomagnetic field can point in any direction. However, due to rotational invariance and time reversal it must be proportional to $\mathbf{S}_q$ [Q 16: why?]. Thus the braket above is proportional to,

$$\mathbf{S}_Q \cdot \mathbf{S}_q \tag{6.162}$$

We pull out this dependence and write,

$$8\mathbf{S}_Q \cdot \mathbf{S}_q \lambda_2(m_Q) = a(\mu) \int d^4x \, \langle H(0)|\bar{Q}_{v_r}\sigma_{\alpha\beta}G^{\alpha\beta}Q_{v_r}|H(0)\rangle \tag{6.163}$$

where the non-perturbative parameter, λ_2 can depend on the heavy quark mass through the logarithmic dependence on $a(\mu)$ (the effect of the higher order corrections in α_s).

We are now in position to write the masses of the heavy quark hadrons. We have,

$$m_H = m_Q + \bar{\Lambda} - \frac{\lambda_1}{2m_Q} + \frac{2\lambda_2}{m_Q}\mathbf{S}_Q \cdot \mathbf{S}_q \tag{6.164}$$

By using $\mathbf{S}_Q \cdot \mathbf{S}_q = (\mathbf{J}^2 - \mathbf{S}_Q^2 - \mathbf{S}_q^2)/2$, we can get relations between the masses. For example for the B meson we have[4],

Meson	$\mathbf{J}^2$	$\mathbf{S}_Q^2$	$\mathbf{S}_q^2$	$\mathbf{S}_Q \cdot \mathbf{S}_q$
B	0	1/2	1/2	−3/4
B^*	1	1/2	1/2	1/4

[4] Recall that $\mathbf{S}^2 = s(s+1)$

$$m_B = m_b + \bar{\Lambda} - \frac{\lambda_1}{2m_b} - \frac{3\lambda_2}{2m_Q} \tag{6.165}$$

$$m_{B^*} = m_b + \bar{\Lambda} - \frac{\lambda_1}{2m_b} + \frac{\lambda_2}{2m_Q} \tag{6.166}$$

Appendix A

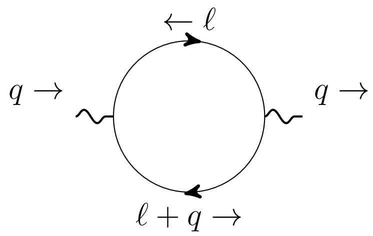
Homework Assignments

A.1 HW 1

The first HW can be found below,

A.1.1 Matching with Massive Electrons

a) The amplitude we want is,

$i\hat{\Pi}^{\mu\nu} =$ 

$$i\hat{\Pi}^{\mu\nu}(q) = (-1)\,(-ie)^2 \int \frac{d^4\ell}{(2\pi)^4} \frac{\mathrm{Tr}\left[\gamma^\mu\left(-\not{\ell}+m\right)\gamma^\nu\left(\not{\ell}+\not{q}+m\right)\right]}{(\ell^2-m^2)\left((\ell+q)^2-m^2\right)} \tag{A.1}$$

The most divergent part is (using a hard cutoff)

$$\int \frac{d^4\ell\,(a\ell^2 g_{\mu\nu} + b\ell_\mu \ell_\nu)}{\ell^4} \sim g_{\mu\nu}\Lambda^2 \tag{A.2}$$

However this object doesn't obey $q^\mu \hat{\Pi}_{\mu\nu} = 0$. Hence this is not true at $\mathcal{O}(\Lambda^2)$! To solve this integral in a gauge invariant way we use dim reg. We can greatly simplify the amount of work that we need to do by identifying that the final answer must be in the form,

$$\hat{\Pi}^{\mu\nu} = (q^2 g^{\mu\nu} - q^\mu q^\nu)\Pi(q^2) \tag{A.3}$$

Thus we just need to calculate either the $q^\mu q^\nu$ or the $g^{\mu\nu}$ part and just infer the other part. As we will see it will be easier to calculate the $q^\mu q^\nu$ contribution. The denomenator is:

$$\frac{1}{\ell^2 - m^2 + i\epsilon} \frac{1}{(\ell + q)^2 - m^2 + i\epsilon} = \int dx \frac{1}{\left[(\ell + qx)^2 - \Delta + i\epsilon\right]^2} \tag{A.4}$$

where

$$\Delta \equiv -q^2 x (1 - x) + m^2 \tag{A.5}$$

We now shift the integration variable and conside the trace:

