COMPOSITE HIGGS AND EFT LECTURE NOTES

Lecture notes are largely based on a lectures series given by Sekhar Chivukula at TASI on composite higgs and effective field theories

Notes Written by: JEFF ASAF DROR

2014

Contents

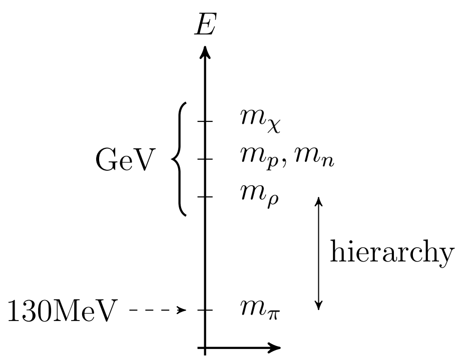
1 Preface

This lecture notes are based on a TASI course given by Sekhar Chivukula If you have any corrections please let me know at ajd268@cornell.edu.

2 Setting the stage

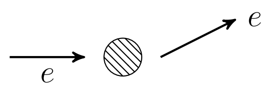
2.1 What is compositeness?

Lets first better understand the theory we are familiar with, QCD. It certainly has compositeness. The low energy degrees of freedom are shown below,



Looking at this spectrum we see a charactoristic mass scale associated with the heavy particles (in this case around 1GeV) and then there is an anomalously light object (in this case the pion). So we see a hierarchy between the two scales. The questions immediately arises, what is the natural scale of the theory? Is it associated with the pions or the high energy states.

One thing we can do is take electrons and scatter them off a pion,



and look at the form factor. The proton isn't actually a point like object, its Rutherford cross section isn't simply the $1/q^2$ you'd expect. Its got a form factor contribution that we can measure. This form factor has a scale in it and its roughly given by $\Lambda \sim$ GeV. This is another reason to beleive that the fundamental scale is around a GeV and the pion is the one that doesn't belong. The fundamental scale is associated with lots and lots of resonances.

The signitures of compositeness are,

- Hierarchy of scales
  - Some light states, why?
- Resonances
  - Many whose scale is of order the fundamental scale of the theory

We'd like to understand to explain the presense of the multitude of states in terms of a few.

Our goal is that we'd like to construct a theory with some fundamental scale, Λ, with a higgs-like object, X, whose mass obeys,

$$\frac{m_X}{\Lambda} \ll 1 \tag{1}$$

The claim is that if you can arrange this hierarchy then the properties of the particle will look a lot like the higg's boson.

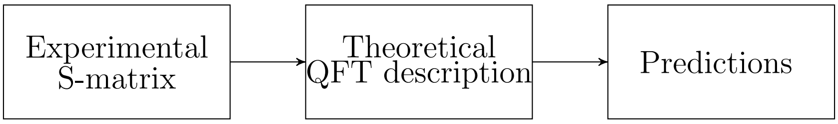
The question then is why would this state be light? There are two known possibilities,

1. Symmetry - if you have an approximate spontaneously broken symmetry we should find pseudogoldstone bosons
2. Tuning - there is some adjustment or coincidence of parameters that causes parameters to be small

2.2 Effective field theory

Before we move on to effective field theory (EFT), we should understand why we use quantum field theories (QFT) in the first place. QFT reconciles quantum mechanics and relativity and produces a unitary scattering matrix (S-matrix). If you start with a QFT that's consistent and you calculate scattering amplitudes you will find an S matrix that is unitary, CPT invariant, etc.

Landau's insight was if we start with an S-matrix, maybe we should be able to go back to some QFT,



The important thing to remember is that the theoretical description is in general, not unique. All descriptions will agree on the symmetries or conserved quantities of the theory, but they don't have to agree on redundant variables (gauge symmetries) and the "fundamental" degrees of freedom (what fields you choose to write your description in). This means that in general the coupling constants that we use are generally not physical.

When we work with for example QED, we talk about the electric charge as though its a physical quantity, but the charge that appears in the calculation is not a physical quantity. It depends on the definition that we use.

The take away moral is that "fundamental" vs "composite" is not a very useful description since what may seem fundamental in one description, may not be fundmental in another. The real criteria should be "strong" vs "weak". Almost all our intuition for QFT has to do with the weakly coupled side of QFT. In a weakly coupled QFT there is a $1-1$ relationship between the fields in the theory and assymptotitc states in the S-matrix. We actually collide and measure objects. If we have a QFT description that corresponds to the states that we see, that's what we normally mean by weakly coupled theory. In a strongly coupled theory there is not necessarily any relationship whatsoever between the particles you write in your Lagrangian and the degrees of freedom that you see.

2.3 QCD

We now see how this works in QCD. QCD is usually written as an $SU(3)_c$ color gauge theory of quarks. This is weakly coupled at energies, $E \gg$ GeV. At low energies, we need to construct an entirely different description of QCD, which is described by,

$$\pi, K, \eta \tag{2}$$

This is the effective chiral Lagrangian of QCD. You can go further and add extra particles if you want such as the vector mesons (e.g., the ρ meson). This description makes sence if $m_{vector}/\Lambda \ll 1$. It isn't apriori clear that this approximation has any validity, but it turns out that this can be done in principle. You can go on and add more and more particles, however as you do that your predictably gets worse and worse since you have to measure more effective coupling constants to make predictions.

Its possible that there is a different description of QCD such as a string theory in $5D$. But these are all the same theory, the theory of the strong interactions.

If we are in the situation where we have an object thats light compared to its fundamental dynamical scale, we'd like to understand what are the general properties of an effective field theory that describes such an object.

