COSMOLOGY LECTURE NOTES

Lecture notes are largely based on a lectures series given by ...

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Contents

Chapter 1

Preface

I have added in exercises in the text and solutions can be found in the appendix. These notes will likely be expanded until I lose interest in neutrinos. If you have any corrections please let me know at ajd268@cornell.edu.

Chapter 2

Introduction

Cosmology has recently become a very popular research topic.

The outline of the course is as follows,

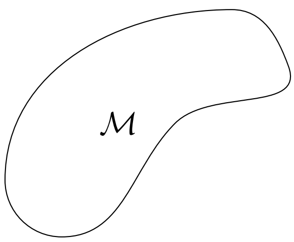
1. Review of Standard Model(SM) and General Relativity(GR)
2. Meet maximally symmetric, FRW, and BH spacetimes
3. Focus in on FRW metric which at first approximation describes our actual universe $\to$ kinematics and dynamics (ΛCDM)
4. Get to know big bang nucleosynthesis(BBN), the cosmic microwave background (CMB), dark matter, dark energy, and matter/antimatter asymmetry
5. Inflation, QFT in curved spacetime, Unruh effect, BH Thermo, cosmological perts.
6. Theory of initial conditions $\to$ Hartle+Hawking

2.1 Differential Geometry

General relavity rests on differential geometry. This is composed of three steps,

1. Introduce manifold and tensors live on this manifold
2. Introduce connections on the manifold through a new covariant derivative
3. Introduce a metric onto your manifold, $g_{\mu\nu}$.

Consider a manifold,



An n dimensional manifold is a bolb such that if you zoom in close enough at any point on the blob it looks like a little patch of n dimensional Euclidean space, $\mathbb{R}^n$. A key feature of a manifold is that it doesn't have any intrinsic way that you are forced to put down coordinates.

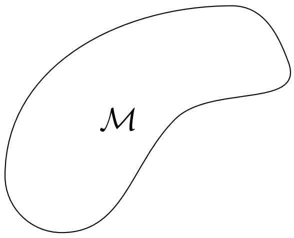
Important objects which can be defined on a manifold are tensors,

$$T^{\alpha_1,\alpha_2,\ldots,\alpha_m}_{\beta_1,\beta_2,\ldots,\beta_n} \tag{2.1}$$

where this is a called a (m,n) tensor. As one changes from the x^μ coordinates to x'^μ coordinate system a tensor must transform as,

$$T^{\alpha_1,\alpha_2,\ldots,\alpha_m}_{\beta_1,\beta_2,\ldots,\beta_n} \to T^{\alpha_1,\alpha_2,\ldots,\alpha_m}_{\beta_1,\beta_2,\ldots,\beta_n} \frac{\partial x'^{\alpha_1}}{\partial x^{\mu_1}} \cdots \frac{\partial x'^{\alpha_n}}{\partial x^{\mu_n}} \frac{\partial x'^{\beta_1}}{\partial x^{\nu_1}} \cdots \frac{\partial x'^{\beta_n}}{\partial x^{\nu_n}} \tag{2.2}$$

If you take a particular case of a $(1,0)$ tensor, v^α, then its called a contravariant vector, while a $(0,1)$ tensor, v_α, is a covariant tensor on a manifold,



We now introduce some notation. We define,

$$\frac{\partial}{\partial x^\mu} \equiv \partial_\mu, \quad \partial_\mu v^\alpha \equiv v^\alpha_{,\mu} \tag{2.3}$$

while covariant derivatives (which we will get to in a moment) are defined as

$$\nabla_\mu v^\alpha \equiv v^\alpha_{;\mu} \tag{2.4}$$

We can always write,

$$v^\mu \partial_\mu = v'^\mu \partial'_\mu = v^\mu \frac{\partial x'^\nu}{\partial x^\mu} \partial'_\nu \tag{2.5}$$

which implies that the coefficients in the primed basis,

$$v'^\nu = \frac{\partial x'^\nu}{\partial x^\mu} v^\mu \tag{2.6}$$

[Q 1: What's the point of this discussion?]

Now consider the derivative of a covariant tensor,

$$\partial_\nu v^\mu \tag{2.7}$$

We would like this object to transform as a $(1,1)$ tensor. However instead we have,

$$\partial'_\nu v'^\mu = \frac{\partial x^\alpha}{\partial x'^\nu} \partial_\alpha \left(\frac{\partial x'^\mu}{\partial x^\beta} v^\mu \right) = \frac{\partial x^\alpha}{\partial x'^\nu} \left(\frac{\partial x'^\mu}{\partial x^\beta} \partial_\alpha v^\mu + v^\mu \frac{\partial^2 x'^\mu}{\partial x^\alpha \partial x^\beta} \right) \tag{2.8}$$

