COLLIDER PHYSICS LECTURE NOTES

Lecture notes based on a course given by Maxim Perelstein at Cornell University on Collider physics. The course goes over the basics of colliders and provides an introduction to Madgraph. The notes focus on the collider aspect of the course. If you find any errors please let me know at *ajd268@cornell.edu*

Presented by: Maxim Perelstein

LaTeX Notes by: JEFF ASAF DROR

Lectures given in 2009
Notes take in 2013
CORNELL UNIVERSITY

Contents

0.1 Preface

This set of notes was written based on lectures given in 2009 by Maxim Perelstein. The course is intended for graduate students which have taken an introductory course to Quantum Field Theory. The aim of the course is to teach about collider physics through analytic and numerical techniques with emphasis on the use of Madgraph[1]. These notes focus primarily on the analytic side do not provide an introduction to Madgraph. The text is composed of material primarily presented in class as well as some small additions. Any long extra derivations done by the author are left to the appendices.

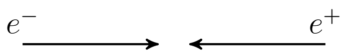
Chapter 1

Introduction

The two main parameters in a collider are

- Type: e^+e^-, pp, $p\bar{p}$, $e^\pm p$
- Center of mass energy - Tells you the scale you are probing in. Another way to look at it is because of $E = mc^2$ the particles you are capable of producing have masses corresponding to this energy.

Consider an electron-positron collision,



we have

$$p_- = E_b\,(1, 0, 0, 1) \tag{1.1}$$
$$p_+ = E_b\,(1, 0, 0 - 1) \tag{1.2}$$

(E_b is the beam energy) which gives

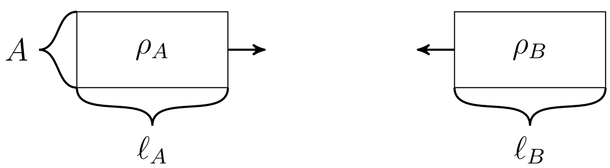
$$s = (p_+ + p_-)^2 = 2p_+ \cdot p_- = 4E_b^2 \tag{1.3}$$

so we have

$$E_{CM} = 2E_b \tag{1.4}$$

The relevant quantities are event rate, luminosity, and cross section

We have



$$\sigma = \frac{\#\text{ of events}}{\rho_A \rho_B \ell_A \ell_B A} \tag{1.5}$$

Normally we don't quote the cross sections like this but instead we define the luminosity,

$$\mathcal{L} = \rho_A \rho_B \ell_A \ell_B A \cdot \frac{\#\text{ of beam crossings}}{\text{second}} \tag{1.6}$$

The event rate is given by

$$R = \mathcal{L}_{int}\sigma \tag{1.7}$$

- $\mathcal{L}_{int}$ is given by accelerator physicists
- σ is given by theorists
- The rates are taken from the experiment

where

$$\mathcal{L}_{int} = \int dt \mathcal{L} \tag{1.8}$$

The units of cross section are in barns, or more commonly in *pb*.

There is a way to calculate the cross section of collision very heuristically without using QFT. The quantum mechanical aspects of a beam are given by its Compton wavelength,

$$\lambda_c \sim \frac{1}{E_b} \tag{1.9}$$

The area of the beam or the "cross section" is thus

$$\sigma \sim 4\pi\lambda_c^2 = \frac{4\pi}{E_b^2} \tag{1.10}$$

This is indeed roughly a maximum value for the cross section. For example at LEP:

$$\frac{4\pi}{(100\text{GeV})^2} \approx 500\text{nb} \tag{1.11}$$

The true value for example

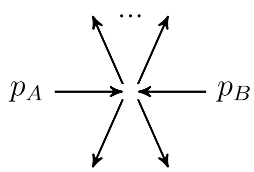
$$\sigma\left(e^+e^- \to Z, E_b \approx M_Z\right) \approx 40\text{nb} \tag{1.12}$$

The reason for the suppression is the fact that the size of the couplings isn't taken into account.

The "rough" expression turns out to have the right scaling with the energy.

1.1 Master Formula

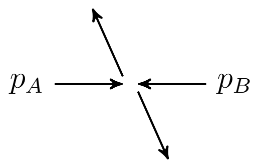
We have all seen the master formula for $2 \to N$ cross section



$$d\sigma = \frac{1}{8E_A E_B} \overbrace{\prod_{i=1}^{N} \frac{d^3 p_i}{(2\pi)^3} \frac{1}{2E_i} (2\pi)^4 \delta^4 \left(p_A + p_B - \sum_{i=1}^{N} p_i \right)}^{\text{phase space}} \cdot \underbrace{\left| \mathcal{M} \left(p_A p_B \to \{p_i\} \right) \right|^2}_{\text{TeV Matrix Element}} \tag{1.13}$$

Number of variables is $3N - 4$.

As a special case we can consider $2 \to 2$ process,



The number of variables is $3(2) - 4 = 2$. However, in practice nothing depends on the azimuthal angle since we can place the collision on a plane.

The task is to find

$$\frac{d\sigma}{d\cos\theta} = \frac{1}{8E_A^2} \prod_{i=1}^{2} \frac{d^3 p_i}{(2\pi)^3} (2\pi)^4 \delta \left(p_A + p_B - p_1 - p_2 \right) \left| \mathcal{M} \right|^2 \tag{1.14}$$

$$= \frac{1}{8E_A^2} \frac{1}{2E_2} \frac{p_1^2 dp_1 d\phi}{(2\pi)^3 2E_1} \delta(E_{\text{cm}} - E_1 - E_2) \left| \mathcal{M} \right|^2 \tag{1.15}$$

The ϕ integral is trivial and we have

$$\frac{d\sigma}{d\cos\theta} = \frac{1}{8E_A^2} \frac{1}{2E_2} \frac{p_1^2 dp_1 d\phi}{(2\pi)^3 2E_1} \delta(E_{\text{cm}} - E_1 - E_2) \left| \mathcal{M} \right|^2 \tag{1.16}$$

$$= \begin{cases} \frac{1}{16\pi} \frac{p_1}{s^{3/2}} \left| \mathcal{M} \right|^2, \sqrt{s} < m_1 + m_2 \\ 0, \quad \text{otherwise} \end{cases} \tag{1.17}$$

If $m_1 = m_2$ then

$$|\mathbf{p}_1| = \frac{\sqrt{s}}{2} \overbrace{\sqrt{1 - \frac{4m_2^2}{s}}}^{\text{velocity of } p_1} \tag{1.18}$$

and

$$\frac{d\sigma}{d\cos\theta} = \frac{1}{32\pi s} \sqrt{1 - \frac{4m_1^2}{s}} \left| \mathcal{M} \right|^2 \tag{1.19}$$

The amplitude is a Lorentz-invariant quantity which depends on the 4-momenta. There are very few Lorentz invariant quantities that you can form out of the momenta. We have

$$\mathcal{M} = f(\{p_i \cdot p_j\}, \{m_k^2\}) \tag{1.20}$$

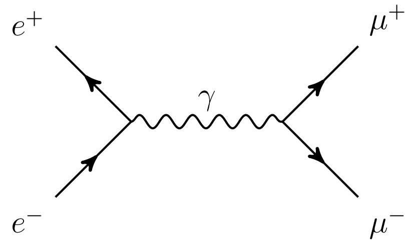
for $N \geq 3$ we can also have $\epsilon_{\alpha\beta\gamma\delta} p_i^\alpha p_j^\beta p_k^\gamma p_\ell^\delta$, but this is equal to zero for $N = 2$ since

$$\epsilon_{\alpha\beta\delta\delta} p_1^\alpha p_2^\beta p_3^\gamma p_4^\delta = \epsilon_{\alpha\beta\gamma\delta} p_1^\alpha p_2^\beta p_3^\gamma (p_1 + p_2 - p_3)^\delta = 0 \tag{1.21}$$

We already know that there is only one variable in the problem, so these must all be related. While the dot-products are fine Lorentz invariant quantities, the more standard thing to do is to define the Mandelstam variables,

$$s = (p_A + p_B)^2 = E_{\text{cm}}^2 = (p_3 + p_4)^2 \tag{1.22}$$
$$t = (p_A - p_1)^2 = (p_B - p_2)^2 \tag{1.23}$$
$$u = (p_A - p_2)^2 = (p_B - p_1)^2 \tag{1.24}$$

1.2 Example 1:$e^+e^- \to \mu^+\mu^-$



We will take $\sqrt{s} \ll M_Z$ so the weak interaction does not come into play and we can consider the single diagram.

This calculation can be done in two different ways. It can be done the standard way using traces and also the more instructive way using helicity. Helicity is an operator which can be defined for any particle with any spin and is given by

$$h = \mathbf{S} \cdot \hat{p} \tag{1.25}$$

where $\hat{p} \equiv \frac{\mathbf{p}}{|\mathbf{p}|}$. For a massless fermion, helicity is roughly equal to chirality.

Recall that the Lorentz group can be broken up as $SO(3,1) \approx SU(2) \times SU(2)$. The Weyl spinors are in $(2,1)$ and $(1,2)$. The four components are natural in QED, however elsewhere it makes more sense to use Weyl spinors since they are the fundamental objects of the Lorentz group. Chirality is what labels our particles. An electron can have left or right handed chirality. s The Lagrangian is given by

$$\mathcal{L}_{int} = \overbrace{g}^{\text{charge}} \bar{e}\gamma^\mu e A_\mu = g\left(e_L^\dagger \sigma^\mu e_L A_\mu + e_R^\dagger \bar{\sigma}^\mu e_R A_\mu\right) \tag{1.26}$$

since QED conserves parity the left and right components have the same coupling. The second equation is really the way you should think about the physics. If we take $|e_L\rangle$ (a state produced by a e_L field) and act with it using the helicity operator,

$$h\,|e_L\rangle = -\frac{1}{2}\,|e_L\rangle\,, \quad h\,|e_R\rangle = +\frac{1}{2}\,|e_R\rangle \tag{1.27}$$

If you work out the action of the creation and annihilation operators you find the confusing part,

$$e_L \ni \left(e_L^-, e_R^+\right), \quad e_R \ni \left(e_R^-, e_L^+\right) \tag{1.28}$$

You may wonder why one needs to worry about these details as we end up summing over spins. However, especially in the weak interactions it is often useful to avoid summing over the spins and calculating matrix elements individually.

Calculating these gives,

$$\mathcal{M}(e_L^- e_R^+ \to \mu_R^- \mu_L^+) = \mathcal{M}(e_L^+ e_R^- \to \mu_R^+ \mu_L^-) = -g^2(1-\cos\theta) \tag{1.29}$$

$$\mathcal{M}(e_L^- e_R^+ \to \mu_L^- \mu_R^+) = \mathcal{M}(e_R^- e_L^+ \to \mu_R^- \mu_L^+) = -g^2\,(1+\cos\theta) \tag{1.30}$$

These are found in Peskin and Schroeder on pages 143-146.