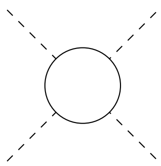
2.4 Scalar

Suppose we have a theory with a light, complex, scalar particle, ϕ, with a fundamental scale, Λ. What is the most general effective theory?

$$\mathcal{L}_\Lambda = \partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi - \lambda (\phi^* \phi)^2 + \frac{\kappa}{\Lambda^2} (\phi^* \phi)^3 + \frac{()}{\Lambda^2} (\phi^* \partial \phi)^2 + ... \tag{3}$$

At first sight this seems hopeless. The most general Lagrangian you can write down has higher and higher derivatives of higher and higher order.

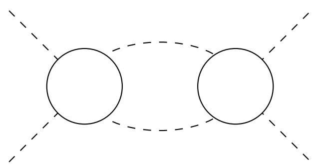
Suppose however, that you were interested in $p \ll \Lambda$. As an example lets consider the calculation of $2 \to 2$ scattering,



$$= \lambda + () \tfrac{p^2}{\Lambda^2} + () \tfrac{p^4}{\Lambda^4} + ...$$

In the limit that the momenta are small, the dominant interaction is given by the self coupling, λ.

However, there is an issue with this analysis. We haven't taken into account contributions from loops. In principle we have diagrams such as,



But the theory is only defined with some highest momenta Λ in it. Since we are interested in $p \ll \Lambda$, we can integrate out all the high energy momentum modes. In reality all the parameters in the Lagrangian depend on how you define the theory,

$$\mathcal{L}_\Lambda = \partial_\mu \phi^* \partial^\mu \phi - m^2(\Lambda) \phi^* \phi - \lambda(\Lambda) (\phi^* \phi)^2 + \frac{\kappa(\Lambda)}{\Lambda^2} (\phi^* \phi)^3 + \frac{()}{\Lambda^2} (\phi^* \partial \phi)^2 + ... \tag{4}$$

The parameters in your theory are defined at momentum scale Λ.

Suppose you imagine that you integrate out the momentum modes between Λ and a smaller scale, $\Lambda' < \Lambda$. Formally we write,

$$e^{S_{\Lambda'}} = \int \mathcal{D}\phi_{\Lambda'<p<\Lambda} e^{S_\Lambda} \tag{5}$$

The arguement that we made earlier for the most general form for the Lagrangian also applies at the scale Λ',

$$\mathcal{L}_{\Lambda'} = \partial^\mu \phi^* \partial_\mu \phi - m^2(\Lambda')\phi^*\phi - \lambda(\Lambda')(\phi^*\phi)^2 + \frac{\kappa(\Lambda')}{\Lambda'^2}(\phi^*\phi)^3 \tag{6}$$

That means that the parameters that define the theory flow,

$$\begin{aligned} &m^2(\Lambda) \to m^2(\Lambda') \\ &\lambda(\Lambda) \to \lambda(\Lambda') \\ &\kappa(\Lambda) \to \kappa(\Lambda') \\ &\qquad\vdots \end{aligned}$$

You can do the functional integral to get $S_{\Lambda'}$ perturbatively, but that's only going to be valid in the weak coupling regime. But there is nothing about this arguement that is perturbative.

We would like to understand what these flows look like. Unfortunately, we have an infinite number of coupling constants which we want to track their value as Λ changes, as so its hard to draw them all at once. For this reason we restrict ourselves to just 3 couplings.

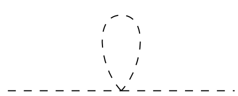
Lets think what happens to each coupling. κ is particularly easy at tree level,

$$\kappa(\Lambda') = \frac{\Lambda'^2}{\Lambda^2}\kappa(\Lambda) + ... \tag{7}$$

The mass squared gets an additive correction. To see this consider the Coleman-Weinberg 1 loop potential. This is computed by calculating all the one loop contributions and adding them to the Lagrangian. In particular the mass squared becomes,

$$m^2(\Lambda^2) + ()\Lambda^2 \tag{8}$$

due to the diagram,



Comparing to the result for Λ' we have,

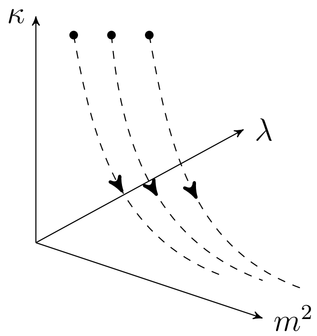
$$m^2(\Lambda'^2) = m^2(\Lambda^2) + ()(\Lambda^2 - \Lambda'^2) \tag{9}$$

The most interesting thing is the behavoir of the coupling at one loop. The coupling goes as,

$$\frac{1}{\lambda(\Lambda')} = \frac{1}{\lambda(\Lambda)} + \frac{b}{(4\pi)^2} \log \frac{\Lambda'}{\Lambda} \tag{10}$$

[Q 1: Confirm this still holds in the Coleman-Weinberg potential.]

In summary as Λ' is decreased (we go to lower energies), the mass increases rapidly, κ drops rapidly, and λ stays roughly the same. Diagramatically we have,



This flow diagram displays a few very general features of flow diagrams,

1. $\kappa(\Lambda) \to 0$ as $\Lambda' \to 0$. These interactions are called irrelevant operators. They are the ones whose effects scale to 0 at low energies.

   We see that we started out with the Lagrangian which had an infinite number of parameters. If we scale far enough we can parametrize everything in terms of just two parameters, m^2 and λ. This is known as renormalizability (universality in condensed matter literature).

   You're quantum field theory is determined at some point in an infinity dimensional coupling space. But luckily, you don't live at the cutoff, but at some low energy. And all the coupling flow to some low energy (Λ') manifold. The dimensionality of the manifold is precisely the same as the number of renormalization constants in the theory. This is a non-perturbative view of a QFT.