We have an extra term which we don't want that spoils the transformation law. The solution is to introduce a covariant deritive,

$$\nabla_\nu v^\mu \equiv \partial_\nu v^\mu + \Gamma^\mu_{\nu\alpha} v^\alpha \tag{2.9}$$

where to "fix the problem" we must have,

$$\Gamma'^\alpha_{\beta\gamma} = \frac{\partial x'^\alpha}{\partial x^\gamma}\frac{\partial x^\mu}{\partial x'^\beta}\frac{\partial x^\rho}{\partial x'^\gamma}\Gamma^\mu_{\nu\rho} - \frac{\partial^2 x'^\alpha}{\partial x^\rho \partial x^\tau}\frac{\partial x^\rho}{\partial x'^\beta}\frac{\partial x^\tau}{\partial x'^\gamma} \tag{2.10}$$

where the second term cancels away the term you don't want. To see how this works consider the transformation of the covariant derivative,

$$\nabla_\nu v^\mu = \partial_\nu v^\mu + \Gamma^\mu_{\nu\lambda} v^\lambda \tag{2.11}$$

$$\rightarrow \frac{\partial x^\beta}{\partial x'^\nu}\frac{\partial}{\partial x^\beta}\left(\frac{\partial x'^\mu}{\partial x^\rho} v^\rho\right) + \frac{\partial x'^\mu}{\partial x^\gamma}\frac{\partial x^\rho}{\partial x'^\nu}\frac{\partial x^\sigma}{\partial x'^\lambda}\Gamma^\mu_{\nu\rho}\frac{\partial x'^\lambda}{\partial x^\tau} v^\tau - \frac{\partial^2 x'^\mu}{\partial x^\rho \partial x^\tau}\frac{\partial x^\rho}{\partial x'^\nu}\frac{\partial x^\tau}{\partial x'^\lambda}\frac{\partial x'^\lambda}{\partial x^\beta} v^\beta \tag{2.12}$$

Using,

$$\frac{\partial x^\tau}{\partial x'^\lambda}\frac{\partial x'^\lambda}{\partial x^\beta} = \delta^\tau_\beta \tag{2.13}$$

we can write the last term as,

$$\frac{\partial^2 x'^\mu}{\partial x^\beta \partial x^\tau}\frac{\partial x^\beta}{\partial x'^\nu} v^\rho \tag{2.14}$$

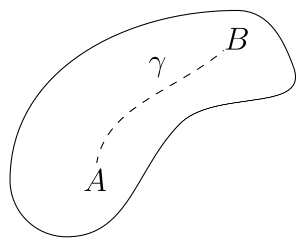
which is equal to the troublesome term on the left. Finally we get (using a similar identity for the Christoffel symbol term),

$$\nabla_\nu v^\mu \rightarrow \frac{\partial x^\beta}{\partial x'^\nu}\frac{\partial x'^\mu}{\partial x^\rho}\left(\nabla_\beta v^\rho\right) \tag{2.15}$$

as required.

At the moment the Γ can be anything we want except that it must transform in this non-tensorial way. These Γ's are often called the "connection" (the derivatives are also often referred to in this way). This is useful since now we can easily build scalars.

Now we have a connection between points A and B on a Manifold:



Before we had vectors at point A and others at B but we had no natural way to connect vectors in one space or another, but now we have a new way to connect them. We define the path γ defined by $\gamma : x^\mu(\tau)$ and $u^\mu \equiv \frac{\partial x^\mu}{\partial \tau}$ is the tangent vector along that curve.

We say that a tensor doesn't change along a curve, or parallel transported, if

$$u^\mu \nabla_\mu T^{\alpha_1,\ldots,\alpha_m}_{\beta_1,\ldots,\beta_n} = 0 \tag{2.16}$$

In other words we say that a curve is parallel to itself along the entire curve. We now get a natural definition of a geodesic:

$$u^\mu \nabla_\mu u^\nu = 0 \tag{2.17}$$
$$u^\mu \partial_\mu u^\nu + u^\mu \Gamma^\nu_{\mu\alpha} u^\alpha = 0 \tag{2.18}$$
$$\frac{\partial x^\mu}{\partial \tau}\frac{\partial x^\nu}{\partial x^\mu \partial \tau} + \Gamma^\nu_{\mu\alpha}\frac{\partial x^\mu}{\partial \tau}\frac{\partial x^\alpha}{\partial \tau} = 0 \tag{2.19}$$
$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma^\nu_{\mu\alpha}\frac{\partial x^\mu}{\partial \tau}\frac{\partial x^\alpha}{\partial \tau} = 0 \tag{2.20}$$

This is the equation of motion of a geodesic.

A nice feature of derivatives is that they commute,

$$[\partial_\mu, \partial_\nu] f(x) = 0 \tag{2.21}$$

However, this is no longer true for covariant derivatives. We now show this explicitly. We have,

$$(\nabla_\mu)^\alpha{}_\beta (\nabla_\nu)^\beta{}_\gamma v^\gamma = (\partial_\mu g^\alpha{}_\beta + \Gamma^\alpha_{\mu\beta})(\partial_\nu g^\beta{}_\gamma + \Gamma^\beta_{\nu\gamma}) v^\gamma \tag{2.22}$$
$$= \partial_\mu \partial_\nu v^\alpha + \Gamma^\alpha_{\mu\gamma}\partial_\nu v^\gamma + \Gamma^\alpha_{\nu\gamma}\partial_\mu v^\gamma + (\partial_\mu \Gamma^\alpha_{\nu\gamma}) v^\gamma + \Gamma^\alpha_{\mu\beta}\Gamma^\beta_{\nu\gamma} v^\gamma \tag{2.23}$$