Its interesting to note that not all the possible combinations of collisions are possible. e.g. the amplitude for two left handed particles colliding or being emitted is zero (in reality this is only true for $m_\mu = m_e = 0$). The final state polarization isn't observable in current detectors. For an unpolarized beam,

$$\frac{1}{4}\sum_{\text{helicity}} |\mathcal{M}|^2 = g^4\left(1+\cos^4\theta\right) \tag{1.31}$$

and

$$\frac{d\sigma}{d\cos\theta} = \frac{g^4}{32\pi s}\left(1+\cos^2\theta\right) \tag{1.32}$$

$$= \frac{\pi\alpha^2}{2s}\left(1+\cos^2\theta\right) \tag{1.33}$$

$$\tag{1.34}$$

Integration of $\cos\theta$ gives the final cross section,

$$\sigma_{e^+e^-\to\mu^+\mu^-} = \frac{4\pi\alpha^2}{3s} \tag{1.35}$$

We will refer to this cross section several times throughout the lectures.

We now move on to calculate the same expression but for scalar muons. We have

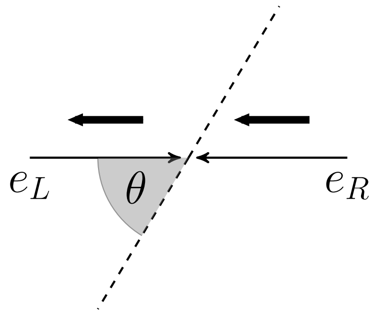
$$\mathcal{L} = gA_\mu\left(e_L^\dagger\sigma^\mu e_L + e_R^\dagger\bar{\sigma}^\mu e_R\right) \tag{1.36}$$

The notation is a little bit unfortunate. For the negatively charged particle, $e_L^\dagger$ creates a left handed electron and $e_R^\dagger$ creates a right handed electron, while for the positively charged particle the opposite is true.

The final result is given by

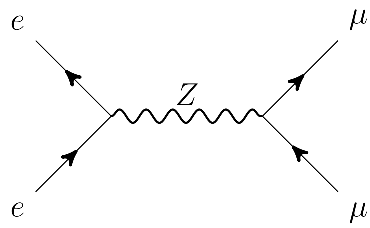
$$\frac{d\sigma}{d\cos\theta^*} = \frac{g^4}{32\pi s}\sin^2\theta \tag{1.37}$$

This can be understood solely on angular momentum arguments. We have



If the outgoing particles are coming out at very small θ then they have $L = 0$ (since orbital angular momentum is $\mathbf{r} \times \mathbf{p}$). In the massless limit we have that chirality is the same as helicity and hence the initial state has $J_z = -1$. Since the muons are scalar and have no spin, we must get a vanishing amplitude at $\theta = 0$.

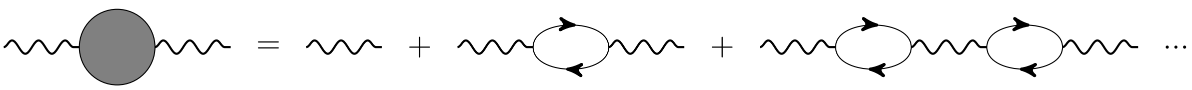
Thus far we have completely ignored the Z boson. We now restore it,



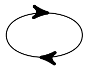
This is a pretty simple addition however there are a couple of issues that come up with these additions. With the photons they were always off-shell since the photon "mass", p^2 was the center of mass energy. With the Z it can be on-shell and the particle can be thought of as a 2 to 1 particle creation process followed by a particle decay. The naive propagator of the vector particle is

$$\frac{-ig_{\mu\nu}}{p^2 - M_Z^2 + i\epsilon} \tag{1.38}$$

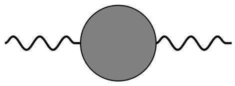
However, what happens if $\sqrt{s} = M_Z$ (our particle is on-shell)? We have a singularity that we need to make sense of. This can be done using higher order corrections. We have



This infinite series can be resummed. If you define

$$i\Pi(p^2) = $$  (1.39)

you have



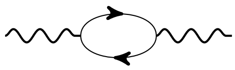
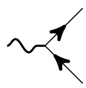
$$= \frac{-ig_{\mu\nu}}{p^2 - M_Z^2 - \Pi(p^2)} \tag{1.40}$$

The physical mass is at

$$p^2 - M_Z^2 - \Pi(p^2)\Big|_{p^2=M_Z^2} = 0 \tag{1.41}$$

In QED that's all there is to it and is done when renormalizing the theory. However, when discussing the Z boson, $\Pi(p^2)$ has both a real and imaginary part. This can be best understood using the optical theorem.

Consider the following,

$$\mathrm{Im}\Big($$  $$\Big) \sim \sum_f \int d\Pi_2 d\Pi_2 \Big|$$  $$\Big|^2$$

Now the difference between the photon and the Z is that for the photon, it can't decay, so the diagram that is begin "squared" on the RHS of the optical theorem doesn't contribute and the imaginary part of Π is zero. However, since the Z boson can decay this is no longer zero. Then the equation for the Z mass is strange since equation 1.41 can't be solved for real p^2. So what people do to define the Z mass to be when

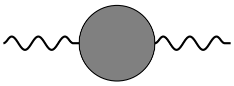
$$p^2 - M_Z^2 - \mathrm{Re}\Pi(p^2)\Big|_{p^2=M_Z^2} = 0 \tag{1.42}$$

Now the propagator no longer vanishes at $p^2 = M_{phys}^2$.

We can go a bit further and expand the real part of$\Pi(p^2)$ around the physical mass (the pole), M_Z. We define the original boson mass above as $M_{Z,0}$ and the physical mass as M_Z. The denominator is then given by

$$\begin{aligned}
&\overbrace{p^2 - M_{Z,0}^2 - \Big(\mathrm{Re}\Pi(p^2)\Big|_{p^2=M_Z^2}}^{p^2-M_Z^2} + \frac{d}{dp^2}\mathrm{Re}\Pi(p^2)\Big|_{p^2=M_Z^2} \cdot (p^2 - M_Z^2) + ... + i\mathrm{Im}\Pi(p^2)\Big) \\
&\quad = \underbrace{\Big(1 + \frac{d}{dp^2}\mathrm{Re}\Pi\Big)}_{Z^{-1}} \big(p^2 - M_Z^2\big) + i\mathrm{Im}\Pi(p^2)
\end{aligned} \tag{1.43}$$

we have the normal expansion of Π and in addition we have the extra imaginary part. Further we denote Z as the wave function renormalization. So to first order (ignoring the "dots" above) we have,



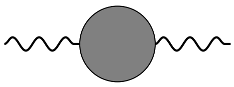
$$= \frac{-g^{\mu\nu} Z}{p^2 - M_Z^2 + iZ\mathrm{Im}\Pi(p^2)} \tag{1.44}$$

So to finish we need to calculate the imaginary part of Π. Roughly speaking looking at the optical theorem we see that it is given by the width. By the definition of the particle width we have

$$\mathrm{Im}\Pi(p^2) = \frac{Z^{-1}}{2} \int d\Pi_1 d\Pi_2 \left|\mathcal{M}(Z \to p_1 p_2)\right|^2 \tag{1.45}$$

$$= Z^{-1} M_Z \Gamma_Z \tag{1.46}$$

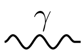
[Q 1: where did the Z^{-1} factor come from? Was it because we have one external Z boson?] so if you are doing this experiment near the Z pole then you need to correct your propagator by writing



$$= \frac{-g^{\mu\nu}}{p^2 - M_Z^2 + iM_Z\Gamma_Z} \tag{1.47}$$

This is the famous Breit-Wigner shape.

Now if $s \approx M_Z^2$ then



$$\sim \frac{1}{s} \sim \frac{1}{M_Z^2} \tag{1.48}$$

$$\sim \frac{1}{M_Z \Gamma_Z} \tag{1.49}$$

So if the width of the Z boson is small and you are colliding particles around the Z mass, to first approximation only the Z boson contributes. This is the case for the real Z boson, where $\Gamma_Z \approx 2$GeV.

We have

$$\mathcal{L} = g_L e_L^\dagger \sigma^\mu e_L Z_\mu + g_R e_R^\dagger \bar{\sigma}^\mu e_R Z_\mu \tag{1.50}$$

where

$$g_L = \frac{e}{s_w c_w}\left(-\frac{1}{2} + s_w^2\right), \quad g_R = \frac{e s_w}{c_w} \tag{1.51}$$

Recall that for $e^+e^- \to \gamma^\mu \to \mu^+\mu^-$ we have

e^-	e^+	μ^-	μ^+	$\mathcal{M}/e^2$
L	R	L	R	$1 + \cos\theta$
L	R	R	L	$1 - \cos\theta$
R	L	L	R	$1 - \cos\theta$
R	L	R	L	$1 + \cos\theta$

Everything is the same for the Z boson. The only change in the different couplings and the different Breit Wigner shape. We get,

e^-	e^+	μ^-	μ^+	$\mathcal{M}$
L	R	L	R	$g_L^2(1+\cos\theta)(sf_{BW})$
L	R	R	L	$g_L g_R(1-\cos\theta)(sf_{BW})$
R	L	L	R	$g_L g_R(1-\cos\theta)(sf_{BW})$
R	L	R	L	$g_R^2(1+\cos\theta)(sf_{BW})$

where

$$f_{BW} = \frac{1}{s - M_Z^2 + iM_Z\Gamma_Z} \tag{1.52}$$

To find the cross sections we square the amplitudes and we have

$$\frac{d\sigma}{d\cos\theta} = \frac{1}{32\pi s}\left|sf_{BW}\right|^2 \left\{(g_L^4 + g_R^4)(1+\cos\theta)^2 + 2g_L^2 g_R^2(1-\cos\theta)^2\right\} \tag{1.53}$$

Unlike in QED, we now get a term linear in $\cos\theta$ given by

$$2(g_L^2 - g_R^2)^2\cos\theta \tag{1.54}$$

From this one can define what is called a forward-backward asymmetry,

$$A_{fb} = \frac{\sigma(\cos\theta > 0) - \sigma(\cos\theta < 0)}{\sigma(\cos\theta > 0) + \sigma(\cos\theta < 0)} \tag{1.55}$$

Now we continue with studying the structure of the diagram in the Z channel. Now that the Z is on-shell, we can think of the process as producing a Z which subsequently decays. This can be done with as many particles you produce. We will now address the question of when you are allowed to split up a diagram. Consider the matrix element for the process above,

$$\mathcal{M} = [v_B\Gamma^\mu u_A]\,\frac{-g_{\mu\nu} + \frac{p_\mu p_\nu}{M_Z^2}}{p^2 - M_Z^2 + iM_Z p_Z}\,[\bar{u}_1\Gamma^\nu u_2] \tag{1.56}$$

where Γ^μ denotes some vertex factors. Now recall the spin sum formula,

$$-g_{\mu\nu} + \frac{p_\mu p_\nu}{M_Z^2} \to \sum_{a=-1}^{1}\epsilon_\mu^{*a}(p)\epsilon_\nu^a(p) \tag{1.57}$$