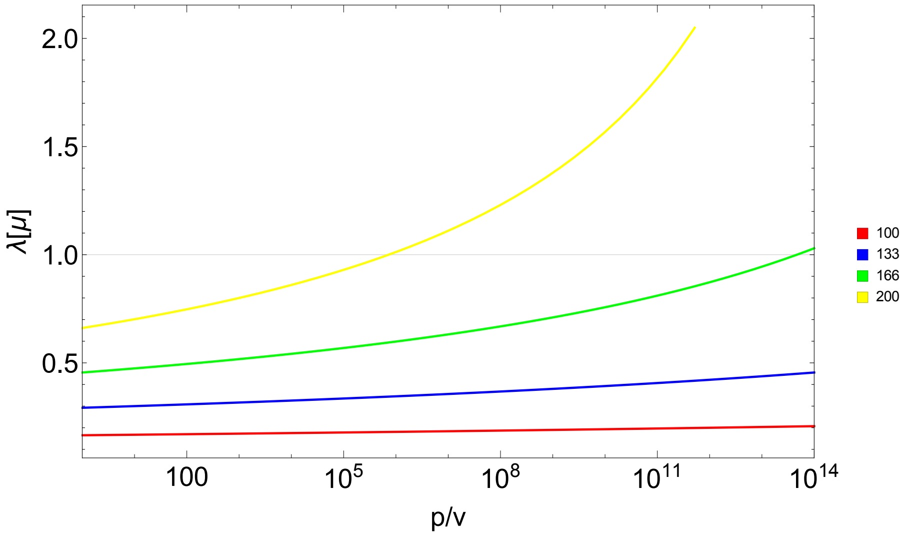
2. Until now we've been thinking about Λ fixed and p going to zero. However, we can also think about p fixed and Λ going to infinity. In this limit $\lambda \to 0$ in the infrared. This is called triviality. This means that a self-interacting scalar theory doesn't have a continuum limit, or more precisely that the continuum limit is free field theoy.

   This says that the higgs theory is an effective low energy theory whose cutoff depends inversely exponentially on the mass of the higgs. In particular, if you want to higgs coupling to remain coupling positive up to the GUT scale, then the higgs better be lighter then 180GeV. Because the running is exponential instead of algebraic, triviality is a question that can be pushed very far away.

Its easier to see why this is an issue if we consider starting at electroweak energies and extrapolating the coupling to high energies,

$$\lambda(\mu) = \frac{\lambda(v)}{1 - \frac{3\lambda(v)}{(4\pi)^2} \log \frac{p^2}{v^2}} \tag{11}$$

The running is shown below,



3. The $m^2(\Lambda)$ becomes very large. This is known as the hieararchy problem. The flip side of this problem is the fine-tuning problem.

   Naturalness means if you start at some generic point in the renormalization group parameter space, where do you end? A natural theory says that no matter where you are in this parameter space you end up at around the same point. But, since $m^2 \to \infty$, $m = 125\text{GeV}$ does not look natural.

   If for example, $\Lambda \to 100\text{TeV}$, the natural mass scale is 100TeV. If you want $m = 125\text{GeV}$, it seems that you are forced to have $\Lambda \approx 100\text{GeV}$.

   If on the other hand you demand that $\Lambda = 10^{15}\text{GeV}$, for example. Then in order to get to 125GeV at low energies you have to start in a very particular point in parameter space. This is usually quantified by saying the spread of available points at high energies that will give the correct higgs mass if very small,

$$\frac{\Delta m^2}{m^2} \ll 1 \tag{12}$$

2.5 The higgs

All descriptions of nature are effective field theories. We are going to construct them order by order in expansions in the couplings, $(p^2/\Lambda^2)^\alpha$. The big question will be what is the relevant scale?

Consider the higgs boson,

$$\mathcal{L} = D^\mu\phi^\dagger D_\mu\phi - \frac{\lambda}{4}\left(\phi^\dagger\phi - \frac{v^2}{2}\right)^2 \tag{13}$$

We have an object ϕ that is $2_{+1/2}$ under $SU(2)_L \times U(1)_Y$. Viewing this as an effective theory we should be able to add other terms,

$$\mathcal{L} = D^\mu\phi^\dagger D_\mu\phi - \frac{\lambda}{4}\left(\phi^\dagger\phi - \frac{v^2}{2}\right)^2 + ()\frac{\phi^\dagger\phi}{\Lambda^2} + ()\frac{(\phi^\dagger D^\mu\phi)^2}{\Lambda^2} \tag{14}$$

If we take the point of view that this object that we've seen is part of an $SU(2)\times U(1)_Y$ scalar doublet then we are not just going to have a dimension 4 piece, we're going to have higher order corrections as well.

This way of writing the Lagrangian doesn't impose all the relevant symmetries that we need to keep track of. In particular it doesn't impose the custodial symmetry. All the dimension 4 pieces in the limit that we set $g, g' \to 0$ there is a larger global symmetry, $SU(2)\times SU(2)$ symmetry. Its useful to rewrite the Lagrangian in a different way to show that.

A simple way to see that is to write the higgs as,

$$\phi = \begin{pmatrix} \pi_1 + i\pi_2 \\ \phi_3 + i\pi_4 \end{pmatrix} \tag{15}$$

Then you see that,

$$\phi^\dagger\phi = \pi_1^2 + \pi_2^2 + \pi_3^2 + \pi_4^2 \tag{16}$$

This is an $O(4)$ symmetry, which is equivalent to $SU(2)\times SU(2)$. To rewrite this we can introduce, $\tilde{\phi} = i\sigma_2\phi^*$. Because the **2** of $SU(2)$ is a pseudoreal representation (or equivalently because σ_2 commutes only with itself and not σ_1 or σ_3, you can show that $\tilde{\phi}$ transforms as a $2_{-1/2}$ under $SU(2)\times U(1)$. Since they are both doublets, you can put them together into a 2×2 complex matrix that is a doublet as well,

$$\Phi \equiv \begin{pmatrix} \tilde{\phi} & \phi \end{pmatrix} \tag{17}$$

We can perform a $SU(2)$ transformation through,

$$\Phi \to L\Phi \tag{18}$$

where L is an $SU(2)_L$ transformation matrix in the same way we did before. Furthermore, we introduce a second matrix to perform the $U(1)_Y$ phase rotation. Since ϕ and $\tilde{\phi}$ have opposite hypercharges, we can use the T_3 component of an additional $SU(2)$ transformation, R.