The terms $\partial_\mu\partial_\nu v^\alpha$ and $\Gamma^\alpha_{\mu\gamma}\partial_\nu v^\gamma + \Gamma^\alpha_{\nu\gamma}\partial_\mu v^\gamma$ are symmetric under the interchange of μ and ν. Therefore we have the commutator is,

$$[\nabla_\mu, \nabla_\nu] v^\gamma = \underbrace{\left[(\partial_\mu \Gamma^\alpha_{\nu\gamma}) - (\partial_\nu \Gamma^\alpha_{\mu\gamma}) + \Gamma^\alpha_{\nu\beta}\Gamma^\beta_{\mu\gamma} - \Gamma^\alpha_{\mu\beta}\Gamma^\beta_{\nu\gamma}\right]}_{R^\alpha_{\mu\gamma\nu}} v^\gamma \tag{2.24}$$

where $R^\alpha_{\mu\gamma\nu}$ is defined as the Reimann curvature. This is called the Ricci identity. Further, we define the Ricci curvature by,

$$R_{\beta\delta} \equiv R^\alpha_{\beta\alpha\delta} \tag{2.25}$$

We don't need a metric to do this contraction since one index is already up and another one is down.

This still doesn't quite smell like the universe we live in. The final stage is to introduce a metric on the manifold. The metric defines a dot product in your space:

$$v^2 \equiv g_{\alpha\beta} v^\alpha v^\mu, \quad v \cdot u \equiv g_{\alpha\beta} v^\alpha u_\beta \tag{2.26}$$

For two infinestimal nearby points, the metric lets you determine the distance between those two points,

$$ds^2 = g_{\mu\nu}(x) dx^\mu dx^\nu \tag{2.27}$$

The distance between two points is given by,

$$\int_{\gamma,A}^{B} ds \tag{2.28}$$

We can also now make a scalar from the Ricci curvature,

$$R \equiv g^{\beta\delta} R_{\beta\delta} \tag{2.29}$$

At the moment it looks like we specified an arbitrary connection, $\Gamma^{\alpha}_{\beta\gamma}$ and an unrelated $g_{\mu\nu}$. However, there is one unique connection that makes the metric parallel to itself along the manifold,

$$g_{\mu\nu;\alpha} = 0 \tag{2.30}$$

Such a connection is often called the Levi-Civita connection. It is given by,

$$\Gamma^{\alpha}_{\beta\gamma} = \frac{1}{2} g^{\alpha\delta} \left(g_{\beta\delta,\gamma} + g_{\gamma\delta,\beta} - g_{\beta\gamma,\delta} \right) \tag{2.31}$$

[Q 2: Exercise!]

Now that curves have length we get a new notion of a geodesic. In the case of Reimannian geometry, where there is only space and not spacetime, every direction you move away from a straight line produces a longer path. More generally in spacetime they are the curves of extremal length. In other words the curves which to first order don't change the length. At first glance this seems to be in contradicatory with the definition we gave before: The curves that are as straight as possible and parallel transport their own tangent vector,

$$u^{\mu} \nabla_{\mu} u_{\nu} = 0 \tag{2.32}$$

However, these two definitions are the same. To find the geodesic in the second way you define a Lagrangian,

$$L \equiv \frac{1}{2} g_{\mu\nu} \dot{x}^{\mu} \dot{x}^{\nu} \tag{2.33}$$

where $\dot{x}^{\mu} \equiv \partial x^{\mu} / \partial \tau$. Finding the equation of motion gives,

$$\partial_{\mu} L = \frac{d}{d\tau} \frac{\partial L}{\partial \dot{x}^{\mu}} \tag{2.34}$$

This definition is in practice much easier to work with. These arise from extremizing the length of the curve.

Thus far we have only studied differential geometry. We now move onto general relativity.

The simplest scalars we can write down is the Ricci scalar and just a constant. As an an satz we take that the be the action of the our theory,

$$S = \int d^4 x \sqrt{-g} \frac{1}{16\pi G_N} \left(R - 2\Lambda \right) \tag{2.35}$$

where $\sqrt{-g}$ is the square root of the negative determinate of the metric. This is called the Einstein Hilbert action.

We now do the following,

1. Calculate the Einstein equations from this action.

2. Derive a nifty formula for the stress energy tensor

In principle we can get the Einstein equations in the typical way by varying the metric,

$$\frac{\delta S}{\delta g^{\mu\nu}} = 0 \tag{2.36}$$

However, this turns out to be a big mess. Part of the reason that it is a big mess is because the Einstein Hilbert action involves first and second derivatives of the metric. Thankfully, there is a better way to do it called the Palatini method. The basis of this method is the observation that the connection and the metric are completely independent objects. The fact that they have a relation between them isn't obvious ahead of time. What we can do is vary both the metric and the connection simultaneously.

We write,

$$S(g, \Gamma) \tag{2.37}$$

When we vary with respect to the connection,

$$\frac{\delta S}{\delta \Gamma} = 0 \tag{2.38}$$

we will get the "equation of motion" that is just the equation above that relates the connection to the metric and when we vary with respect to the metric,

$$\frac{\delta S}{\delta g^{\mu\nu}} = 0 \tag{2.39}$$

we'll get the Einstein equations.