$$\begin{aligned}
\mathrm{Tr}[...] &\to \mathrm{Tr}\left[\left(-\not{q} + \not{\ell} + \not{q}x + m\right)\gamma_\nu \left(-\not{\ell} - \not{q}x + m\right)\gamma_\mu\right] && \text{(A.6)}\\
&= -\mathrm{Tr}\left[\left(\not{\ell} + \not{q}(x-1)\right)\gamma_\nu\left(\not{\ell} + \not{q}x\right)\gamma_\mu\right] + m^2 \mathrm{Tr}\left[\gamma_\mu\gamma_\nu\right] && \text{(A.7)}\\
&= -\mathrm{Tr}\left[\not{\ell}\gamma_\nu\not{\ell}\gamma_\mu\right] - \mathrm{Tr}\left[\not{q}(x-1)\gamma_\nu\not{q}x\gamma_\mu\right] + m^2\mathrm{Tr}\left[\gamma_\mu\gamma_\nu\right] + \text{linear in } \ell && \text{(A.8)}\\
&= 4\left(2\ell^\mu\ell^\nu - \ell^2 g^{\mu\nu}\right) + 4x(1-x)\left(2q^\mu q^\nu - q^2 g^{\mu\nu}\right) && \text{(A.9)}
\end{aligned}$$

Δ only depends on q^2 so there is only one term that has a $q^\mu q^\nu$ contribution. We work out this term and infer the value of the rest of the terms. We need the integral,

$$\begin{aligned}
\int \frac{d^4\ell}{(2\pi)^4} \frac{1}{\left[\ell^2 - \Delta\right]^2} &= \frac{i}{(4\pi)^2}(1 + \epsilon \log 4\pi)\left(\frac{2}{\epsilon} - \gamma\right)(1 - \epsilon\log\Delta) && \text{(A.10)}\\
&= \frac{i}{16\pi^2}\left(\frac{2}{\epsilon} - \log\frac{\Delta}{4\pi} - \gamma\right) && \text{(A.11)}
\end{aligned}$$

Thus

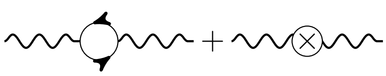
$$\hat{\Pi}_{\mu\nu} = q^\mu q^\nu \left(-\frac{e^2}{4\pi^2}\int dx 2x(1-x)\left(\frac{2}{\epsilon} - \log\frac{\Delta}{4\pi} - \gamma\right)\right) + (...)g^{\mu\nu} \tag{A.12}$$

$$\Rightarrow \quad \Pi(q^2) = \frac{2\alpha}{\pi}\int dx(1-x)\left(\frac{2}{\epsilon} - \log\frac{\Delta}{4\pi} - \gamma\right) \tag{A.13}$$

b) The QED Lagrangian is given by

$$\begin{aligned}
\mathcal{L} = &-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\not{\partial} - m)\psi - eA_\mu\bar{\psi}\gamma^\mu\psi\\
&+ \delta_3\left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right) + \delta_1\bar{\psi}i\not{\partial}\psi - \delta_m\bar{\psi}\psi - \delta_1 eA^\mu\bar{\psi}\gamma_\mu\psi && \text{(A.14)}
\end{aligned}$$

Calculating the first order correction to the propagator gives,



$$= i\overbrace{\frac{2\alpha}{\pi}\int dx\, x(1-x)\left(\frac{2}{\epsilon} - \log\Delta(q^2) + \log 4\pi - \gamma\right)}^{\Pi(q^2)} - i\delta_3$$

where $\Delta = -q^2 x(1-x) + m^2$.

The renormalization condition is

$$\Pi(0) = \delta_3 \tag{A.15}$$

which gives,

$$\mathcal{M}(q^2) = -\frac{2\alpha}{\pi}\int dx\, x(1-x)\log\left(-\frac{q^2}{m^2}x(1-x)+1\right)\left(q^2 g^{\mu\nu} - q^\mu q^\nu\right) \tag{A.16}$$

$$= \frac{2\alpha}{\pi}\int dx\, x(1-x)\left(\frac{q^2}{m^2}x(1-x) + \left(\frac{q^2}{m^2}\right)^2 \frac{x^2(1-x)^2}{2} + \dots\right)\left(q^2 g^{\mu\nu} - q^\mu q^\nu\right) \tag{A.17}$$

$$= \frac{2\alpha}{\pi}\left(\frac{q^2}{m^2}\frac{1}{30} + \left(\frac{q^2}{m^2}\right)^2 \frac{1}{280} + \dots\right)\left(q^2 g^{\mu\nu} - q^\mu q^\nu\right) \tag{A.18}$$