We can write this transformation as,

$$\Phi \to L\Phi R^\dagger \tag{19}$$

An arbitrary 2×2 matrix has 8 degrees of freedom while this one only has 4, so there is a very special constraint that is satisfied by this,

$$\Phi^\dagger \Phi = \phi^\dagger \phi \mathbb{1} \tag{20}$$

So this 2×2 complex matrix actually can be written as,

$$\Phi = \rho(x)\Sigma(x) \tag{21}$$

where, $\rho(x)$ is a real positive scalar and $\Sigma(x)$ is a 2×2 $SU(2)$ matrix. The degrees of freedom add up since an $SU(2)$ matrix has 3 degrees of freedom.

With this machinery the kinetic Lagrangian can be written,

$$D^\mu \phi^\dagger D_\mu \phi = \frac{1}{2} \mathrm{Tr} D^\mu \Phi^\dagger D_\mu \Phi \tag{22}$$

where,

$$D_\mu \Phi \equiv (\partial_\mu + g W_\mu^a T^a)\Phi(g' B_\mu T_3) \tag{23}$$

[Q 2: Check this.]

Furthermore we have,

$$(\phi^\dagger \phi - \frac{v^2}{2})^2 = \frac{1}{2} \mathrm{Tr} \left(\Phi^\dagger \Phi - \frac{v^2}{2} \right)^2 \tag{24}$$

Having written it in this way you can see that this is actually invariant under a full set of global $SU(2) \times SU(2)$ transformations,

$$\Phi \to L \Phi R^\dagger \tag{25}$$

Further, if we choose the right basis we can write,

$$\langle \Phi \rangle \propto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{26}$$

Which means that $SU(2)_L \times SU(2)_R$ breaks to a vector global symmetry that remains, $SU(2)_V$. Its the $SU(2)_V$ symmetry that protects all the low energy quantities, but in order to get this symmetry you need $SU(2)_R$ to start with.

3 Effective models

3.1 Nambu-Tona-Lasinio model

The Nambu-Tona-Lasinio (NJL) model was originally a model of chiral symmetry breaking in QCD but we'll describe it as a theory of EWSB. Consider the Lagrangian,

$$\mathcal{L}_\Lambda = \mathcal{L}_{gauge} + \bar{\psi}_L i \not{D} \psi_L + \bar{t}_R i \not{D} t_R - \frac{4\pi^2 \kappa}{\Lambda^2} \left(\bar{\psi}_L^i t_R \right) \left(\bar{t}_R \psi_L^i \right) \tag{27}$$

where $\psi_L \equiv (t_L, b_L)^T$ is the $SU(2)$ 3rd generation doublet (quantum numbers, $2_{+1/6}$)and t_R is the right handed top (quantum numbers, $1_{+2/3}$). The local $SU(3)$ gluon interaction has been "integrated out" and absorbed in the four-fermi operator. This Lagrangian is only invariant under $SU(2) \times U(1)$!

This is an easy theory to solve this model in the large N_c approximation. Its convenient to redefine $\kappa \to \kappa/N_c$ such that,

$$\mathcal{L}_\Lambda = \mathcal{L}_{gauge} + \bar{\psi}_L i\not{D}\psi_L + \bar{t}_R i\not{D} t_R - \frac{4\pi^2\kappa}{\Lambda^2 N_c}\left(\bar{\psi}_L^i t_R\right)\left(\bar{t}_R \psi_L^i\right) \tag{28}$$

We would like to do this such that one-loop contributions will have the same size as tree level contributions. So $\kappa \sim 1$ is strong coupling (1 loop processes are the same order as tree level processes).

We are going to solve this by "bosonization" - introduce an auxilary field, ϕ, which is a higgs-doublet like object given by

$$\frac{(4\pi)^2}{\Lambda^2}\sqrt{\frac{\kappa}{N_c}}\bar{t}_R\psi_L^i \tag{29}$$

(quantum numbers, $2_{-1/2}$). We can then rewrite the interaction Lagrangian for the field as [1]

$$\mathcal{L}_{int} = -\frac{\Lambda^2}{4\pi^2}\phi^\dagger\phi + \sqrt{\frac{\kappa}{N_c}}\bar{\psi}_L^i t_R \phi_i + h.c. \tag{31}$$

At this point we haven't done anything.

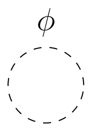
Its interesting to note that initially the Lagrangian appeared nonrenormalizable and now suddenly it looks renormalizable. But this is deceiving.

This is because we need to differentiate between mass dimension and scaling dimension. Scaling dimension is really what counts, its what determines the order of divergence of loops in perturbation theory.

Mass dimension of this field is clearly equal to 1. However, it has a different scaling dimenson. The scaling dimension is determined by how fast its propagator falls at high momentum.

When we say a scalar field has scaling dimension 1, what we really mean that in perturbation theory at high momentum its propagator falls like $1/p^2$. For an ordinary scalar field its mass dimension is equal to its mass dimension.