The first step is to split the Reimann curvature into the Ricci scalar and the metric,

$$S = \int d^4x \sqrt{-g} \left\{ \frac{R_{\alpha\beta} g^{\alpha\beta} - 2\Lambda}{16\pi G_N} + \mathcal{L}_m(g_{\mu\nu}, \phi_i, \partial_\mu \phi_i) \right\} \tag{2.40}$$

where ϕ_i denote the matter fields. Here we have assumed renormalizabiliity of the Lagrangian and hence there are no matter fields couple to the Ricci curvature and hence $\mathcal{L}_m$ doesn't depend on the connection.

We first do the variation with respect the connection. The connection only appears in the Ricci curvature which is given explicitly by,

$$R_{\alpha\beta} = R^{\mu}_{\alpha\mu\beta} = \Gamma^{\gamma}_{\alpha\beta,\gamma} + \Gamma^{\rho}_{\alpha\gamma}\Gamma^{\gamma}_{\rho\beta} - \left(\Gamma^{\gamma}_{\alpha\gamma,\beta} + \Gamma^{\gamma}_{\gamma\rho}\Gamma^{\rho}_{\alpha\beta} \right) \tag{2.41}$$

and now vary this and keep only parts that are first order in $\delta\Gamma$.

We have,

$$\delta \left(\Gamma^{\mu}_{\alpha\beta,\mu} + \Gamma^{\mu}_{\alpha\gamma}\Gamma^{\mu}_{\mu\beta} \right) = \delta\Gamma^{\mu}_{\alpha\beta,\mu} - (\delta\Gamma^{\gamma}_{\alpha\mu})\Gamma^{\mu}_{\gamma\beta} - \Gamma^{\mu}_{\gamma\beta}(\delta\Gamma^{\gamma}_{\alpha\mu}) \tag{2.42}$$

We define,

$$\delta\left(\Gamma^{\mu}_{\alpha\beta;\mu}\right) = \delta\Gamma^{\mu}_{\alpha\beta,\mu} - (\delta\Gamma^{\mu}_{\mu\rho})\Gamma^{\rho}_{\alpha\beta} - \Gamma^{\mu}_{\mu\rho}\delta\Gamma^{\rho}_{\alpha\beta} \tag{2.43}$$

which finally gives,

$$\delta R_{\alpha\beta} = \delta\left(\Gamma^{\mu}_{\alpha\beta;\mu}\right) - \delta\left(\Gamma^{\gamma}_{\alpha\gamma;\beta}\right) \tag{2.44}$$

Inserting into the variation of the action,

$$\delta S = \int d^4x\sqrt{-g}\left\{\frac{g^{\alpha\beta}(\delta\Gamma^{\gamma}_{\alpha\beta;\gamma} - \delta\Gamma^{\gamma}_{\alpha\gamma;\beta})}{16\pi G_N}\right\} \tag{2.45}$$

We want to write this as some expression of the form,

$$\int d^4x\sqrt{-g}\,\{...\}\,\delta\Gamma^{\alpha}_{\beta\gamma} \tag{2.46}$$

which would imply that due to the calculus of variation that the term in curly brackets is equal to zero.

We don't have this form yet since we have the covariant derivatives in Γ. We need to integrate by parts. There is a tricky and not tricky way to go about this. We will do it the rudimentary way. To the integration by parts we just need to know how to evaluate,

$$\left(\sqrt{-g}g^{\alpha\beta}\right)_{,\gamma} \tag{2.47}$$

You need to figure out what you mean by the covariant derivative of g.

To do this we need the following two identities. Suppose we start with a metric and we vary it a little,

$$g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu} \tag{2.48}$$

We want to know how this effects the inverse metric,

$$g^{\mu\nu} \to g^{\mu\nu} + \delta g^{\mu\nu} \tag{2.49}$$

and how will the determinate vary,

$$g \to g + \delta g \tag{2.50}$$

Its easy to know what happens to the inverse since we have,

$$g_{\alpha\beta}g^{\beta\gamma} = \delta^{\gamma}_{\alpha} \tag{2.51}$$

$$g_{\alpha\beta}\delta g^{\beta\gamma} + \delta g_{\alpha\beta}g^{\beta\gamma} = 0 \tag{2.52}$$

which implies that

$$\delta g_{\alpha\beta} = -g_{\alpha\gamma}g_{\beta\delta}\delta g^{\gamma\delta} \tag{2.53}$$

Notice that there is a key minus sign which shows that in GR the metric and its inverse are two different tensors, not the same one with their indices raised and lowered.

There is a famous formula in linear algebra known as the Jacobi identity. It gives the derivative of a determinant:

$$\frac{\delta \det M}{\delta M_{ij}} = \frac{\det(M+\delta M) - \det M}{\delta M} \tag{2.54}$$

$$= \frac{\det M}{\delta M_{ij}}\left(\det\left(1+M^{-1}\delta M\right) - 1\right) \tag{2.55}$$

but we can expand,

$$\det(1+M^{-1}\delta M) = \exp\left(\mathrm{Tr}\log\left(1+M^{-1}\delta M\right)\right) \tag{2.56}$$

$$\approx \exp\left(\mathrm{Tr}M^{-1}\delta M\right) \tag{2.57}$$

$$\approx 1 + \mathrm{Tr}M^{-1}\delta M \tag{2.58}$$

Therefore,

$$\frac{\delta \det M}{\delta M_{ij}} = \frac{\det M}{\delta M_{ij}}\mathrm{Tr}M^{-1}\delta M \tag{2.59}$$