This requires the particle being studied to be on-shell. If this is the case then we have

$$\mathcal{M} = \sum_a [v_B\Gamma^\mu u_A]\,\frac{\epsilon_\mu^{*a}(p)\epsilon_\nu^a(p)}{p^2 - M_Z^2 + iM_Z\Gamma_Z}\,[\bar{u}_1\Gamma^\nu u_2] \tag{1.58}$$

$$= \sum_a \mathcal{M}(e^+e^- \to Z^a) f_{BW}\mathcal{M}(Z^a \to \mu^-\mu^+) \tag{1.59}$$

This process is known as "factorization of the matrix element". This gets us half of the way to factorizing the cross section, we still need to factorize the phase space factor. The cross section is

$$d\sigma = \frac{1}{8E_AE_B} d\Pi_1 d\Pi_2 (2\pi)^4 \delta^4(p_A + p_B - p_1 - p_2) \left|\mathcal{M}\right|^2 \tag{1.60}$$

To factorize this we need to insert a 1,

$$1 = \int d\Pi_Z (2\pi)^3 \delta^3(\mathbf{p}_A + \mathbf{p}_B - \mathbf{p}_Z) 2M_Z \tag{1.61}$$

Its straightforward to check that this is equal to one. Inserting in this relation,

$$\begin{aligned} d\sigma = \frac{1}{2s} \sum_{ab} \int & d\Pi_Z (2\pi)^3 \delta^3(\mathbf{p}_A + \mathbf{p}_B - \mathbf{p}_Z) \mathcal{M}(e^+e^- \to Z^a) \mathcal{M}^*(e^-e^+ \to Z^b) \\ & 2M_z \left|f_{BW}(s)\right|^2 d\Pi_1 d\Pi_2 (2\pi)^4 \delta^4(p_Z - p_1 - p_2) \mathcal{M}^*(Z^a \to \mu^+\mu^-) \mathcal{M}(Z^b \to \mu^+\mu^-) \end{aligned} \tag{1.62}$$

Now this is almost factorized. However there are two "problems". There is a missing delta function for the energy of the Z boson and there is also the propagator factor. It turns out that in the limit of narrow width these two problems cancel each other. This is called the "narrow-width approximation". It consists of taking the Breit Wigner shape and approximating it as,

$$M_Z \left|f_w(s)\right|^2 = \frac{M_Z}{(s - M_Z)^2 + \Gamma^2 M_Z^2} \approx \frac{\pi}{\Gamma_Z M_Z} \delta(E_A + E_B - M_Z) \tag{1.63}$$

The coefficient in front can be derived by integrating the Breit Wigner shape from $-\infty \to \infty$.

Applying the narrow with approximation gives

$$\begin{aligned} d\sigma = \sum_{ab} \Bigg(& \frac{1}{8E_AE_B} \int d\Pi_z (2\pi)^4 \delta^4(p_A + p_B - p_Z) \mathcal{M}(e^+e^- \to Z^a) \mathcal{M}^\dagger(e^+e^- \to Z^b) \\ & \frac{1}{\Gamma_Z} \frac{1}{2M_Z} d\Pi_1 d\Pi_2 (2\pi)^4 \delta^4(p_Z - p_1 - p_2) \mathcal{M}^*(Z^a \to \mu^+\mu^-) \mathcal{M}(Z^b \to \mu^+\mu^-) \Bigg) \end{aligned} \tag{1.64}$$

$$\equiv \frac{1}{\Gamma_Z} \mathrm{tr}\left[P^{ab} dD^{ab}\right] \tag{1.65}$$

here we defined P as the production or "Spin Density Matrix" and D is the decay matrix which are given by

$$P^{ab} \equiv \frac{1}{8E_AE_B} \int d\Pi_z (2\pi)^4 \delta^4(p_A + p_B - p_z) \mathcal{M}(e^+e^- \to Z^a) \mathcal{M}^\dagger(e^+e^- \to Z^b) \tag{1.66}$$

$$dD^{ab} \equiv \frac{1}{2M_Z} d\Pi_1 d\Pi_2 (2\pi)^4 \delta^4(p_Z - p_1 - p_2) \mathcal{M}^\dagger(Z^a \to \mu^+\mu^-) \mathcal{M}(Z^b \to \mu^+\mu^-) \tag{1.67}$$

Note that these are "almost" the expressions for the cross section for production and decay distributions. If you have unpolarized production (This is NOT the case for the Z!) then

$$d\sigma = \sigma(e^+e^- \to Z)\frac{1}{\Gamma_Z}d\Gamma(Z \to \mu^+\mu^-) \tag{1.68}$$

$$\sigma = \sigma(e^+e^- \to Z)\mathrm{Br}(Z \to \mu^+\mu^-) \tag{1.69}$$

This is okay for total rates, but not distributions.

We now work an example. Recall that we have

$$\mathcal{L} = g_L e_L^\dagger \sigma^\mu e_L Z_\mu + g_R e_R^\dagger \bar{\sigma}^\mu e_R Z_\mu \tag{1.70}$$

We have two possible initial states, $e_L^+ e_R^-$ and $e_R^+ e_L^-$. Lets look at angular momentum,

This are the only two possibilities. Because of this the spin-density matrix is actually pretty simple. The off-diagonal matrix elements are zero. We only have $P^{--} \propto g_L^2$ and $P^{++} \propto g_R^2$. In the electron positron collisions if you are given unpolarized beams then you will be dominated with outgoing left-handed particles since in the SM $g_L \gg g_R$. We did one part of the calculation, now we need to calculate the decay distributions,

$$\frac{dD^{--}}{d\cos\theta} = g_L^2(1+\cos\theta)^2 + g_R^2(1-\cos\theta)^2 \tag{1.71}$$

$$\frac{dD^{++}}{d\cos\theta} = g_R^2(1+\cos\theta)^2 + g_L^2(1-\cos\theta)^2 \tag{1.72}$$

and we have

$$\sigma_-\frac{dD^{--}}{d\cos\theta} + \sigma_+\frac{dD^{++}}{d\cos\theta} \propto (g_L^4 + g_R^4)(`1+\cos\theta)^2 + 2g_L^2 g_R^2(1-\cos\theta)^2 \tag{1.73}$$

as expected.

Now imagine what would happen if we forgot about the polarization of the beam. For an unpolarized beam you would have

$$\frac{d\Gamma}{d\cos\theta} = \frac{1}{2}\left(\frac{dD^{++}}{d\cos\theta} + \frac{dD^{--}}{d\cos\theta}\right) \tag{1.74}$$

$$= \left(g_L^2 + g_R^2\right)\left(1+\cos^2\theta\right) \tag{1.75}$$

and you would conclude that $A_{fb} = 0$ which is clearly wrong.

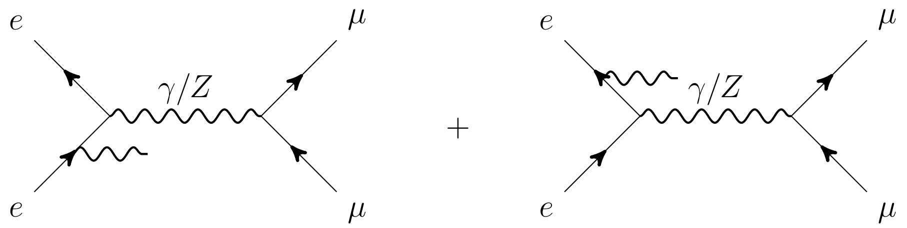
In general you don't need to factorize your amplitudes. Alternatively you just use Breit Wigner shapes for your intermediate particles. However, this greatly simplifies the calculations. For example, if you are working at the Tevatron and calculating the cross section for $q\bar{q} \to g \to t\bar{t} \to b\bar{b}\ell\bar{\ell}\nu\bar{\nu}$. Then without factorization you will be working with a 6 body final state!

Chapter 2

QED

2.1 Initial State Radiation

So far we have been considering the $2 \to 2$ process. What can we do different? We can consider the $2 \to 3$ process,



In general we must include the diagrams for the photon emitted by the muons in our discussion. However, when dealing with Feynman diagrams you can get physically interesting results if you are dealing with a subset of the diagrams **IF** the subset is gauge invariant. This doesn't give the correct results since there are interference terms, however the result can hold in some approximation. It just so happens that each of these diagrams is gauge invariant on its own (we will check this later on).

We begin with a naive guess of the cross section.

$$\sigma_{2\to3} \sim \frac{\alpha}{2\pi}\sigma_{2\to2} \sim \frac{1}{300}\sigma_{2\to2} \tag{2.1}$$

This is very small. So the naive guess about such processes is that they have little importance except for precision measurements. However, it turns out that this guess isn't quite correct. The reason for this is because of the propagator of the electron (or anti-electron) between the emission of the photon and the annihilation can blow up.

Consider the first diagram. Take the incoming electron momentum to be p_A and the photon momentum to be p_γ. Then

$$\mathcal{M} \propto \frac{1}{(p_A - p_\gamma)^2 - m_e^2} = \frac{1}{-2p_A \cdot p_\gamma} \tag{2.2}$$

We take the four momentum to be (for now we ignore m_e though we will put it back in later),

$$p_A = (E, 0, 0, E) \tag{2.3}$$

$$p_\gamma = (zE, \mathbf{p}_\perp, \sqrt{z^2E^2 - p_\perp^2}) \tag{2.4}$$

where we have defined z as the fraction of the electron energy taken by the photon. The dot product is given by

$$p_A \cdot p_\gamma = zE^2(1 - \sqrt{1 - \frac{p_\perp^2}{z^2E^2}}) \tag{2.5}$$

This goes to zero in two different limits.

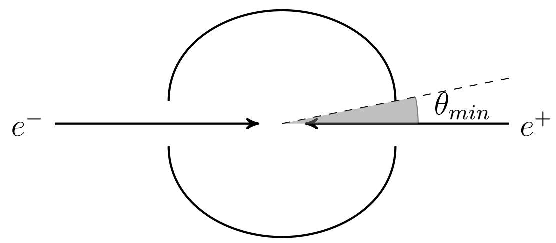
1. When $z \to 0$. This limit is a bit subtle. We have $0 < p_\perp < zE$. So we can write $p_\perp = xzE$ where x is the fraction of energy in the transverse direction (this takes care of the fact that $p_\perp$ isn't really independent of z. It is then obvious that in the $z \to 0$ limit the dot product vanishes. This corresponds to a photon that carries very little energy and is known as a "soft singularity"

2. When $p_\perp \to 0$. This means the photon is emitted parallel to the electron motion. This is called a "collinear singularity"

So this is bad. If you try to calculate the cross section for these tree-level Feynman diagram you integrate over all phase space configurations and you get results that don't make sense. The resolution comes at two different levels.

1. Soft photons are drowned out in detector noise.

2. In practice you measure muons using the muon detectors and photons using the EM Calorimeter. The EM calorimeter cannot cover the entire solid angle of the interaction point. There needs to be holes where the beams get in,



If γ are emitted with $\theta < \theta_{min}$ then they won't be observed. This is nice since most of these photons are unobservable and we don't need to worry about them. This deals with collinear photons.