If we look at the propagator for ϕ, its given by, $\sim 4\pi^2/\Lambda^2$. If we put it in a loop,



$\sim \frac{4\pi^2}{\Lambda^2}\int d^4k$

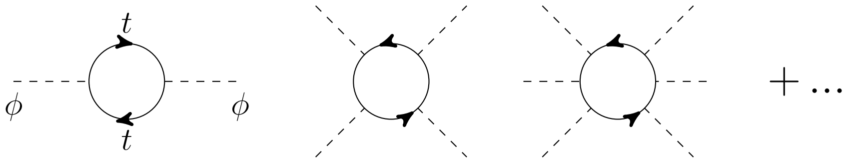
[1] Finding the equation of motion gives,

$$\frac{\Lambda^2}{4\pi^2}\phi^\dagger\phi = \sqrt{\frac{\kappa}{N_c}}\frac{1}{2}\left(\bar{\psi}t_R\phi + h.c.\right) = \frac{4\pi^2}{\Lambda^2}\frac{\kappa}{N_c}\bar{\psi}t_R\bar{t}_R\psi \tag{30}$$

Then its quartically divergent.

Now we are going to solve the interaction in the limit that $N_c \to \infty$, so we only keep the leading order terms.

The simplest thing to compute is the effective potential for the field, ϕ. You can convince yourself that the only diagrams that contribute are,



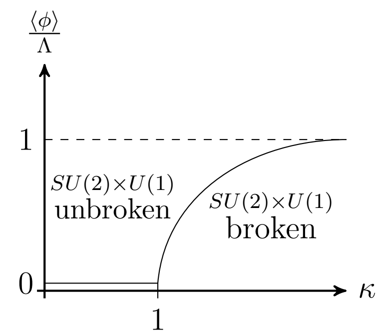
Thats because if we try to put any other ϕ across the top loop, then you introduce additional powers of N_c.

We can calculate this result exactly and is given by [Q 3: calculate this.],

$$V_{eff}(\phi) = \frac{\Lambda^2}{4\pi^2}(1-\kappa)\phi^\dagger\phi + \frac{\kappa^2}{4\pi^2}(\phi^\dagger\phi)^2\left[\log\left(\frac{N_c\Lambda^2}{\phi^\dagger\phi}\right) + \frac{1}{2}\right] \tag{32}$$

This is regardless of the value of κ, the expansion is controlled by $1/N_c$.

If you look at this potential, it has a very interesting property. You can look at the value for the expectation value of the field as a function of coupling strength [Q 4: Calculate this.],

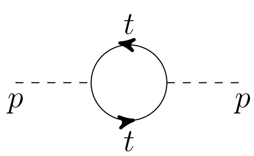


In the language of condensed matter field theory this is a 2^{nd} order phase transition, meaning that the order parameter (in our case κ) changes continuously but its derivative does not. When ϕ gets a VEV, this gives the top a mass. Thus κ parameterizes the $SU(2) \times U(1)$ phase transition. In the broken symmetry region,

$$\langle\phi\rangle \propto \Lambda\sqrt{\kappa - 1} \tag{33}$$

What we want to understand is what is the effective field theory near $\kappa \approx 1^+$ (for κ slightly less than 1)?

Lets consider the full two-point function for ϕ,



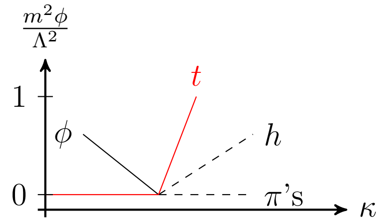
Computing this one can show that the wavefunction renormalization function for ϕ is given by,

$$Z_\phi \approx \frac{\kappa}{4\pi^2} \log\left|\frac{1}{\kappa - 1}\right| \tag{34}$$

[Q 5: show this.] Having wavefunction renormalization, implies that we have a kinetic energy contribution! We started with a field ϕ which had no momentum term, but we've generated a kinetic energy for ϕ with momentum. We have a kinetic term,

$$\mathcal{L}_{eff}^{kin} \sim \frac{\kappa}{4\pi^2}\partial_\mu \phi \partial^\mu \phi \tag{35}$$

Now lets see the spectrum of the theory as a function of κ:



Below $\kappa = 1$ (when $\langle \phi \rangle = 0$), we have the full higgs-like doublet. Above this scale, the doublet gets a VEV which breaks the symmetry. On the symmetry broken side we have a single scalar with 3 goldstone bosons. Near the phase transition we have a light scalar. [Q 6: Expand the logarithm properly and find the π masses.] This VEV also gives a mass to the top quark.

We see that the spectrum varies continuously. This is a hallmark of a second order phase transition. If we tune κ close to 1 but slightly greater than 1. Then we find,

$$\frac{\Delta\kappa}{\kappa} = \frac{\kappa - 1}{\kappa} \approx \mathcal{O}\left(\frac{m_h^2}{\Lambda^2}\right) \tag{36}$$

So if we want $m_h \sim 125$GeV and $\Lambda \sim 10$TeV then we have to adjust κ,

$$\frac{\Delta\kappa}{\kappa} \approx \mathcal{O}\left(\frac{100\text{GeV}}{10\text{TeV}}\right)^4 \sim 10^{-4} \tag{37}$$

So with this complicated mechanism we've reproduced the hierarchy problem.

This is a theory with a composite higgs boson. It is in the same universality class as the SM. You could argue that all we've done is rewrite the SM in a bizarre way.