Now since,

$$\frac{\delta M_{\ell k}}{\delta M_{ij}} = \delta_{\ell i}\delta_{kj} \tag{2.60}$$

we have,

$$\frac{d \det M}{d M_{ij}} = \det M M^{-1}_{ij} \tag{2.61}$$

In our case we have,

$$\delta g = g g^{\alpha\beta}\delta g_{\alpha\beta} \tag{2.62}$$

which using this formula above this can be rewritten,

$$\delta g = -g g_{\alpha\beta}\delta g^{\alpha\beta} \tag{2.63}$$

which finally gives,

$$g_{,\gamma} = g g^{\alpha\beta} g_{\alpha\beta,\gamma} = -g g_{\alpha\beta} g^{\alpha\beta}_{,\gamma} \tag{2.64}$$

This formula is really all you need to work out the variation above. We get,

$$\delta S = \int \frac{d^4x\sqrt{-g}}{16\pi G_N}\delta\Gamma^{\gamma}_{\alpha\beta}\left\{\left(\frac{1}{2}g^{\alpha\beta}g_{\mu\nu}g^{\mu\nu}_{;\gamma} - g^{\alpha\beta}_{;\gamma}\right) - \delta^{\beta}_{\gamma}\left(\frac{1}{2}g^{\alpha\rho}g_{\mu\nu}g^{\mu\nu}_{;\rho} - g^{\alpha\rho}_{;\rho}\right)\right\} \tag{2.65}$$

$\delta S = 0$ implies that,

$$\frac{1}{2}g^{\alpha\beta}g_{\mu\nu}g^{\mu\nu}_{;\gamma} - g^{\alpha\beta}_{;\gamma} - \delta^{\beta}_{\gamma}\left(\frac{1}{2}g^{\alpha\rho}g_{\mu\nu}g^{\mu\nu}_{;\rho} - g^{\alpha\rho}_{;\rho}\right) = 0 \tag{2.66}$$

which after contracting with δ^{γ}_{β} and $g_{\alpha\beta}$ respectively gives,

$$\frac{1}{2}g^{\alpha\rho}g_{\mu\nu}g^{\mu\nu}_{;\rho} - g^{\alpha\rho} = 0 \tag{2.67}$$

$$\frac{1}{2}g^{\alpha\beta}g_{\mu\nu}g^{\mu\nu}_{;\gamma} - g^{\alpha\beta}_{;\gamma} = 0 \tag{2.68}$$

the bottom equation simplifies to $g^{\alpha\beta}_{;\gamma} = 0$ which shows that varying the Christoffel symbol leads us to the Levi-Civita connection.

If you instead vary the metric one can show that you get[Q 3: exercise],

$$\int d^4x\sqrt{-g}\delta g^{\mu\nu}\left[\frac{R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R}{16\pi G_N}-\frac{1}{2}g_{\mu\nu}\mathcal{L}_m+\frac{\partial\mathcal{L}_m}{\partial g^{\mu\nu}}\right] \tag{2.69}$$

This leads to the Einstein equation,

$$G_{\mu\nu}+\Lambda g_{\mu\nu}=8\pi G_N T_{\mu\nu} \tag{2.70}$$

if you define,

$$G_{\mu\nu}\equiv R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R \tag{2.71}$$

and the energy-momentum tensor,

$$T_{\mu\nu}\equiv g_{\mu\nu}\mathcal{L}_m-2\frac{\partial\mathcal{L}_m}{\partial g^{\mu\nu}} \tag{2.72}$$

For example for a real scalar field,

$$\mathcal{L}_m=-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi-V(\phi) \tag{2.73}$$

gives,

$$T_{\mu\nu}=\phi_{,\mu}\phi_{,\nu}-g^{\mu\nu}\left[\frac{1}{2}g^{\alpha\beta}\phi_{,\alpha}\phi_{,\beta}+V(\phi)\right] \tag{2.74}$$

This theory is particularly important and has many applications in cosmology.

Another important example is if you take the matter part of the action to be the action for a bunch of point particles,

$$S_m=-\sum_j m_j\int d\tau_j \tag{2.75}$$

one can then apply the stress energy tensor here and get the formula for a bunch of point particles in GR.

Finally there is another important stress energy which is a good approximation in many systems in cosmology and that is of a perfect fluid. It is not derived from a Lagrangian but its given by,

$$T_{\mu\nu}=pg_{\mu\nu}+(p+\rho)u_\mu u_\nu \tag{2.76}$$

where all these are functions that depend on spacetime. u_μ is the four-velocity of the perfect fluid. In the rest frame of that fluid ρ is the density and p is the pressure. In this frame we have,

$$T^\mu_\nu=\begin{pmatrix}-\rho & 0 & 0 & 0\\ 0 & \rho & 0 & 0\\ 0 & 0 & \rho & 0\\ 0 & 0 & 0 & \rho\end{pmatrix} \tag{2.77}$$

This tends to be a general result for an isotropic system.