Where do we draw the line between observable and unobservable collinear photons? To decide what kind of photons are "observed" we define a value Q such that if

$$p_T^{\gamma} > Q \tag{2.6}$$

we call it a $2 \to 3$ process. But there is still a problem. What will happen for the events that $p_\perp$ is small? These events will not disappear, they will just look like a $2 \to 2$ process but with one difference, the energy of the electron will be smaller. This is also the case for the collinear photons. In short $p_\perp < Q$ reactions modify the rate of $2 \to 2$ processes. We need to know what this shift is.

The key observation here is that when $p_\perp$ is small, $p_\gamma \approx z p_A$. In this case

$$p_A \cdot p_\gamma = z p_A^2 \approx 0 \tag{2.7}$$

and hence the virtual electron is almost on-shell. This will be useful below.

The amplitude takes the form,

$$\mathcal{M}_{2\to 3} = \bar{v}_B \gamma^\mu \frac{\not{p}_A - \not{p}_\gamma - m_e}{-2 p_A \cdot p_\gamma} (e\gamma^\alpha) u_A \times (...) \tag{2.8}$$

We want take the collinear limit, so

$$p_A \cdot p_\gamma = z E^2 \left(p_\perp^2 / 2 z^2 E^2 \right) = p_\perp^2 / 2z \tag{2.9}$$

and hence we can replace the denominator with the above. Furthermore, the electron is roughly on shell so we can make the spin sum replacement,

$$\mathcal{M}_{2\to 3} \approx \bar{v}_B \gamma^\mu \frac{\sum_\lambda u^\lambda(p_A(1-z)) \bar{u}^\lambda(p_A(1-z))}{-2(p_\perp^2/2z)} (e\gamma^\alpha) u_A \times (...) \tag{2.10}$$

where the momenta of the virtual electron is just given by

$$p_A - p_\gamma = (1-z) p_A \tag{2.11}$$

We now just rewrite the term above as,

$$\mathcal{M}_{2\to 3} = -e \frac{z}{p_\perp^2} \sum_\lambda e \underbrace{\left[\bar{u}^\lambda (p_A(1-z)) \gamma^\alpha u_A \right]}_{\mathcal{M}^\lambda_{e^- \to e^- \gamma}} \underbrace{\left[\bar{v}_B \gamma^\mu u_\lambda (p_A(1-z)) \right] \times (...)}_{\mathcal{M}^\lambda_{2\to 2}(p_A(1-z), p_B \to \{p_i\})} \tag{2.12}$$

The first factor, known as the splitting amplitude doesn't know anything about the rest of the diagram. This result is very general.

We pause for a moment to prove gauge invarience. This element is multiplied by a polarization vector ϵ^α. Switching $\epsilon^\alpha \to p_\gamma^\alpha$ gives,

$$\begin{aligned} \left[\bar{u}^\lambda \not{p}_\gamma u_A \right] &= \left[\bar{u}^\lambda \left(\not{p}_A - \not{p}_{e_\lambda} \right) u_A \right] && (2.13) \\ &= \left[\bar{u}^\lambda \left(m_e - m_e \right) u_A \right] && (2.14) \\ &= 0 && (2.15) \end{aligned}$$

This justifies considering this single diagram in our calculation.

This is what the matrix element look like, the next step is to consider the cross section. We have the usual master formula,

$$d\sigma_{2\to 3} = \frac{1}{8E_A E_B} d\Pi_\gamma \overbrace{\left(\frac{z}{p_\perp^2}\right)^2}^{\text{From } \mathcal{M}_{2\to 3}} \times \frac{1}{2} \sum_{\lambda,\lambda',h,a} \mathcal{M}_{e_h^- \to e_\lambda^- \gamma_a}(z) \mathcal{M}^*_{e_h^- \to e_{\lambda'} \gamma_a}(z) \cdot d\Pi_{1,2}$$
$$\mathcal{M}^{\lambda}_{2\to 2}(s(1-z)) \mathcal{M}^{\lambda',*}_{2\to 2}(s(1-z)) \epsilon^a(p_\gamma) \delta^4(p_A + p_B - p_\gamma - p_1 - p_2)(2\pi)^4 \quad (2.16)$$

where λ and λ' are the spins from the matrix element, h is the spin of the electron, and a is the spin of the photon. Furthermore, s is the Mandelstam variable. Notice the matching of the spin indices, λ, λ', between the different contributions. This is the same result we saw in our first example of factorization above. The $2 \to 2$ amplitude will depend only on s as we are working in the s channel. However the effective "s" is modified since

$$s_{eff} = (p_B + (1-z)p_A)^2 \quad (2.17)$$
$$= 2p_A \cdot p_B \, (1-z) \quad (2.18)$$
$$= s\,(1-z) \quad (2.19)$$

Note that

$$\mathcal{M}(e_L^- \to e_R^- \gamma) = \mathcal{M}(e_R^- \to e_L^- \gamma) = 0 \quad (2.20)$$

This is easy to see from the structure of the $e^- \to e^- \gamma$ amplitude since,

$$\left[u^{\lambda,\dagger} P_L \gamma^0 \gamma^\alpha P_R u_A\right] = 0 \quad (2.21)$$

Note that if the initial state radiation was through a parity violating vertex this would not have been true.

This means we only have non-zero amplitudes if

$$\lambda = h \quad (2.22)$$
$$\lambda' = h \quad (2.23)$$

This is convenient since that means we can just sum the spins in the usual way,

$$d\sigma_{2\to 3} = \frac{1}{2s} d\Pi_\gamma \left(\frac{z}{p_\perp^2}\right)^2 \frac{1}{2} \sum_{h,a} |\mathcal{M}_{hha}(z)|^2 \, d\Pi_{1,2} \sum_{h'} \left|\mathcal{M}^{h'}_{2\to 2}(s(1-z))\right|^2 \times \delta^4(...)(2\pi)^4 \quad (2.24)$$

Doing the explicit calculation gives the same result for $h = +1/2$ and $h = -1/2$. This calculation is done in the appendix (2.A) and gives,

$$\frac{1}{2} \sum_{h,a} |\mathcal{M}_{hha}(z)|^2 = \frac{2e^2 p_\perp^2}{z(1-z)} \left[\frac{1 + (1-z)^2}{z}\right] \quad (2.25)$$

Note that a similar calculation is given in Peskin and Schroeder, 576-578.

Inserting in this result gives,

$$d\sigma_{2\to3} = \frac{1}{2s} d\Pi_\gamma \left(\frac{z}{p_\perp^2}\right)^2 \frac{2e^2 p_\perp^2}{z(1-z)} \left[\frac{1+(1-z)^2}{z}\right] d\Pi_{1,2} \sum_h \left|\mathcal{M}_{2\to2}^h(s(1-z))\right|^2 \tag{2.26}$$

$$= d\Pi_\gamma \left(\frac{z}{p_\perp^2}\right)^2 \frac{2e^2 p_\perp^2}{z} \left[\frac{1+(1-z)^2}{z}\right] \times 2 \times d\sigma_{2\to2}(s(1-z)) \tag{2.27}$$

where we have used

$$\sigma_{2\to2}\left(s(1-z)\right) = \frac{1}{2s(1-z)} d\Pi_{1,2} \frac{1}{2} \sum_h \left|\mathcal{M}_{2\to2}\right|^2 \tag{2.28}$$

Now we have

$$d\Pi_\gamma = \frac{d^3 p_\gamma}{(2\pi)^3} \frac{1}{2E_\gamma} = \frac{d^2 p_\perp d(zE_A)}{(2\pi)^3} \frac{1}{2zE_A} = \frac{p_\perp dp_\perp}{8\pi^2} \frac{dz}{z} \tag{2.29}$$

So we have

$$d\sigma_{2\to3} = \int_0^1 \frac{dz}{z} \int_{0\to m_e}^Q \frac{p_\perp dp_\perp}{8\pi^2} \left(\frac{z}{p_\perp^2}\right)^2 \frac{4e^2 p_\perp^2}{z} \left[\frac{1+(1-z)^2}{z}\right] \tag{2.30}$$

$$= \int_0^1 dz \frac{\alpha}{\pi} \left[\frac{1+(1-z)^2}{z}\right] \log\frac{Q}{m_e} \tag{2.31}$$

Notice that above we changed the limit of integration from $0 \to m_e$. This is because it turns out that a massive electron cannot omit perfectly collinear photons (this can be checked using momentum conservation in a few lines). [Q 2: How do we know the limit changes exactly to m_e, or is this an approximation?] We define

$$f_e(1-z, Q) \equiv \frac{\alpha}{\pi} \left[\frac{1+(1-z)^2}{z}\right] \log\frac{Q}{m_e} \tag{2.32}$$

and so

$$d\sigma_{2\to3}(p_\perp < Q) = \int_0^1 dz f_e(1-z, Q) \times d\sigma_{2\to2}(s(1-z)) \tag{2.33}$$

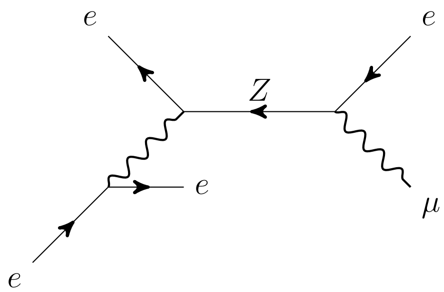
This formula is very simple and has a beautiful interpretation. $f_e(x, Q)$ is the probability to find an electron of energy xE in a beam supposedly consisting of electrons of energy E! This is analogous to the parton distribution functions of QCD. So inside a moving electron there is in fact a distribution of electrons with lower energy due to initial state radiation. However, this assumes that the electrons emitted a photon. This is of course not always true. To include the correction for when we have a true $2 \to 2$ process we write,

$$f_e(x, Q) = \delta(1-x) + \frac{\alpha}{\pi} \frac{1+x^2}{1-x} \log\frac{Q}{m_e} \tag{2.34}$$

Note that this equation is obviously not normalized at this point. The delta function is normalized and there is an extra correction which will not integrate to zero. We will remedy this soon.

However, there is an issue with this formula. It is not well defined when $z \to 0$ (or equivalently, when $1 - x \to 0$). f_e is actually a divergent integral, and so is ill-defined we will return to this in a moment.

Before we go on, consider the following point. In our example, $e^- \to e^-\gamma$ was followed by $e^-e^+ \to X$ but the collinear photon itself can also enter the subsequent reactions, e.g.



The cross section for this would be very similar,

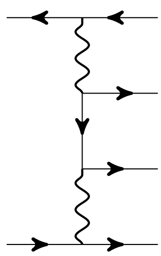
$$d\sigma_{2\to3}(p_\perp^e < Q) = \int dz f_\gamma(z, Q) d\sigma_{2\to2}(zs) \tag{2.35}$$

where to this order in perturbation theory we have (by energy conservation)

$$f_\gamma(z, Q) = f_e(1 - z, Q) \tag{2.36}$$

The form of the z dependence arises from the fact that we still want to define z as the energy fraction taken by the photon. Before we wrote $d\sigma_{2\to2}(zs)$, where the term in parenthesis was the energy the center of mass energy of the $2 \to 2$ collision. In the new $2 \to 2$ collision the energy of the $2 \to 2$ collision is given by $\sqrt{(1-z)s}$.