One may worry about the phenomenology of such a modification to the SM. The most important low energy effects are always going to be symmetry violating effects. In particular, our composite higgs model has a term which violates custodial symmetry,

$$\sqrt{\frac{\kappa}{N_c}}\bar{\psi}_L^i t_R \phi_i \tag{38}$$

The measure of custodial symmetry in the SM is given by the ρ parameter:

$$\rho \equiv \frac{M_W^2}{M_Z^2 \cos^2\theta_w} \tag{39}$$

The higher dimensional operator,

$$\frac{(\phi^\dagger D_\mu \phi)^2}{\Lambda^2} \tag{40}$$

gives rise to violations of the $\rho = 1$ relation,

$$\Delta r \equiv \rho - 1 \approx \mathcal{O}\left(\frac{v^2}{\Lambda^2}\right) \tag{41}$$

The ρ parameter is known to about 0.1% giving $\Lambda \gtrsim 10$TeV. This was the reason we chose $\Lambda \approx 10$TeV.

Lets now match parameters,

1. Scale Λ: Λ, κ
2.

3.2 QCD chiral Lagrangian

The QCD Lagrangian for the 3 light quarks is given by,

$$\mathcal{L} =_{QCD}= \mathcal{L}_{gauge} + \bar{\psi}_L i \not{D} \psi_L + \bar{\psi}_R i \not{D} \psi_R - \bar{\psi}_L M \psi_R - \bar{\psi}_R M^\dagger \psi_L \tag{42}$$

where $\psi \equiv (u, d, s)^T$. We chose to write the mass matrix as,

$$M = \begin{pmatrix} m_u & & \\ & m_d & \\ & & m_s \end{pmatrix} \tag{43}$$

In the limit that M goes to zero we have a new symmetry,

$$SU(3)_L \times SU(3)_R \tag{44}$$

This is known as a chiral symmetry. One of the amazing properties of QCD is that the vacuum of QCD changes non-perturbatively. We're used to the of a changing vacuum in the higgs model. In QCD we have,

$$\langle \bar{u}_L u_R \rangle = \langle \bar{d}_L d_R \rangle = \langle \bar{s}_L s_R \rangle \neq 0 \tag{45}$$

Note that nothing distinguishes the different flavors as $M \to 0$ so the expectation values of the different flavors must be the same.

This means that spontaneously the dynamics of QCD breaks the chiral symmetry,

$$SU(3)_L \times SU(3)_R \to SU(3)_V \equiv SU(3)_{L+R} \tag{46}$$

This is an $SU(3)$ symmetry that transforms the left and right handed quarks in the same way.

The initial theory has $8+8=16$ preserved generators, while the broken theory only has 8. Thus we expect there to be 8 massless goldstone bosons in the limit that $M \to 0$.

Recall that we can write a complex scalar matrix Φ with $9*2=18$ degrees of freedom as a radial hermitian part multiplied by a unitary matrix [2],

$$\Phi = H(x)\Sigma(x) \tag{47}$$

If you really look at this theory naively, it seems to have an additional axial $U(1)$ symmetry. This axial $U(1)$ symmetry is explicitly broken due to an anomaly in QCD. With that in mind we can set the degrees of freedom in $H(x)$ to be infinitely massive. We could try to write down a description of these, but there is no gaurentee these even exist. We have no reason to expect these to be light. The lowest order term is then,

$$\mathcal{L} = \frac{f^2}{4}\mathrm{Tr}\partial_\mu\Sigma^\dagger\partial^\mu\Sigma \tag{48}$$

The only restriction on $\Sigma(x)$ is that it must be unitary. A convenient representation is,

$$\Sigma(x) = \exp\left(\frac{2i\pi^a T^a}{f}\right) \tag{49}$$

where T^a are the broken generators of the theory and obey, $\mathrm{Tr}T^aT^b = \frac{1}{2}\delta^{ab}$.

We're used to the idea of spontaneous symmetry breaking in the higgs potential. There we have a mexican hat potential, which has a radial direction and an angular direction. The angular direction, is the vacuum manifold of the theory. The goldstone bosons parametrize coordinates along the vacuum manifold since moving along those direction don't cost any energy.

The goldstones bosons in general are fields which map from our space time into whatever the vacuum manifold is for whichever symmetry we are talking about. In our case they map,

$$\mathbb{R}^{3,1} \to \frac{SU(3)_L \times SU(3)_R}{SU(3)_V} \tag{50}$$

This is a very special example of a general phenomena of what's known as a non-linear sigma model. In such models you have fields, $\{\pi^i\}$, which parameterize coordinates of a vacuum manifold.

The most general Lagrangian you can write down is,

$$\mathcal{L} = \eta^{\mu\nu}g_{ij}(\pi)\partial_\mu\pi^i\partial_\nu\pi^j \tag{51}$$

In our representation the unitary field tranforms linearly,

$$\Sigma \to L\Sigma R^\dagger \tag{52}$$

[2] Since each part has 9 degrees of freedom, this is the most general definition one can make.

This makes it easy to write down and figure out what terms are invariant.

One may wonder why we didn't just write out the transformations in terms of the fields themselves. That's because the golstones don't transform linearly. On the other hand for the preserved symmetry, the π's transform homogeneously,

$$\pi^a T^a \to V(\pi^a T^a)V^\dagger \tag{53}$$

But if you think about doing an infinitesimal left handed transformation by a parameter ϵ_a, given by,

$$L = \exp\left(\frac{2i\epsilon^a T^a}{f}\right) \tag{54}$$

To leading order in ϵ_a we have,

$$L\Sigma(x) = \left(1 + \frac{2i\epsilon^a T^a}{f}\right)\left(1 + \frac{2i\pi^a T^a}{f}\right) + ... \tag{55}$$

which corresponds to a shift in the goldstones,

$$\pi^a \to \pi^a + \epsilon^a \tag{56}$$

This is one of the charactoristic features of the goldstone boson. Under the broken global symmetry, the π^a transforms inhomogeneously through a displacement. This symmetry is what forbids writing down a mass for the goldstone bosons.