2.1.1 The *connection* between GR and gauge theories

Recall that we can write a tensor in differential geometry but its derivative is no longer a tensor and so we need an extra peice,

$$\nabla_\mu v^\alpha = (\partial_\mu v^\alpha + \Gamma_{\mu\beta}{}^{\alpha})v^\beta \tag{2.78}$$

and under a coordinate transformation we have,

$$v^\alpha \to v'^\alpha = \frac{\partial x'^\alpha}{\partial x^\beta} v^\beta \tag{2.79}$$

Similarly in gauge theory we have,

$$\Phi^a \to U(x)^a{}_b \Phi^b \tag{2.80}$$

We similarly need to introduce a "connection" for gauge theories,

$$D_\mu \Phi^a = \partial_\mu \Phi^a + A_\mu{}^a{}_b \Phi^b \tag{2.81}$$

In this case a and b are indices in the gauge space and not in space time as μ.

In differential geometry to get a tensor from the Crystoff symbols,Γ, we take a commutator of the covariant derivatives. We do the same thing in gauge theories,

$$\left[D_\mu, D_\nu\right] \Phi^a = i(F^a{}_b)_{\mu\nu} \Phi^b \tag{2.82}$$

We now want to build an action from the tensors that we have defined. In the case of GR the simplest thing we can write down is just a constant,

$$S_{simp} = -2\Lambda \tag{2.83}$$

where we have added a minus two for later convenience. The next simplest object we can write down is

$$g^{\beta\nu} R^\alpha{}_{\beta\alpha\nu} \tag{2.84}$$

conventionally we normally write,

$$L_{GR} = \frac{1}{16\pi G_N}\left(-2\Lambda + g^{\beta\nu} R^\alpha{}_{\beta\alpha\nu}\right) \tag{2.85}$$

In gauge theories we can't form a scalar with just a single $F_{\mu\nu}$ since they are traceless. So the simplest thing you can write down is

$$L_{YM} = -\frac{1}{4}\mathrm{Tr}\left[F_{\mu\nu}F^{\mu\nu}\right] \tag{2.86}$$

Notice that the structure of these two theories is parallel.

2.1.2 Weak field limit

As an example lets consider the weak field limit. In this case we take the metric of the form,

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{2.87}$$

where $h_{\mu\nu}$ is small. The gravitational potential in Newton's theory is a scalar, ϕ. This suggests the metric,

$$g_{00} = -(1+2\phi) \quad g_{ij} = \delta_{ij} \tag{2.88}$$

where the 2 is just a useful convention. We now need to calculate the connection. We have,

$$\Gamma^0_{00} = \frac{1}{2} g^{0\rho} \left(\partial_0 g_{\rho 0} + \partial_0 g_{\rho 0} - \partial_\rho g_{00}\right) \tag{2.89}$$

We calculate each term seperately,

$$g^{0\rho}\partial_0 g_{\rho 0} = \left(\eta^{00} + h^{00}\right)(-2\partial_0\phi) = 0 \tag{2.90}$$

where we have assumed a static gravitational source. Furthermore we have,

$$g^{0\rho}\partial_0\partial_\rho\phi = g^{0i}\partial_0\partial_i\phi = 0 \tag{2.91}$$

since $g^{0i} = 0$ and so $\Gamma^0_{00} = 0$.

Next we have,

$$\Gamma^0_{0i} = \frac{1}{2} g^{0\rho} \left(\partial_0 g_{\rho i} + \partial_i g_{\rho 0} - \partial_\rho g_{0i}\right) \tag{2.92}$$

The first and last term are zero because the problem is static and metric is diagonal while the the middle term gives,

$$\Gamma^0_{0i} = -\partial_i\phi \tag{2.93}$$

Continuing,

$$\Gamma^i_{00} = \frac{1}{2} g^{i\rho}(-\partial_\rho g_{00}) = g^{ij}\partial_j\phi = \partial_j\phi \tag{2.94}$$

$$\Gamma^0_{ij} = \Gamma^j_{i0} = \Gamma^j_{0i} = \Gamma^k_{ij} = 0 \tag{2.95}$$

We now can find how a geodesic looks like in this geometry using the equations of motion,

$$\frac{d^2x^i}{d\tau^2} + \partial_i\phi\frac{\partial t}{\partial\tau}\frac{\partial t}{\partial\tau} = 0 \quad \frac{d^2x^0}{d\tau^2} - \partial_i\phi\frac{\partial x^i}{\partial\tau}\frac{\partial t}{\partial\tau} = 0 \tag{2.96}$$

If we take the nonrelativitstic limit we can set $\tau = t$ and we have,

$$\frac{d^2\mathbf{x}}{d\tau^2} = -\nabla\phi \quad (\nabla\phi)\cdot\mathbf{v} = 0 \tag{2.97}$$

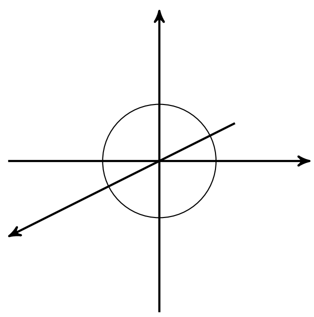
where the first equation is the typical equation of motion. The second equation shows that for a geodescic the particle moves perpencular to the direction of motion, i.e., in a circle.