Much more complicated processes are also possible. e.g.



As you go to higher and higher energies it becomes more and more important to include the effect that an "electron" beam isn't just electrons.

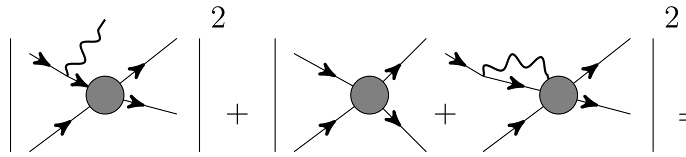
Thus far we have only dealt with the collinear divergence and we found that it is not a true divergence and it goes like $\sim \log \frac{Q}{m_e}$.

We now confront the $z \to 0$ divergence. When $z \to 0$,

$$E_\gamma \to zE_A \to 0 \tag{2.37}$$

and we have "soft photons". Recall from QFT that the soft-photon divergence cancels between diagrams with real soft photon emission and virtual photons with the virtual momentum, $\ell \to 0$,

$\int_0 d\Pi_\gamma$



$= \text{finite}$

In order to solve for the effect of soft photon emission you need to calculate a complicated loop diagram. Instead, we can use a much easier way, and that's by using normalization,

$$\int_0^1 f_e(x, Q) = 1 \tag{2.38}$$

The expression we wrote above for f_e is clearly not normalized. By matching orders of α it has a chance at being normalized if we write the more appropriate form,

$$f_e(x, Q) = (1 + \alpha A + ...)\delta(1-x) + \frac{\alpha}{2\pi}\frac{1+x^2}{1-x}\log\frac{Q}{m_e} + \mathcal{O}(\alpha^2) \tag{2.39}$$

The difficulty is that

$$\int_0^1 \frac{dx}{1-x} \tag{2.40}$$

is not well defined. Mathematically you can define

$$\frac{1}{(1-x)_+} = \frac{1}{1-x}, \forall x \neq 1 \tag{2.41}$$

and

$$\int_0^1 \frac{f(x)}{(1-x)_+}dx = \int_0^1 \frac{f(x) - f(1)}{1-x}dx \tag{2.42}$$

In other words we are subtracting away the singularity. Now we have

$$\int_0^1 \frac{dx(1+x^2)}{(1-x)_+} = \int_0^1 \frac{x^2-1}{1-x} = -\int_0^1 (x+1)dx = -\frac{3}{2} \tag{2.43}$$

This implies that

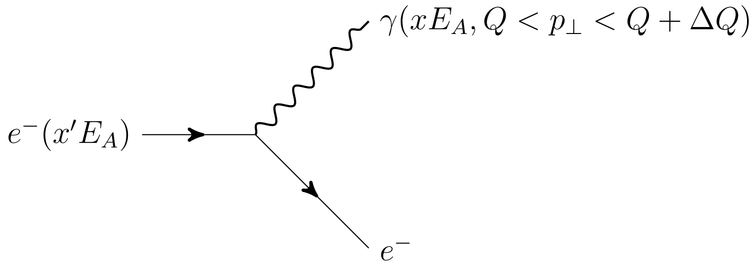
$$A = \frac{3}{2}\frac{\alpha}{2\pi}\log\frac{Q}{m_e} \tag{2.44}$$

and

$$f_e(x, Q) = \delta(1-x) + \frac{\alpha}{2\pi}\left[\frac{1+x^2}{(1-x)_+} + \frac{3}{2}\delta(1-x)\right]\log\frac{Q}{m_e} \tag{2.45}$$

This expression is well defined as $x \to 1$ but it is still highly singular. As $x \to 1$ the photons get softer and softer and the probability of emitting another photon is getting larger and larger. Perturbation theory in this sense is breaking down. The computation is pretty good for most values of x but as you get within a few percent of 1 it breaks down. When that happens you need to try to understand the systematics of the calculation and make some approximation to get the leading behaviour of the process. If you can then sum up all the multiple photons contributions then you can achieve this. We will not do the rigorous calculation, instead we use a trick to make the calculation straightforward. In QED this is just a beautiful trick. In QCD it is necessary.

The idea to find equations for how f_e changes as a function of Q. These will not be done perturbatively. When we talk about these probability distribution functions we talk about their dependence on their energy, but they also have a second important dependence on Q. At first this dependence may seem strange since why would the composition of an electron beam dependence on this value that we choose Q. However, thinking more critically Q really tells us when we have or don't have a photon. So depending on what energies we still count as photons we have a difference beam composition. i.e. this value tells you when you call the photon part of the beam and when you call it a true emission. It's value is up to us. When we want to study a $2 \to 2$ process, we can decide what Q is. Imagine that we know f_e for some value of Q and then we increase $Q \to Q + \Delta Q$. For simplicity we focus on the photons. Consider



where the splitting fraction is then $z = \frac{x}{x'} < 1$.

$$f_\gamma(x, Q + \Delta Q) = f_\gamma(x, Q) + \int_0^1 dx' f_e(x', Q) \int_0^1 dz \delta(x - zx') \frac{\alpha}{\pi} \frac{1 + (1-z)^2}{z} \log \frac{Q + \Delta Q}{Q} \tag{2.46}$$

The first term is the probability of finding a photon with $p_\perp < Q$ and the second term is just the probability of finding a photon with $Q < p_\perp < Q + \Delta Q$. The second term can be understood as follows.

$$dz\delta(x - zx') \frac{\alpha}{\pi} \frac{1 + (1-z)^2}{z} \log \frac{Q + \Delta Q}{Q} = f_\gamma(1 - z, Q + \Delta Q) - f_\gamma(1 - z, Q) \tag{2.47}$$

is the probability of finding an electron beam with energy fraction x' having a photon with energy fraction x between Q and $Q + \Delta Q$, $dx' f_e(x', Q)$ is the probability of finding an electron beam with energy fraction x'.

We have,

$$f_\gamma(x, Q + \Delta Q) = f_\gamma(x, Q) + \frac{\alpha}{\pi} \int_x^1 \frac{dz}{z} f_e\left(\frac{x}{z}, Q\right) \frac{1 + (1-z)^2}{z} \log \frac{Q + \Delta Q}{Q} \tag{2.48}$$

where we have used the fact that in order for the delta function to not have been zero we require, $\frac{x}{x'} < z < 1$. Furthemore,

$$\frac{df_\gamma}{d\log Q} = \frac{\alpha}{\pi}\int_x^1 \frac{dz}{z} f_e\left(\frac{x}{z}, Q\right)\frac{1+(1-z)^2}{z} \tag{2.49}$$

where in the last step we took the infinitesimal limit for ΔQ. We define a "splitting function" $P_{e^-\to\gamma}(z)$ and we can write the final result as

$$\frac{df_\gamma}{d\log Q} = \frac{\alpha}{\pi}\int_x^1 \frac{dz}{z} P_{e^-\to\gamma}(z) f_e\left(\frac{x}{z}, Q\right) \tag{2.50}$$

In the treatment above we didn't include all the possibilities. You could also have photons coming from other photons and photons coming from positrons. We worked out $P_{e^-\to\gamma}$, the complete set is in Peskin and Schroeder, p.587. Including all the possibilities you get,

$$\begin{aligned}\frac{df_\gamma(x,Q)}{d\log Q} = \frac{\alpha}{\pi}\int_x^1 \frac{dz}{z}\Big(& P_{e^-\to\gamma}(z) f_{e^-}(\frac{x}{z}, Q) + P_{\gamma\to\gamma}(z) f_\gamma(\frac{x}{z}, Q) \\ & + P_{e^+\to\gamma}(z) f_{e^+}(\frac{x}{z}, Q)\Big)\end{aligned} \tag{2.51}$$

These are known as the Gribov-Lipatov equations. In principle you can now calculate all the f_i's at a given energy and solve the differential equation to the get the distributions to all orders in Q.

At $Q = m_e$ you must just have

$$f_{e^-}(x, m_e) = \delta(1-x) \tag{2.52}$$

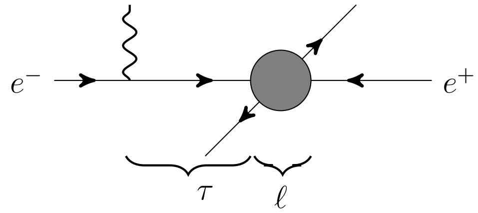
$$f_\gamma(x, m_e) = f_{e^+}(x, m_e) = 0 \tag{2.53}$$

2.2 Final State Radiation

Recall that we have

$$d\sigma_{2\to3} = \frac{\alpha}{\pi}\frac{dz}{z}(1+(1-z))^2\frac{dp_\perp}{p_\perp} d\sigma_{2\to2}(s(1-z)) \tag{2.54}$$

Consider a collision occurring in space-time:



The factorization can happen if you have physics at two separate length scales. The typical length of decay and the energy of the collision. Where the photon will be emitted is dependent on the energy of the photon emitted. We have

$$\tau \sim \frac{1}{\sqrt{-(p_A - p_\gamma)^2}} \tag{2.55}$$

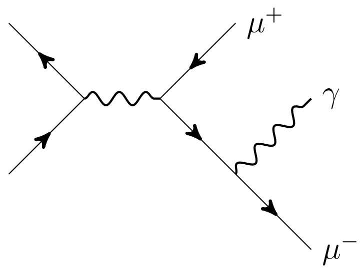
$$\ell \sim \frac{1}{\sqrt{s}} \tag{2.56}$$

where the value of τ was found from taking the Fourier Transform of roughly on-shell propagator of the virtual electron. If $\tau \gg \ell$ then we have factorization. Essentially all factorization examples on a fundamental level involve separation of length scales.

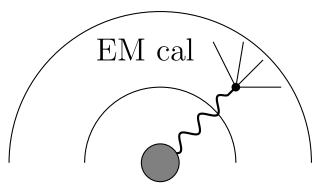
The same considerations apply to final state radiation (FSR). As an example consider

$$e^+e^- \to \mu^+\mu^-\gamma \tag{2.57}$$

or diagrammatically,



One physical difference between IRS is that a collinear γ **can** be detected in this case!



A soft γ is still undetectable though. Set E_γ^{min}.

We won't go through the detailed calculation, but the ideas are the same here as they were in the ISR case. We use equation (2.54) as a template. The primary contribution will be at small x since that is where we now have our soft divergence.