3.3 Spurion analysis

A spurion analysis is just a manifestation of the Wigner Eckart theorem. All it says is that the mass term breaks transformations independently under the left and right handed rotations, but if we consider the transformation $L \in SU(3)_L, R \in SU(3)_R$ and we replace,

$$M \to LMR^\dagger \tag{57}$$

all at the same time then the form of the Lagrangian should not change. All we're doing is trying to consistently figure out how M breaks the $SU(3)_L \times SU(3)_R$ symmetry and put that back into the Lagrangian. We're going to assume that M is small, so the leading terms will be ones with the smallest number of terms.

Said another way, we know the Lagrangian should be invariant under $SU(3)_L \times SU(3)_R$ when M transforms. To keep this behavoir in the low energy theory we only write down terms that our invariant under this transformation.

The symmetry breaking terms are given by,

$$\mathcal{L}_{SB} = \frac{\mu f^2}{2}\mathrm{Tr}\Sigma^\dagger M + h.c. \tag{58}$$

The pion field is given by,

$$\pi = \pi^a T^a = \begin{pmatrix} \frac{\pi^0}{2} + \frac{\eta}{\sqrt{2}} & \frac{\pi^+}{\sqrt{2}} & \frac{K^+}{\sqrt{2}} \\ \frac{\pi^-}{\sqrt{2}} - \frac{\pi^0}{2} & \frac{\eta}{\sqrt{2}} & \frac{K^0}{\sqrt{2}} \\ \frac{K^-}{\sqrt{2}} & \frac{\bar{K}^0}{\sqrt{2}} & -\frac{\eta}{3} \end{pmatrix} \tag{59}$$

The 8 light mesons are an isotriplet, $\{\pi^0, \pi^+, \pi^-\}$, an isosinglet, η, and a complex doublet, $\{K^+, K^-, K^0, \bar{K}^0\}$.

Plugging this into $\mathcal{L}_{SB}$, you can calculate the masses of the pions, eta, and kaons. Working in the limit that $m_u = m_d \equiv m$ we have,

$$m_\pi^2 = \mu(2m) \tag{60}$$

$$m_\eta^2 = \frac{4\mu}{3}(m_s + ...) \tag{61}$$

$$m_K^2 = \mu(m_s + m) \tag{62}$$

If you work in this limit you find that π^0 and π^+ have the same mass. This isn't true in reality due to an essential missing ingredient, electromagnetism.

3.4 Electromagnetism

Firstly we note that electromagnetism breaks $SU(3)_L \times SU(3)_R$, due to the different charges of the quarks. In mass basis, the charge operator is,

$$Q = \begin{pmatrix} 2/3 & & \\ & -1/3 & \\ & & 2/3 \end{pmatrix} \tag{63}$$

To add electromagnetism we need to modify the covariant derivative,

$$\bar{\psi} i \not{D} \psi \rightarrow \bar{\psi}_L \left(\partial^\mu - ieA^\mu Q_L\right) \psi_L + \bar{\psi}_R \left(\partial^\mu - ieA^\mu Q_R\right) \psi_R ... \tag{64}$$

As numerical constants, Q_L and Q_R are the same. As symmetry breaking parameters, they transform differently. Q_L is embedded in $SU(3)_L$ and says the left quarks feel EM. Q_R is embedded in $SU(3)_R$ and says the right quarks feel EM. To maintain $SU(3)_L \times SU(3)_R$ invariance we must allow $Q_{L/R}$ to rotate,

$$Q_L \rightarrow L Q_L L^\dagger \quad Q_R \rightarrow R Q_R R^\dagger \tag{65}$$

This tells us how to write our covariant derivative for our Σ field,

$$D^\mu \Sigma = \partial^\mu \Sigma + ieQ_L A^\mu \Sigma - ieA^\mu \Sigma Q_R \tag{66}$$

since in this way, $Q_L \Sigma \rightarrow L Q_L \Sigma R^\dagger$, the same transformation as for $\partial_\mu \Sigma$ (and similarly for Q_R). We have now successfully gave our pions a charge. How do we get a mass from this charge? The lowest order invariant is,

$$e^2 f^4 \mathrm{Tr}\left[Q_L \Sigma Q_R \Sigma^\dagger\right] \tag{67}$$

[Q 7: What about the invariant, $\mathrm{Tr} Q_L \Sigma Q_R M$?]

This term gives you,

$$\Delta m_\pi^2 = m_{\pi^+}^2 - m_{\pi^0}^2 = \mathcal{O}(e^2 f^2) \tag{68}$$

One can show that the left and right handed chiral currents have the form,

$$j^a_{\mu L} = \frac{f^2}{()} \mathrm{Tr} T^a \Sigma \partial_\mu \Sigma^\dagger \tag{69}$$

$$j^a_{\mu R} = \frac{f^2}{()} \mathrm{Tr} T^a \Sigma^\dagger \partial_\mu \Sigma \tag{70}$$

The left handed chiral current governs, $\pi^+ \to \mu\nu$. This comes from four-fermi operator,

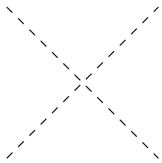
$$G_F \cos\theta_c \bar{u}_L \gamma^\mu d_L \bar{\mu}_L \gamma_\mu \nu_L \tag{71}$$

You can replace the left handed current, $\bar{u}_L \gamma^\mu d_L$ by the low energy chiral current, $j^a_{\mu L}$. This is related to the pion decay constant.