2.1.3 Building a metric

There are three most important metrics,

1. Maximully symmetric
   - Minkowski (E_n), M_n
   - Sphere (S_n), dS_n
   - Antidisitter (H_n), AdS_n
2. FRW
3. Black hole
   - Schwarzschild (static blackhole)
   - Kerr (for a spinning blackhole)

As a warm up lets start with trying to describe universe on the surface of a $2D$ sphere,



To describe the motion on the sphere we have to embed it in a universe of 2+1 dimensions. We only two coordinatese to describe the motion on the sphere, but we need one extra coordinate when working in the embedding space.The coordinates on the sphere, x_1 and x_2, are restricted by

$$x_1^2 + x_2^2 = R^2 - x_3^2 \tag{2.98}$$

If we assume the embedding space is completely flat (as it would be if we try to describe motion on say a tennis ball in our flat 3 dimensional world) we have the metric in that space given by,

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2.99}$$

and so we have the invariant interval,

$$ds^2 = dx_1^2 + dx_2^2 + dx_3^2 \tag{2.100}$$

Inserting in the constraint above we have the invariant interval on the sphere,

$$ds^2 = dx_1^2 + dx_2^2 + \frac{x_1^2 dx_1^2 + x_2^2 dx_2^2 + 2x_1 x_2 dx_1 dx_2}{R^2 - x_1^2 - x_2^2} \tag{2.101}$$

To extract the metric we need to write this in the form $g_{ab}(x)dx^a dx^b$. We make the definitions,

$$x_a \equiv \delta_{ab} x^b \tag{2.102}$$
$$x^2 \equiv x_a x^a \equiv \delta_{ab} x^a x^b = x_1^2 + x_2^2 \tag{2.103}$$

then we have,

$$ds^2 = \delta_{ab} dx^a dx^b + \frac{\delta_{db}\delta_{ca} x^d x^c dx^a dx^b}{R^2 - x^2} \tag{2.104}$$
$$= \left(\delta_{ab} + \frac{x_a x_b}{R^2 - x^2}\right) dx^a dx^b \tag{2.105}$$
$$\tag{2.106}$$

Therefore, we have the metric,

$$g_{ab} = \delta_{ab} + \frac{x_a x_b}{R^2 - x^2} \tag{2.107}$$

Notice that this offdiagonal metric is not that we are used to for spherical coordinates. This is because we are using quasicartesion coordinates. To fix this we can make the coordinate redefinition,

$$x^1 = R \sin\theta \cos\phi \tag{2.108}$$
$$x^2 = R \sin\theta \sin\phi \tag{2.109}$$

Then we have,

$$dx^1 = R(c_\theta c_\phi d\theta - s_\theta s_\phi d\phi) \tag{2.110}$$
$$dx^2 = R(s_\theta c_\phi d\phi + c_\theta s_\phi d\theta) \tag{2.111}$$
$$dx_a dx^a = R^2 \left(s_\theta^2 d\phi^2 + c_\theta^2 d\theta^2\right) \tag{2.112}$$
$$x^2 = R^2 \sin^2\theta \tag{2.113}$$

and so,

$$ds^2 = R^2 \left(\sin^2\theta d\phi^2 + d\theta^2\right) \tag{2.114}$$

or a metric,

$$g = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix} \tag{2.115}$$

We see now that the transformation above simply diagonalized the previous version of the metric.

Now that we saw the toy example we now move onto studying

The general construction is really just a trivial modification of this case. Suppose instead that we embed a n dimensional manifold into a $n+1$ dimensional manifold with signitures,

$$(p, q) \tag{2.116}$$

and curvatures that can be positive, zero, or negative, which we denote by variable,

$$K = \begin{cases} +1 \\ 0 \\ -1 \end{cases} \tag{2.117}$$

[Q 4: Why not embed a curved manifold into a flat space?]

We invariant interval in the embedding space is modified as follows,

$$\begin{aligned} ds^2 &= \delta_{ab} dx^a dx^b + (dx^{n+1})^2 && (2.118) \\ &\to \eta_{ab}^{(p,q)} dx^a dx^b + K(dx^{n+1})^2 && (2.119) \end{aligned}$$

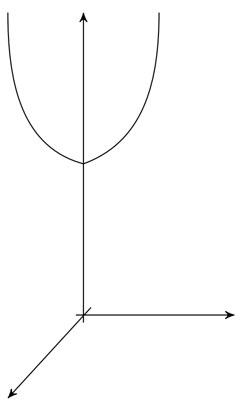
where,

$$\eta_{ab}^{(p,q)} = (\underbrace{+1, ..., +1}_{p}, \underbrace{-1, ..., -1}_{q}) \tag{2.120}$$

To understand why this is the right identification consider the 3+1 dimensional version of this,

$$-t^2 + x_1^2 + x_2^2 = R^2 - K x_3^2 \tag{2.121}$$

If $K = 1$, this is just the equation of a sphere in Minkowski space. If $K = -1$ we have a hyperbola,



This a space of "negative" curvature.