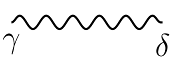
c) The second order term is given by

$$\frac{2\alpha}{\pi}\frac{q^2}{m^2}\frac{1}{30} \tag{A.19}$$

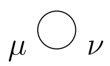
To get this term through a term in the Lagrangian we need a dimension 6 operator. To contract all the indices we need a term:

$$\int d^4x \frac{C}{2m^2}\partial^\mu F_{\mu\nu}\partial^\alpha F_\alpha{}^\nu = \int \frac{d^4q}{(2\pi)^4} q^2(q^2 g_{\mu\nu} - q_\mu q_\nu)\tilde{A}^\mu \tilde{A}^{*\nu} \tag{A.20}$$

where the factor of 2 is just conventional to avoid symmetry factors (the only reason I know its the right choice is because I calculated the Feynman rules explicitly). This gives the Feynman rule,



$$= \frac{-i}{k^2 + i\epsilon}\left(g_{\mu\nu} - (1-\xi)\frac{k_\mu k_\nu}{k^2}\right)$$



$$= -\frac{C}{2m^2}k^2(k^2 g^{\mu\nu} - k^\mu k^\nu)$$

Matching the corrections to the propagator in the effective and high energy theory we have,

$$C = \frac{4\alpha}{30\pi} \approx \frac{1}{5000} \tag{A.21}$$

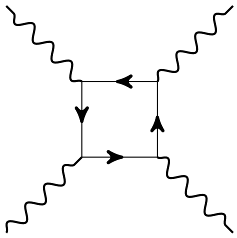
This is a very small correction and doesn't even include momentum suppression! [Q 17: Is this a typical value for the C's? I would have naively expected $C \sim 1$.]

d) Dimension six operators with 3 field strengths is forbidden by the antisymmetry of $F_{\mu\nu}$(and indirectly gauge symmetry as we must use $F_{\mu\nu}$'s) To see this consider

$$\begin{aligned} F_{\alpha\beta}F^{\beta\gamma}F_\gamma{}^\alpha &= -F_{\beta\alpha}F^{\beta\gamma}F_\gamma{}^\alpha && \text{(A.22)} \\ &= -F_{\alpha\beta}F^{\beta\gamma}F_\gamma{}^\alpha && \text{(A.23)} \end{aligned}$$

and thus this operator is zero.

e) Light-light scattering can occur through



$$\sim \int d^4p \frac{1}{16\pi^2}\frac{e^4}{m^4} \sim \frac{p^4\alpha^2}{m^4}$$

The cross-section is given by

$$\sigma \sim \frac{p^6}{m^8}\alpha^4 \sim 10^{-14}\text{GeV}^{-2} \approx 10^{-2}\text{fb} \tag{A.24}$$

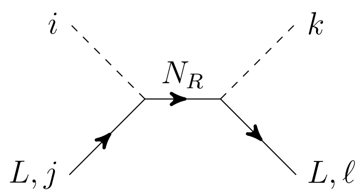
As a point of reference the total integrated luminosity at the LHC is about 10fb.

A.1.2 Right handed neutrinos and proton decay

a) Consider the SM lepton sector with a right handed neutrino,

$$\mathcal{L} = \bar{N}_R i\partial\!\!\!/ N_R - \frac{M}{2}(\bar{N}_R^c N_R + h.c.) + g_\nu\left(\bar{N}_R H_i \epsilon^{ij} L_{L,j} + h.c.\right) \tag{A.25}$$

Now consider the diagram



This gives,

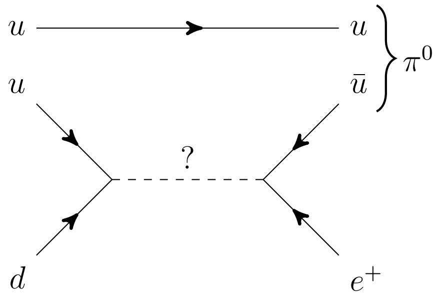
$$i\mathcal{M} = (ig_\nu)^2\epsilon^{ij}\epsilon^{\ell k}\bar{u}_\ell \frac{p\!\!\!/ - M}{p^2 - M^2}u_j \approx -\frac{g^2\nu}{M}\epsilon^{ij}\epsilon^{\ell k}\bar{u}_\ell u_j \tag{A.26}$$