Here we take $z \to x$[Q 3: I'm not sure what is the reason to make this substitution]. Furthermore, we work in the limit of small x so $1 + (1 - x) \xrightarrow{x \ll 1} 2$.

$$\left.\frac{d\sigma_{2\to 3}}{dx}\right|_{x \ll 1} = \frac{2\alpha}{\pi}\frac{1}{x}\int_{m_\mu}^{xE_\mu} \frac{dp_\perp}{p_\perp} d\sigma_{2\to 2} \tag{2.58}$$

As before we let the limit of the $p_\perp$ integral have a lower bound of the particle emitting the particle. Furthermore, the maximum $p_\perp$ is just the energy of the photon.

We have

$$\sigma_{2\to 3} \approx \frac{2\alpha}{\pi}\int_{x_{min}}^{\mathcal{O}(1)}\frac{dx}{x}\int_{m_\mu}^{xE_\mu}\frac{dp_\perp}{p_\perp}d\sigma_{2\to 2} \tag{2.59}$$

where $x_{min} = E_\gamma^{min}/E_\mu$. We don't really know the upper-bound since we are assuming that x is small. So where the cut-off for "small"? But we are still okay since we have a logarithmic integral over x. A $\log(\mathcal{O}(1)) \approx 0$ so we ignore the extra term and we have

$$d\sigma_{2\to 3} = \frac{2\alpha}{\pi}\left(\log\frac{E_\mu}{E_\gamma^{min}}\right)\left(\log\frac{E_\mu}{m_\mu}\right)\overbrace{\sigma_{2\to 2}^{LO}}^{\equiv\sigma_0} + \begin{pmatrix}\text{terms with 1}\\ \text{log or no log}\end{pmatrix} \tag{2.60}$$

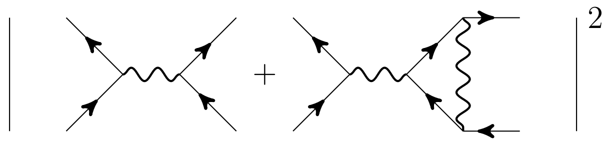
$$= \frac{\alpha}{2\pi}\left(\log\frac{s}{(E_\gamma^{min})^2}\right)\left(\log\frac{s}{m_\mu^2}\right)\sigma_0 + \begin{pmatrix}\text{terms with 1}\\ \text{log or no log}\end{pmatrix} \tag{2.61}$$

The two log form is called the "Sudakov double Logarithm". The photons with energies lower then E_γ^{min} still get counted in the detector but they are detected as $2 \to 2$ collisions. We now want to quantify the effect of these photons.

The rate of γ's with $E_\gamma < E_\gamma^{min}$ (which we call "real γ" to distinguish it from diagrams with virtual photons) is IR divergent as $x \to 0$ (as the energy of the photon goes to zero). This comes up since we are only working in leading order. The solution is to introduce a "regulator" μ that represents the photon mass, a trick often used in a QFT class. We now take the upper-bound to be E_γ^{min} and the lower bound to be μ. We have the same calculation as before.

$$\Delta\sigma_{2\to 2}^{\text{real }\gamma} = \frac{\alpha}{2\pi}\log\frac{(E_\gamma^{min})^2}{\mu^2}\log\frac{s}{m_\mu^2}\sigma_0 + ... \tag{2.62}$$

On top of this contribution there are also contributions from



Calculating these diagrams to leading order and next-to-leading order result

$$\Delta\sigma_{2\to 2}^{LO+NLO} = \sigma_0\left(1 - \frac{\alpha}{2\pi}\log\frac{s}{\mu^2}\log\frac{s}{m_\mu^2} + ...\right) \tag{2.63}$$

To get the full $2 \to 2$ rate we need to add this result with the calculation above for the soft photons that we miss from the $2 \to 3$ calculation. This gives,

$$\sigma_{2\to 2} = \sigma_0\left(1 - \frac{\alpha}{2\pi}\log\frac{s}{(E_\gamma^{min})^2}\log\frac{s}{m_\mu^2}\right) \tag{2.64}$$

The total result is well defined (independent of regulator)!

There are several interesting facts that we can glean from this calculation.

1. Radiative correction to $\sigma_{2\to 2}$(no γ's of $E > E_\gamma^{min}$) is proportional to $\frac{\sigma}{\pi}\log^2$. For example, for $\sqrt{s} = 100\text{GeV}, E_\gamma^{min} = 1\text{GeV}$, we have $\frac{\Delta\sigma_{NLO}}{\sigma_{LO}} \approx 8\%$ instead of the naive guess of $\frac{\alpha}{\pi} \approx 0.3\%$. So the radiative corrections are not small due to the logarithmic factors.

   Any kind of precision result requires resummation: this is achieved by techniques similar to GL equation for ISR in the previous lecture. We will not do this here but the answer is

$$\sigma_{2\to 2} = \sigma_0 \exp\left(-\frac{\alpha}{2\pi}\log\frac{s}{(E_\gamma^{min})^2}\log\frac{s}{m_\mu^2}\right) + (\text{no } log^2) \tag{2.65}$$

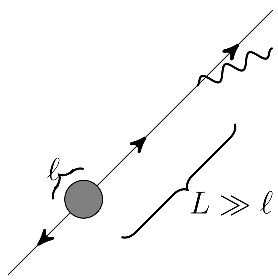
   The lesson here is when comparing different orders in perturbation theory you need to be careful due to the potential for large logs.

2. You may think that we are in deep trouble, since you need to go to higher and higher orders in perturbation theory to get precision results and this was in QED! The situation is even more bleak in QCD. Fortunately, leading log corrections cancel in **"inclusive"** cross section,

$$\sigma(\mu^+\mu^- \text{ or } \mu^+\mu^-\gamma) = \sigma_{2\to 2}(\text{no } \gamma \text{ in } E_\gamma < E_\gamma^{min}) + \sigma_{2\to 3}(E_\gamma > E_\gamma^{min}) \tag{2.66}$$
$$= \sigma_0 + (\text{no } \log^2\ ) \tag{2.67}$$

Although we have not yet shown in it turns out that there are no single log terms either.

This result is quite general: inclusive observables behave better in perturbation theory then exclusive. It has a simple physical reason, which is easy to see in the space-time picture.



σ_0 describes the production process (shown as a gray circle). If the produced muon happens to be a little bit off-shell then it splits.

We have

$$\sigma_{2\to 3} \sim Prob\left(\mu^- \to \mu^-\gamma\right)\sigma_0 \tag{2.68}$$
$$\sigma_{2\to 2} \sim Prob\,(\text{no splitting})\,\sigma_0 \tag{2.69}$$

so we have

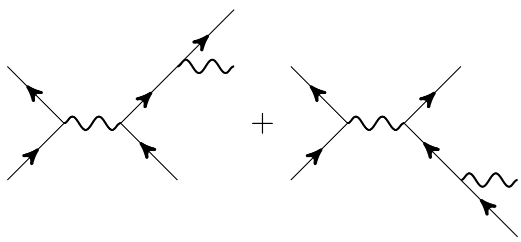
$$\sigma^{tot} \sim \overbrace{\left\{Prob(\mu^- \to \mu^-\gamma) + Prob(\text{no split})\right\}}^{1} \sigma_0 \tag{2.70}$$

$$\sim \sigma_0\,! \tag{2.71}$$

The only caveat is that this again requires separation of scales. The production scale must be much smaller then the decay length. The factorization and conservation of probability is behind this cancellation. In practical terms this is very nice. If you are worried about large logs, you could either do the resummation which takes a lot of work or you could take the cheap way out and just work with inclusive variables.

2.3 $e^+e^- \to \mu^+\mu^-\gamma$

Up to now we have avoided discussion of any interference terms. While this was wrong we got sensible results since we considered a gauge invariant subset of diagrams. We will now outline the necessary steps to do the exact calculations without making approximations. We consider $e^+e^- \to \gamma \to \mu^-\mu^+\gamma$ however, we will cheat a little. In general we should also have interference terms between initial and final state radiation. We omit the initial state radiation diagrams. The justification for this is that we do the calculations as a model for $e^+e^- \to \gamma \to q\bar{q}g$. These diagrams do not have any initial state gluon radiation. We will just go over the basic ideas and we will fill them in in the HW. We have two diagrams



and as we just learned there are also these virtual diagrams.

The available phase space is

$$3 \cdot 3 - 4 - 1 = 4 \tag{2.72}$$

since the naive phase space is $3 \cdot 3$, we have 1 energy conserving delta function which takes away 4 degrees of freedom. Furthermore, we will work in the center of mass frame. This constrains the decay products to be in a plane as their momenta need to add to zero. This plane can be in any orientation without a change in the cross section. This invarience leads to a variable that we can integrate over leaving one less degree of freedom.

We have

$$q \equiv p_A + p_B = \left(\sqrt{s}, \mathbf{0}\right) \quad \to \quad x_i \equiv \frac{2p_i \cdot q}{q^2} = \frac{2E_i}{\sqrt{s}} \tag{2.73}$$

Energy conservation constrains the possible energy fractions of the final states,

$$\sum_i E_i = \sqrt{s} \quad \Rightarrow \quad x_1 + x_2 + x_3 = 2 \tag{2.74}$$

where $0 \leq x_i \leq 1$. We can use the last condition on x_i to restrict the free parameters. This "restriction" is different from the phase space degrees of freedom calculation above. There we argued that the cross section will be invariant under a rotation. The condition on the x_i's is built into the momentum conserving delta function. Knowing these conditions helps us choose appropriate parameters to use in our cross section. We choose x_1, x_2 and the direction of e.g. γ with respect to e^- as our free parameters. The three body phase space can be split up as follows,

$$\int \prod_{i=1}^{3} d\Pi_i (2\pi)^4 \delta^{(4)}(q - \sum_i p_i) = \frac{s}{128\pi^3} \int dx_1 dx_2 d\Omega \tag{2.75}$$

where $d\Omega$ is the overall rotation of the final state particles. This expression is derived in appendix 2.B.2.

Now,

$$\frac{1}{4} \sum_{spins} |\mathcal{M}|^2 = L_{\mu\nu} M^{\mu\nu} \tag{2.76}$$

where $L_{\mu\nu} (M_{\mu\nu})$ involves the electron (muon) trace.

There is a useful trick we can use. By Lorentz invarience the structure of $M^{\mu\nu}$ is very simple. After integration over all angles the only vector left in the problem is the momenta going into the right vertex, q^μ, so we can write,

$$\int d\Omega M^{\mu\nu} = (A g^{\mu\nu} + B q^\mu q^\nu) \cdot f(x_1, x_2) \tag{2.77}$$

(obviously $L^{\mu\nu}$, the electron contribution, will not depend on Ω)

One can show that

$$q_\mu M^{\mu\nu} = q_\nu M^{\mu\nu} = 0 \tag{2.78}$$

This implies that $A = -B$ (these are secretly just Ward Identities).

So we have

$$\frac{1}{4} \sum_{spins} |\mathcal{M}|^2 = (g^{\mu\nu} L_{\mu\nu} - q^\mu q^\nu L_{\mu\nu}) A f(x_1, x_2) \tag{2.79}$$

where

$$f(x_1, x_2) = \frac{1}{3} \int d\Omega g_{\mu\nu} \mathcal{M}^{\mu\nu} \tag{2.80}$$

So in fact you don't need to calculate $M^{\mu\nu}$ but just its trace.