Now recall the kinetic term for the goldstones,

$$\frac{f^2}{4} \mathrm{Tr} \partial_\mu \Sigma^\dagger \partial^\mu \Sigma \supset (\partial^\mu \pi)^2 + () \frac{[\pi, \partial^\mu \pi]}{f^2} + \ldots \tag{72}$$

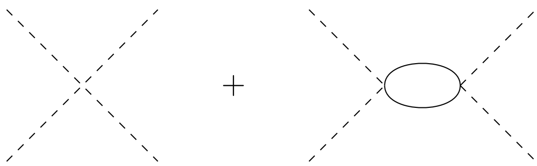
You can use this, as well as the mass term to calculate the pion decay constant,



$$\mathcal{A}(\pi\pi \to \pi\pi) = () \tfrac{s}{f^2} + () \tfrac{m_\pi^2}{f_\pi^2}$$

Notice that the interactions of the goldstone bosons are suppressed. In the limit the that symmetry breaking is zero ($m_\pi \to 0$), the amplitude is proportional to the momenta. This is a general property of goldstone bosons and their shift symmetry.

We further note that the chiral Lagrangian is nonrenormalizable. So if we naively compute the one loop correction to $2 \to 2$ scattering,



This four-point scattering amplitude is given by,

$$\sim \frac{s}{f_\pi^2} + \frac{s^2}{16\pi^2 f_\pi^4} \log \frac{s}{\mu^2} \tag{73}$$

We need a dimension 4 counterterm such as,

$$C.T.(\mu) = \frac{f^2}{\Lambda^2} \mathrm{Tr}(\delta^\mu \Sigma^\dagger \partial_\mu \Sigma)^2 \tag{74}$$

where the coefficient on the f^2 on the outside is necessary to correspond to the f dependence in the original kinetic term [Q 8: I don't understand where this comes form?] It must be of this form since we need to get s^2 dependence.

Notice however that the structure of the smplitude can be rewritten,

$$\frac{s}{f_\pi^2}\left(1+\frac{s}{16\pi^2 f_\pi^2}\log\frac{s}{\mu^2}\right)+C.T.(\mu) \tag{75}$$

The correction is suppressed by $s/(4\pi f_\pi)^2$.

We should also ask ourselves, where does the scale Λ come from? Apriori we have no idea. However, we do have a consistency arguement that tells us that Λ can't be any larger then a certain amount. This is done using naive dimensional analysiss (NDA).

The true amplitude is μ invariant. If we change the renormalization point we change the amplitude in such a way that the physical amplitude remains invariant. If we take μ and shift if by a factor of 2, then it shifts the difference contributions to the amplitude such that it remains invariant:

$$\frac{s}{16\pi^2 f_\pi^2}\log\frac{s}{\mu}\leftrightarrow C.T.(\mu) \tag{76}$$

Suppose we say that we know that $\Lambda=\infty$. If this was the case we wouldn't have a dimension 4 term. But then μ becomes undefined. Shifting μ by order one means that Λ goes from ∞ to $4\pi f_\pi$. So even if we Λ happened to be infinity that if we change μ, Λ would be less then this value.

NDA is the assumption that $\Lambda\approx 4\pi f_\pi$ and all other constants are assumed to be $\mathcal{O}(1)$. The amazing thing that as silly as this assumption seems, it works well in most cases.

4 Composite models

Consider a QCD-like theory. We're going to recycle the notation we are used to for QCD. We have,

$$\begin{pmatrix} U \\ D \end{pmatrix}_L, \quad S_L, \quad \begin{pmatrix} U \\ D \end{pmatrix}_R, \quad S_R \tag{77}$$

We're going to choose both the left and right handed doublets to be $2_{1/6}$ under the weak interactions. This is the deviation from the SM. We are going to chose both S_L and S_R to be $1_{-1/3}$.

This has an $SU(3)_L\times SU(3)_R$ approximate chiral symmetry. The T_L^a and T_R^a are associated with the $SU(2)$ weak interactions and $T_{8,L}$ and $T_{8,R}$ are associated with $U(1)_Y$.

The reason for doing this is if you look at the kaons in,

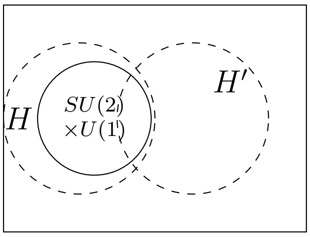
$$\pi=\pi^a T^a=\begin{pmatrix} \frac{\pi^0}{2}+\frac{\eta}{\sqrt{2}} & \frac{\pi^+}{\sqrt{2}} & \frac{K^+}{\sqrt{2}} \\ \frac{\pi^-}{\sqrt{2}}-\frac{\pi^0}{2} & \frac{\eta}{\sqrt{2}} & \frac{K^0}{\sqrt{2}} \\ \frac{K^-}{\sqrt{2}} & \frac{\bar{K}^0}{\sqrt{2}} & -\frac{\eta}{3} \end{pmatrix} \tag{78}$$

We see that they are doublets under the weak interactions with hypercharge, 1/2. The kaons have the quantum numbers of the higgs. What was the kaons in QCD,

$$\begin{pmatrix} K^+ \\ K^0 \end{pmatrix} \tag{79}$$

becomes the higgs doublet of the weak interactions, ϕ with $2_{1/2}$. We've managed to create a theory of a composite higgs with the correct qunatum numbers. What's left is we need to give it a VEV.

To do this we will detour into what's known as vacuum misalignment. If we have some strongly interacting theory with global symmetry, G, it will in general spontaneously break into some subgroup, H.



What we mean by our unbroken group isn't clear. Its possible that $SU(2) \times U(1)$ won't be properly contained within the broken group H. We'd idealy like to break such that the only unbroken piece is the $SU(2) \times U(1)$. We'd like to add additional terms that shift the vacuum. The easiest way to do this is to add an additional axial $U(1)$:

$$SU(2)_W \times U(1)_Y \times U(1)_A \tag{80}$$

We now compute the low energy consequences of this.

4.1 Banks model

4.2 Little Higgs