This arises from the non-renormalizable operator,

$$\Delta\mathcal{L} = \frac{ig^2}{M}\left(H_i\epsilon_{ij}L_j\right)\left(H_k\epsilon_{k\ell}L_\ell\right) \tag{A.27}$$

which is just the Weinberg operator as expected.

b) The proton is the lightest baryon and so if baryon number is exactly conserved then it should be perfectly stable. However, non-renormalizable operators can contribute to proton decay. One potential decay route for the proton is into a pion and an electron:



where the dashed indicates some unknown particle. The effective operator for this process will have $2u$'s, a d, and a e (with no bars on top of these). First note that this interaction is gauge invariant under QED which is a good start, but we want $SU(2) \times U(1)_Y$ invariance as well. Each of these particles can be either right or left handed. We tabulate the possible hypercharges as follows:

$$\begin{array}{cccc} u & u & d & e \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{3} & -1 \\ \frac{4}{3} & \frac{4}{3} & -\frac{2}{3} & -2 \end{array}$$

where the dashed line indicate a set of hypercharges that add up to 0. Thus a gauge invariant term is:

$$Q^T \epsilon Q u_R e_R = u_L d_L u_R e_R - d_L u_L u_R e_R = 0 \tag{A.28}$$

but this is unfortunately zero due to the antisymmetry of ϵ. Instead we can consider the $SU(2)$ singlet contribution:

$$u_R d_R u_R e_R \tag{A.29}$$

A.1.3 Electric Dipole Moments

a) The neutron dipole moment is given by,

$$\mathbf{d} = \int \mathbf{x} q(\mathbf{x}) \, d^3x \tag{A.30}$$

For a particle with a magnetic moment this will be proportional to the spin of the particle. To see why this is the case, suppose you have a particle with spin $\mathbf{S}$ with energy E_0 (this arguement follows from Ref. [2]). If it is placed in a magnetic field its two energy levels will be split by the magnetic field:

$$E_0 \to \begin{cases} E_0 + \mu B \\ E_0 - \mu B \end{cases} \tag{A.31}$$

The energy of the neutron is solely dependent on the angle between the spin and the magnetic field. Now suppose the particle also has a electric dipole moment, $\mathbf{d}$. then a second vector exists to denote its state. but since the energy is only dependent on the angle between $\mathbf{S}$ and $\mathbf{B}$, every configuration of $\mathbf{d}$ must give the same energy. However, this contradicts the assumption that the ground state is non-degenerate. To avoid this contribution we must have that,

$$\mathbf{d} = d\frac{\mathbf{S}}{S} \tag{A.32}$$

Now consider the operation of time reversal on such a particle. Under time reversal magnetic fields change direction but electric fields do not. Thus the existence of an electric dipole moment for a spin particle violates time-reversal and hence violates CP.

Thus QED cannot provide a neutron dipole moment. We can however, get one from the interaction,

$$\Delta\mathcal{L} = ic\bar{u}\sigma^{\mu\nu}uF_{\mu\nu} \tag{A.33}$$

Under CP we have,

$$\bar{u}\sigma^{\mu\nu}u \to (-1)(-1)^{\mu}(-1)^{\nu} \tag{A.34}$$

where following notation Peskin and Schroeder (see pg. 71) we define $(-1)^{\mu}$ to be $+1$ for $\mu = 0$ and -1 for $\mu \neq 0$. Furthermore, by looking at the matrix structure of $F_{\mu\nu}$ and remembering that the photon is its own antiparticle and odd under Parity we have,

$$F_{\mu\nu} \to (-1)^{\mu}(-1)^{\nu} \tag{A.35}$$

under CP. Thus we have,

$$\bar{u}\sigma^{\mu\nu}uF_{\mu\nu} \to -\bar{u}\sigma^{\mu\nu}uF_{\mu\nu} \tag{A.36}$$

which violates CP and so this term could contribute the neutron dipole moment.

b)

A.1.4 Field Redefinitions

Consider the Lagrangian,

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4 + \frac{\eta g_1}{6!}\phi^6 + \frac{\eta g_2}{3!}\phi^3\partial^2\phi \tag{A.37}$$

with $\eta \ll 1$.