Doing the calculation gives

$$\frac{d\sigma}{dx_1 dx_2} = \frac{1}{2s} g_{\mu\nu} L^{\mu\nu} \cdot f(x_1, x_2) = \underbrace{\frac{4\pi\alpha^2}{3s}}_{\sigma_0} \frac{\alpha}{2\pi} \frac{x_1^2 + x_2^2}{(1 - x_1)(1 - x_2)} \tag{2.81}$$

There are two singularities

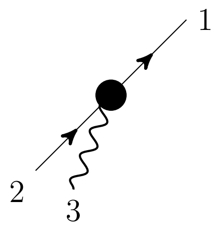
1. $x_1 = \frac{2p_1\cdot q}{q^2} \to 1$. This occurs when $2p_1\cdot q = q^2$, or equivalently,

$$(p_2+p_3)^2 = (q-p_1)^2 = q^2 - 2q\cdot p_1 \xrightarrow{x_1\to 1} 0 \tag{2.82}$$

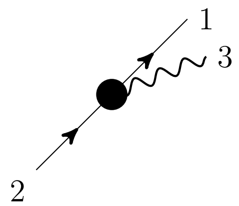
Hence, when the invariant mass of 1 and 2 is zero. If two massless momenta, k_1 and k_2 are collinear then,

$$m_{inv}^2 \equiv (k_1+k_2)^2 = (E_1+E_2)^2 - (E_1+E_2)^2 = 0 \tag{2.83}$$

So the first singularity corresponds to particle two being roughly collinear with the photon.



2. Likewise for the second singularity we have $x_2 \to 1$ and particle 1 is roughly collinear with the photon.



These are our old friends, the collinear singularities.

We need to regulate these singularities. There are two ways,

- We can do what we just did for the singularity calculation and use small photon mass, μ (changes $x_{min} \neq 0$) and reintroduce m_μ.
- Alternatively you can solve the problem by dimensional regularization: go to $d = 4-2\epsilon$. At this point the calculation gets complicated enough that you probably want to use dim-reg.

Repeating the calculation in dim-reg brings about a few changes. The result is,

$$\frac{d\sigma}{dx_1 dx_2} = \sigma_0 \frac{\alpha}{2\pi} \underbrace{\frac{3(1-\epsilon)^2}{(3-2\epsilon)\Gamma(2-2\epsilon)}}_{H(\epsilon)} \cdot \frac{x_1^2+x_2^2-\epsilon(2-x_1-x_2)}{(1-x_1)^{1+\epsilon}(1-x_2)^{1+\epsilon}} \tag{2.84}$$

Integration gives

$$\sigma(\epsilon) = \sigma_0 \frac{\alpha}{2\pi} H(\epsilon) \cdot \left[\frac{2}{\epsilon^2} + \frac{3}{\epsilon} + \frac{19}{2} + \mathcal{O}(\epsilon)\right] \tag{2.85}$$

The $\frac{2}{\epsilon^2}$ term corresponds to the $\log\frac{s}{\mu^2}\log\frac{s}{\mu^2}$ divergence while the $\frac{3}{\epsilon}$ corresponds to the single logs.

Also in dim-reg one can show that the contribution to the $2 \to 2$ cross section from NLO calculations is,

$$\sigma^{NLO}_{2\to 2}(\epsilon) = \sigma_0 \frac{\alpha}{2\pi} H(\epsilon) \cdot \left[-\frac{2}{\epsilon^2} - \frac{3}{\epsilon} - 8 + \mathcal{O}(\epsilon)\right] \tag{2.86}$$

which implies that

$$\sigma_{inclusive} = \sigma_{2\to 2} + \sigma^{\text{real } \gamma}_{2\to 3} = \sigma_0 \left(1 + \frac{\alpha}{2\pi}\left(\frac{19}{2} - 8\right) + \mathcal{O}(\alpha^2)\right) \tag{2.87}$$

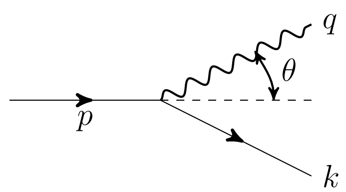
$$= \sigma_0 \left(1 + \frac{3}{4}\frac{\alpha}{\pi} + \mathcal{O}(\alpha^2)\right) \tag{2.88}$$

This is of order 0.3% as promised.

2.A Calculating the Splitting Probability

[Q 4: This section is under construction]

We now calculate the averaged splitting probability. The derivation for this result was omitted in Maxim's lectures but done in Peskin and Schroeder. We want to be consistent in our calculation and approximate the final result to order $p_\perp^2$. We are considering the process,



We want to consider a real electron which splits off into an on-shell photon and a final electron to be slightly off-shell. We use a small angle approximation,

$$\theta \approx \frac{p_\perp}{zp} \tag{2.89}$$

This gives

$$p_\mu = p(1, 0, 0, 1) \tag{2.90}$$

$$q_\mu = zp\left(1, \frac{p_\perp}{zp}, 0, 1 - \frac{p_\perp^2}{2z^2p^2}\right) \tag{2.91}$$

$$k_\mu = \left((1-z)p, -p_\perp, 0, (1-z)p + \frac{p_\perp^2}{2zp}\right) \tag{2.92}$$

The difficulty in this calculation is that since the electron is off-shell we can't use the usual spin-sum formula. Instead we construct the spinors explicitly. For the initial electron travelling in the z direction we have,

$$u^s(p) = \sqrt{2p}\begin{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \xi^s \\ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \xi^s \end{pmatrix} \tag{2.93}$$

$$= \left\{ \sqrt{2p}\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \sqrt{2p}\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\} \tag{2.94}$$

where ξ^s is the spinor and is $(1,0)^T$ for a right handed particle and $(0,1)^T$ for a left-handed one. To calculate the final electron spinor we take the spinor above and rotate it in the xz plane. This rotation is accomplished using,

$$e^{-\theta S^{1,3}} = \begin{pmatrix} \cos\frac{\theta}{2} - i\sigma^2 \sin\frac{\theta}{2} & 0 \\ 0 & \cos\frac{\theta}{2} - i\sigma^2 \sin\frac{\theta}{2} \end{pmatrix} \tag{2.95}$$

We now make a small angle approximation:

$$\sin\frac{\theta}{2} \approx \tan\frac{\theta}{2} \approx -\frac{p_\perp}{2p(1-z)}, \quad \cos\theta \approx 1 \tag{2.96}$$

Acting on the spinor above with energy $2p(1-z)$ gives,

$$u^s = \left\{ \sqrt{2p(1-z)}\begin{pmatrix} \frac{p_L}{2p(1-z)} \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \sqrt{2p(1-z)}\begin{pmatrix} 0 \\ 0 \\ 1 \\ \frac{p_\perp}{2p(1-z)} \end{pmatrix} \right\} \tag{2.97}$$

QED interactions conserve chirality and a left-handed electron only interacts with a left-handed electron. So we only have either a left-left interaction or right-right. We begin by studying the left-left interaction.The last ingredient we need is the polarization vector. The polarization vector must be perpendicular to the direction of motion of the photon. We must have

$$\boldsymbol{\epsilon} \cdot \mathbf{q} = 0 \tag{2.98}$$

(and $\epsilon_0 = 0$). Our condition implies

$$p_\perp \epsilon_1 + \frac{2z^2p^2 - p_\perp^2}{2zp}\epsilon_3 = 0 \tag{2.99}$$

$$\epsilon_3 = \frac{p_\perp^2 - 2z^2p^2}{2zpp_\perp}\epsilon_1 \tag{2.100}$$

and hence

$$\epsilon = \left(1, a, \frac{p_\perp^2 - 2z^2p^2}{2zpp_\perp}\right) \tag{2.101}$$

$$= \left(1, a, -\frac{zp}{p_\perp}\right) + \mathcal{O}(\frac{p_\perp^2}{z^2p^2}) \tag{2.102}$$

where a is arbitrary. A convenient choice turns out to be $a = i$. We have also yet to normalize the polarization vector such that $\boldsymbol{\epsilon}^2 = 1$. This gives the final result

$$\boldsymbol{\epsilon}_L^* = \frac{1}{\sqrt{2}}\left(1, i, -\frac{p_\perp}{zp}\right) + \mathcal{O}(\frac{p_\perp^2}{z^2p^2}) \quad , \quad \boldsymbol{\epsilon}_R^* = \frac{1}{\sqrt{2}}\left(1, -i, -\frac{p_\perp}{zp}\right) + \mathcal{O}(\frac{p_\perp^2}{z^2p^2}) \tag{2.103}$$

where we chose our convention of ϵ_L having a $-i$ to conform with normal conventions of circular polarization. Furthermore the right handed partner was found as the vector perpendicular to the left handed polarization vector.

We can now finally calculate the amplitude,

$$i\mathcal{M}_{LLL} = [\bar{u}(k)(-ie\gamma^\mu)u(p)]\,\epsilon_\mu^* \tag{2.104}$$

$$\mathcal{M}_{LLL} = -ig\sqrt{2p(1-z)}\begin{pmatrix} 0 & 0 & \frac{p_\perp}{2p(1-z)} & 1 \end{pmatrix}\begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}\sqrt{2p}\frac{1}{\sqrt{2}}\left(1, +i, -\frac{p_\perp}{zp}\right)^i \tag{2.105}$$

$$= i\sqrt{2}ep_\perp\sqrt{1-z}\left(\frac{1}{1-z} + \frac{1}{z}\right) \tag{2.106}$$

$$= iep_\perp\frac{\sqrt{2(1-z)}}{z(1-z)} \underbrace{=}_{\text{by Parity}} i\mathcal{M}_{RRR} \tag{2.107}$$

A similar calculation yields,

$$i\mathcal{M}_{LLR} = iep_\perp\frac{2(1-z)}{z} \overbrace{=}^{\text{by Parity}} i\mathcal{M}_{RRL} \tag{2.108}$$

Furthermore, we have,

$$i\mathcal{M}_{RLL} = \mathcal{M}_{RLR} = \mathcal{M}_{LRL} = \mathcal{M}_{LRR} = 0 \tag{2.109}$$

Summing over all spins:

$$\overline{|\mathcal{M}|^2} = \frac{1}{2}\sum_{pol}|\mathcal{M}| \tag{2.110}$$

$$= 2e^2p_\perp^2\left(\frac{1+(1-z)^2}{z^2(1-z)}\right) \tag{2.111}$$

2.B Phase Space

In this section we calculate the three body phase space. The calculation is based on a lovely note by Hitoshi Murayama[2].