We take the embedding equation to be the same,

$$K\rho^2 = \eta_{ab}^{(p,q)} x^a x^b + K(x^{n+1})^2 \tag{2.122}$$

We find,

$$ds^2 = \left(\eta_{ab}^{(p,q)} + \frac{K x_a x_b}{\rho^2 - x^2}\right) dx^a dx^b \tag{2.123}$$

where in analogy to before,

$$x^2 = \eta_{ab}^{(p,q)} x^a x^b \tag{2.124}$$

$$x_a = \eta_{ab}^{(p,q)} x^b \tag{2.125}$$

This notation is such that,

$$x^a_{,b} = \delta^a_b, x_{a,b} = \eta_{ab}, x^2_{,c} = 2x_c \tag{2.126}$$

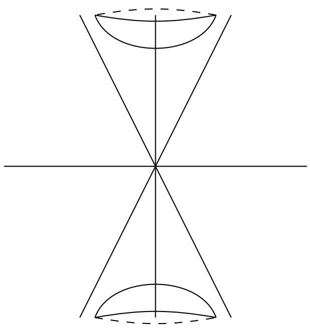
Examining the equation for ds^2 we see that,

$$g_{ab}^{(p,q)} = \eta_{ab}^{(p,q)} + K\frac{x_a x_b}{\rho^2 - Kx^2} \tag{2.127}$$

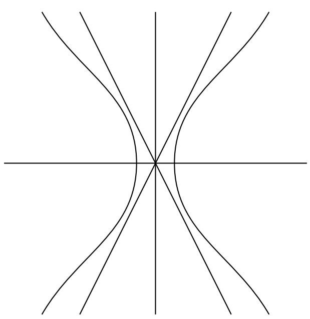
This is the metric for every maximally symmetric case. We tabulate all the important cases below,

	$+1$	0	-1
$(n, 0)$	Sphere(S_n)	Euclidean(E_n)	Hyperboloid(H_n)
$(n-1, 1)$	de Sitter(dS_n)	Mikowski(M_n)	Anti de Sitter(AdS_n)

We already are familiar with Minkowski and euclidean space. We also just saw how a spherical space looks like with positive curvature. The light cone is,



The last case has a purely negative curvature. The light cone is,



The claim is that the inverse is given by,

$$\eta^{ab}_{(p,q)} - K\frac{x^a x^b}{\rho^2} \tag{2.128}$$

To see that this is true we can just do it explicitly,

$$\begin{aligned} g^{ab}g_{bc} &= \left(\eta^{ab}_{(p,q)} - K\frac{x^a x^b}{\rho^2}\right)\left(\eta_{bc} + K\frac{x_b x_c}{\rho^2 - Kx^2}\right) & (2.129)\\ &= \delta^a_c + K\left(\frac{x^a x_c}{\rho^2 - Kx^2} - \frac{x^a x_c}{\rho^2}\right) - K^2\frac{x^2 x^a x_c}{\rho^2(\rho^2 - Kx^2)} & (2.130)\\ &= \delta^a_c + Kx^a x_c\frac{Kx^2}{\rho^2(\rho^2 - Kx^2)} - K^2\frac{x^2 x^a x_c}{\rho^2(\rho^2 - Kx^2)} & (2.131)\\ &= \delta^a_c & (2.132) \end{aligned}$$

as required. They key formula to remember for maximally symmetric spaces is the one for the Reimann curvature with an index lowered,

$$R_{abcd} = g_{ae}R^e_{bcd} = \frac{K}{\rho^2}\left(g_{ac}g_{bd} - g_{ad}g_{bc}\right) \tag{2.133}$$

Above we discussed maximally symmetric spacetimes. We now consider FRW spaces. FRW spacetime is a one such that if you freeze time and explore the geometry at some time the space looks like a maximally symmetric spacetime, but there is no particular symmetry in the temporal direction. This is described by the interval,

$$ds^2 = -dt^2 + a^2(t)g^{(p,q),K}_{ab}dx^a dx^b \tag{2.134}$$

where $a(t)$ is called the scale factor. If we just take our metric that we derived previously and plug it in we get,

$$ds^2 = -dt^2 + a^2(t)\left[\delta_{ab} + K\frac{x_a x_b}{\rho^2 - Kx^2}\right]dx^a dx^b \tag{2.135}$$

We have two quantities here $a(t)$ and ρ. However, these are not independent of one another. It is conventional to define,

$$x^a\rho\tilde{x}^a, a = \tilde{a}\rho \tag{2.136}$$

which gives,

$$ds^2 = -dt^2 + \tilde{a}^2(t)\left[\delta_{ab} + K\frac{\tilde{x}_a\tilde{x}_b}{1 - K\tilde{x}^2}\right]d\tilde{x}^a d\tilde{x}^b \tag{2.137}$$

We will henceforth forget that we had these tildes.

The Freidmann equation is given by,

$$H^2 = \frac{16\pi G_N\rho + 2\Lambda}{n(n-1)} - \frac{K}{a^2} \tag{2.138}$$

where,

$$H(t) \equiv \frac{da/dt}{a} \equiv \frac{\dot{a}}{a} = \frac{d(\log a)}{dt} \tag{2.139}$$

This arises from the G_{00} component of Einsteins equation.

We can then derive that,

$$\dot{\rho} = -nH(p + \rho) \tag{2.140}$$

which is derived from the G_{ii} component of Einsteins equation. Lets consider the case where the pressure is negligable when compared to the energy density[1]. Then we have,

$$\frac{\dot{\rho}}{\rho} = -3\frac{\dot{a}}{a} \tag{2.141}$$

which means that for a 3 dimensional universe,

$$\rho \propto a^{-3} \tag{2.142}$$

[1]This is often true because rest mass is so large compared to thermal energies.