a) We assume a field redefinition of the form,

$$\phi \to \phi + \alpha\eta\phi^3 \tag{A.38}$$

This gives,

$$\frac{1}{2}(\partial_\mu\phi)^2 \to \frac{1}{2}(\partial_\mu\phi)^2 - \alpha\eta\phi^3\partial^2\phi \tag{A.39}$$

For this to cancel the $\phi^3\partial^2\phi$ term in the Lagrangian requires $\alpha = \frac{g_2}{3!}$. The rest of the shifts give,

$$\frac{1}{2}m^2\phi^2 \to \frac{1}{2}m^2\phi^2 + m^2\frac{g_2}{3!}\eta\phi^4 \tag{A.40}$$

$$\frac{\lambda}{4!}\phi^4 \to \frac{\lambda}{4!}\phi^4 + \frac{20\lambda g_2}{6!}\eta\phi^6 \tag{A.41}$$

$$\frac{\eta g_1}{6!}\phi^6 \to \frac{\eta g_1}{6!}\phi^6 \tag{A.42}$$

$$\frac{\eta g_2}{3!}\phi^3\partial^2\phi \to \frac{\eta g_2}{3!}\phi^3\partial^2\phi \tag{A.43}$$

which gives the Lagrangian

$$\mathcal{L} \to \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4 + \frac{\eta g_1'}{6!}\phi^6 \tag{A.44}$$

where $\lambda' \equiv \lambda + 4m^2\eta g_2$ and $g_1' \equiv g_1 - 20\lambda g_2$.

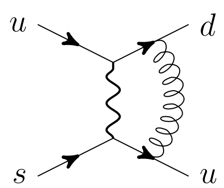
A.2 HW 2

A.2.1 Mixing and 4-Quark Operators

The second HW can be found below:

a) To do the calculation we need to calculate the six diagrams that were show above. Note that in the diagrams it is implied that we have right moving W boson. We show this explicitly to avoid confusion but take its mass to be much larger then the momentum transfer.

We begin with the gluon loop appearing between the final states,



Since the operator does not have any derivatives in it we don't expect its renoraml-ization to depend on the external momenta and we can set the external momenta to zero. Furthermore, we can set the masses to zero as long as we work in energies above the masses of the external particles. We define Ω to be the Weak interaction vertex factor. The diagram is given by

$$= \Omega(-ig)^2 \int \text{đ}^d\ell \frac{-i}{\ell^2} \left[\bar{d}_L \gamma_\mu T^a \frac{i\not{\ell}}{\ell^2} \gamma_\nu u_L\right] \left[\bar{u}_L \gamma^\mu T^a \frac{-\not{\ell}}{\ell^2} \gamma^\nu s_L\right] \tag{A.45}$$

$$= i\Omega g^2 \int \text{đ}^d\ell \frac{\ell^\sigma \ell_\rho}{\ell^6} \left[\bar{d}_L \gamma_\mu T^a \gamma^\sigma g_\nu u_L\right] \left[\bar{u}_L \gamma^\mu T^a \gamma^\rho \gamma^\nu s_L\right] \tag{A.46}$$

$$= -\frac{\Omega g^2/2}{16\pi^2\epsilon} \left[\bar{d}_L \gamma_\mu T^a \gamma_\sigma \gamma_\nu u_L\right] \left[\bar{u}_L \gamma^\nu T^a \gamma^\sigma \gamma^\nu s_L\right] \tag{A.47}$$

$$= -\frac{\Omega g^2/2}{16\pi^2\epsilon} \Big\{ \left[\bar{d}_{L,i} \gamma_\mu \gamma_\sigma \gamma_\nu u_{L,j}\right] \left[\bar{u}_{L,j} \gamma^\mu \gamma^\sigma \gamma^\nu s_{L,i}\right]$$

$$- \frac{1}{N} \left[\bar{d}_{L,i} \gamma_\mu \gamma_\sigma \gamma_\nu u_{L,i}\right] \left[\bar{u}_{L,j} \gamma^\mu \gamma^\sigma \gamma^\nu s_{L,j}\right] \Big\} \tag{A.48}$$

where we have used,

$$\int \text{đ}^d\ell \frac{\ell^\mu \ell^\nu}{\ell^6} = \frac{ig^{\mu\nu}/2}{16\pi^2\epsilon} + ... \tag{A.49}$$

The second term is already in a familiar form of the first operator. However, the second one has the "wrong" index structure. To simplify it we need to use the Fierz identities. Its straight forward to show that

$$\left[\bar{d}_L \gamma_\mu \gamma_\sigma \gamma_\nu u_L\right] \left[u_L \gamma^\mu \gamma^\sigma \gamma^\nu s_L\right] = 16 \left[\bar{d}_L \gamma^\nu s_L\right] \left[\bar{u}_L \gamma_\nu u_L\right] \tag{A.50}$$

after anticommuting the color wavefunctions across. For the second diagram we have,

$$\left[\bar{d}_{L,i} \gamma_\mu \gamma_\sigma \gamma_\nu u_{L,i}\right] \left[\bar{u}_{L,j} \gamma^\mu \gamma^\sigma \gamma^\nu s_{L,j}\right] = 16 \left[\bar{d}_L \gamma_\nu u_L\right] \left[\bar{u}_L \gamma^\nu s_L\right] \tag{A.51}$$

Therefore the diagram is given by,

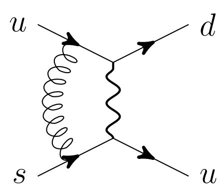
$$= -\frac{\Omega g^2/2}{16\pi^2\epsilon} \left(\mathcal{O}_2 - \frac{1}{N}\mathcal{O}_1\right) \tag{A.52}$$

where $\mathcal{O}_{1,2}$ are defind as

$$\mathcal{O}_1 \equiv \left[\bar{d}_L \gamma_\nu u_L\right] \left[\bar{u}_L \gamma^\nu s_L\right] \tag{A.53}$$

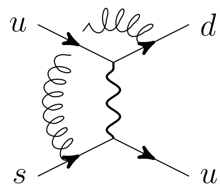
$$\mathcal{O}_2 \equiv \left[\bar{d}_L \gamma_\nu s_L\right] \left[\bar{u}_L \gamma^\nu u_L\right] \tag{A.54}$$

Its easy to see that,



gives the same result.

Next consider the following diagram,



It is given by,

$$= \Omega \int \bar{d}^d\ell(-ig)^2 \left[\bar{d}_{L,i}\gamma_\mu \frac{\not{\ell}}{\ell^2}\gamma_\nu u_{L,j}\right]\left[\bar{u}_{L,k}\gamma^\nu \frac{\not{\ell}}{\ell^2}\gamma^\mu s_{L,\ell}\right](T^a)_{ij}(T^a)_{\ell,k} i^2 \frac{-i}{\ell^2} \tag{A.55}$$

$$= \frac{\Omega g^2/2}{16\pi^2\epsilon} 2\left\{\mathcal{O}_1 - \frac{1}{N}\mathcal{O}_2\right\} \tag{A.56}$$

which gives the same contribution as the other "cross" diagram.

Therefore the sum of all four diagrams gives,

$$-\frac{3\Omega g^2}{16\pi^2\epsilon}\left(\mathcal{O}_2 - \frac{1}{N}\mathcal{O}_1\right) \tag{A.57}$$

The wavefunction renormalization cancels away the final two diagrams. We have,

$$\frac{3\Omega g^2}{16\pi^2\epsilon}\left(\mathcal{O}_2 - \frac{1}{N}\mathcal{O}_1\right) = (1 - Z_{11}^{-1})\mathcal{O}_1 + (1 - Z_{12}^{-1})\mathcal{O}_2 \tag{A.58}$$

which gives,

$$Z_{11}^{-1} = 1 - \frac{3\Omega g^2}{16\pi^2\epsilon}\frac{1}{N} \tag{A.59}$$

$$Z_{12}^{-1} = 1 - \frac{3\Omega g^2}{16\pi^2\epsilon} \tag{A.60}$$

Repeating the entire calculation but starting with $\mathcal{O}_2$ we get the matrix,

$$Z_{ij}^{-1} = 1 - \frac{\Omega g^2}{16\pi^2\epsilon}\begin{pmatrix} -3/N & 3 \\ 3 & -3/N \end{pmatrix} \tag{A.61}$$

which gives the anomalous dimension matrix,

$$\gamma_{ij} = \frac{g^2}{16\pi^2}\begin{pmatrix} -3/N & 3 \\ 3 & -3/N \end{pmatrix} \tag{A.62}$$

where we have used, $\beta = -\epsilon g/2 + ...$

Bibliography

[1] A. Manohar and M. Wise, *Heavy Quark Physics*. Cambridge Monographs on Particle Physics, Nuclear Physics and Cosmology. Cambridge University Press, 2007.

[2] W. Zheng, *Experiments of search for neutron electric dipole moment and spin-dependent short-range force*, 2012.