2.B.1 Two Body

Here we quickly summarize some properties of two body phase space. In the center of mass frame, two body phase space can always be written in the form,

$$\int d\Phi_2 = \int \frac{d\cos\theta}{2}\frac{d\phi}{2\pi}\frac{\beta}{8\pi} \tag{2.112}$$

where

$$\beta \equiv \sqrt{1 - \frac{2(m_1^2 + m_2)}{s} + \frac{(m_1^2 - m_2^2)^2}{s^2}} \tag{2.113}$$

The energies of the outgoing particles are related to their masses and incoming energies by,

$$E_1 \equiv \frac{\sqrt{s}}{2}\left(1 + \frac{m_1^2}{s} - \frac{m_2^2}{s}\right) \tag{2.114}$$

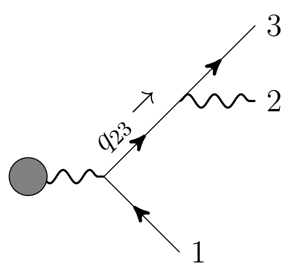
$$E_2 \equiv \frac{\sqrt{s}}{2}\left(1 + \frac{m_2^2}{s} - \frac{m_1^2}{s}\right) \tag{2.115}$$

Furthermore, their momenta are equal and given by

$$p_1 = p_2 = \frac{\sqrt{s}}{2}\bar{\beta} \tag{2.116}$$

2.B.2 Three Body

We can now move onto three body phase space. We begin by introducing some convenient notation. We have,



we depict the second particle as a vector boson, however the results are more general and apply to any 3 body phase space. We take the initial center of mass energy to be $\sqrt{s}$. Further we denote the virtual particle which goes on to produce particles 2 and 3 by q_{23}

and its mass as $s_{23} \equiv (p_2 + p_3)^2$. Recall that we can split up any three body phase space into a product of 2 body phase spaces,

$$\int d\Phi_3 = \int \frac{ds_{23}}{2\pi} d\Phi_2(p_1, q_{23}) d\Phi_2(p_2, p_3) \tag{2.117}$$

The phase spaces are Lorentz invariant quantities. Since the phase space takes a simple form in the zero momentum frame it is convenient to consider $d\Phi_2(q_1, q_{23})$ in the center of mass frame and $d\Phi_2(p_2, p_3)$ in the zero momentum frame for q_{23}.

$$\int d\Phi_3 = \int \frac{ds_{23}}{2\pi} \frac{d\cos\theta_1}{2} \frac{d\phi_1}{2\pi} \frac{\bar{\beta}_1}{8\pi} \frac{d\cos\hat{\theta}_{23}}{2} \frac{d\hat{\phi}_{23}}{2\pi} \frac{\bar{\beta}_{23}}{8\pi} \tag{2.118}$$

where we have

$$\bar{\beta}_1 \equiv \sqrt{1 - \frac{2(m_1^2 + s_{23})}{s} + \frac{(m_1^2 - s_{23})^2}{s^2}} \tag{2.119}$$

$$\bar{\beta}_{23} \equiv \sqrt{1 - \frac{2(m_2^2 + m_3^2)}{s_{23}} + \frac{(m_2^2 - m_3^2)^2}{s_{23}^2}} \tag{2.120}$$

Note that we don't need to worry about the momenta of the virtual particles explicitly yet since we took $d\Phi_2(p_2, p_3)$ to be the zero momentum frame of particle q_{23}.

We define the angle $\hat{\theta}_{23}$ to be the polar angle from $-\mathbf{p}_1$ and θ_1 to be the polar angle of 1 frame some arbitrary axis. Then rotations about p_1 leave the cross section invariant. Hence we can integrate over this angle,

$$\int d\Phi_3 = \int \frac{ds_{23}}{2\pi} \frac{d\cos\theta_1}{2} \frac{d\phi_1}{2\pi} \frac{\bar{\beta}_1}{8\pi} \frac{d\cos\hat{\theta}_{23}}{2} \frac{\bar{\beta}_{23}}{8\pi} \tag{2.121}$$

Now the angles of θ_1 and ϕ_1 move the angle of p_1 around and with it the entire system by the dependence of $\hat{\theta}_{23}$. Everything is usually set by $\hat{\theta}_{23}$ and invariant under rotations of the $\hat{\theta}_1$ and ϕ_1. This holds true if every axis picked out by $\hat{\theta}_1$ is equally valid. This is no longer the case of the particle spin is being measured. We will not make any assumptions about the isotropy until the very end of the calculation.

Our goal now is to try to rewrite the phase space in terms of convenient variables. One such set of variables are the energy fractions of the particles. The energy fractions obey the constraint,

$$x_1 + x_2 + x_3 = 2 \tag{2.122}$$

so they are not independent. We want to rewrite two of the variables above (which we choose to be s_{23} and $\cos\hat{\theta}_{23}$) in terms of x_1 and x_2. We need to find their Jacobian.

From the energies of two particles splittings we know that,

$$x_1 = 1 + \frac{m_1^2}{s} - \frac{s_{23}}{s} \tag{2.123}$$

Now we we just need to find x_2 in terms of s_{23} and $\cos\hat{\theta}_{23}$. To do this we need to relate the quantities in the center of mass frame to the rest frame of q_{23}. Since the final momenta are on a plane we set the y component to be zero, and q_{23} to be along the z axis,

$$q_{23} = \left(E_{23}, 0, 0, \frac{\sqrt{s}}{2}\bar{\beta}_1\right) \tag{2.124}$$

$$p_1 = \left(E_1, 0, 0, -\frac{\sqrt{s}}{2}\bar{\beta}_1\right) \tag{2.125}$$

where

$$E_{23} = \frac{\sqrt{s}}{2}\left(1 + \frac{s_{23}}{s} - \frac{m_1^2}{s}\right) \tag{2.126}$$

We have,

$$\gamma = \frac{E_{23}}{\sqrt{s_{23}}} \tag{2.127}$$

$$\gamma\beta = \frac{p_{23}}{\sqrt{s_{23}}} = \frac{1}{2}\sqrt{\frac{s}{s_{23}}}\bar{\beta}_1 \tag{2.128}$$

and

$$\hat{p}_2 = \left(\hat{E}_2, \hat{p}_2 \sin\hat{\theta}_{23}, 0, \hat{p}_2\cos\hat{\theta}_{23}\right) \tag{2.129}$$

$$\tag{2.130}$$

where

$$\hat{E}_2 \equiv \frac{\sqrt{s_{23}}}{2}\left(1 + \frac{m_2^2}{s_{23}} - \frac{m_3^2}{s_{23}}\right) \tag{2.131}$$

$$\hat{p}_2 \equiv \frac{\sqrt{s_{23}}}{2}\hat{\beta}_{23} \tag{2.132}$$

From here we can finally extract the energy fraction,

$$x_2 = \frac{2}{\sqrt{s}}\left(\gamma\hat{E}_2 + \gamma\beta\hat{p}_2\cos\hat{\theta}_{23}\right) \tag{2.133}$$

where $\gamma, \beta, \hat{E}_2, \hat{p}_2$ each contain s_{23} dependence. Explicitly we have,

$$x_2 = \frac{2}{\sqrt{s}}\left\{\frac{1}{2}\sqrt{\frac{s}{s_{23}}}\left(1 + \frac{s_{23}}{s} - \frac{m_1^2}{s}\right)\frac{\sqrt{s_{23}}}{2}\left(1 + \frac{m_2^2}{s_{23}} - \frac{m_3^2}{s_{23}}\right) + \right. \tag{2.134}$$

$$\frac{1}{2}\sqrt{\frac{s}{s_{23}}}\sqrt{1 - 2\left(\frac{m_1^2 + s_{23}}{s}\right) + \left(\frac{m_1^2 - s_{23}^2}{s}\right)^2} \quad \times \tag{2.135}$$

$$\left. \frac{\sqrt{s_{23}}}{2}\sqrt{1 - 2\left(\frac{m_2^2 + m_3^2}{s_{23}}\right) + \left(\frac{m_2^2 - m_3^2}{s_{23}}\right)^2}\cos\hat{\theta}_{23}\right\} \tag{2.136}$$

We are almost done. We just need to find the Jacobian that relates x_1, x_2 to $\cos\hat{\theta}_{23}$ and s_{23} as well as the ranges of x_1 and x_2. In fact we don't need to find the entire Jacobian. From equation 2.123 we already know the Jacobian has the form

$$\mathcal{J} = \begin{pmatrix} \frac{\partial s_{23}}{\partial x_1} & \frac{\partial s_{23}}{x_2} \\ \frac{\partial \cos\hat{\theta}_{23}}{\partial x_1} & \frac{\partial \cos\hat{\theta}_{23}}{\partial x_2} \end{pmatrix} = \begin{pmatrix} -s & 0 \\ \frac{\partial \cos\hat{\theta}_{23}}{\partial x_1} & \frac{\partial \cos\hat{\theta}_{23}}{\partial x_2} \end{pmatrix} \tag{2.137}$$

Thus to find the determinant we only need $\frac{\partial \cos\hat{\theta}_{23}}{\partial x_2}$.

Unfortunately, the s_{23} dependence of x_2 is a bit nasty. The calculation gets significantly simpler in the massles limit. There we have,

$$x_2 = \frac{2}{\sqrt{s}}\left(\frac{\sqrt{s}}{4}\left(1+\frac{s_{23}}{s}\right)+\frac{\sqrt{s}}{4}\sqrt{1-\frac{2s_{23}}{s}+\frac{s_{23}^2}{s^2}}\cos\hat{\theta}_{23}\right) \tag{2.138}$$

with $\frac{s_{23}}{s} = 1 - x_1$. Inserting in this relation,

$$x_2 = 1 - \frac{x_1}{2} + \frac{x_1}{2}\cos\hat{\theta}_{23} \tag{2.139}$$

Hence we have,

$$\frac{\partial \cos\hat{\theta}_{23}}{\partial x_2} = \frac{2}{x_1} \tag{2.140}$$

and the determinant of the Jacobian is,

$$|\det \mathcal{J}| = \frac{2s}{x_1} \tag{2.141}$$

Lastly, we have the range of integration.

$$1 - x_1 < x_2 < 1 \tag{2.142}$$

Finally we have,

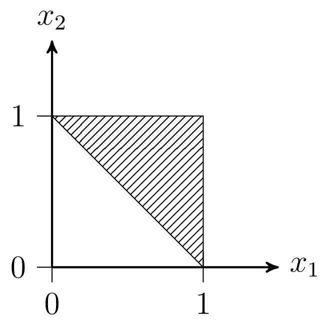
$$\int d\Phi_3 = \int_0^1 dx_1 \int_{1-x_1}^1 dx_2 x_1 \frac{2s}{x_1}\frac{1}{2\pi}\frac{1}{2}\frac{1}{(8\pi)^2}\int \frac{d\cos\theta_1}{2}\frac{d\phi_1}{2\pi} \tag{2.143}$$

$$= \frac{s}{128\pi^3}\int_0^1 dx_1 \int_{1-x}^1 dx_2 \int \frac{d\Omega}{4\pi} \tag{2.144}$$

We are left with an integral over a solid angle. We usually do not measure the spins of the final state particles. In this case the overall rotation of the final particles can be dropped. Finally we have,

$$\int d\Phi_3 = \frac{s}{128\pi^3}\int_0^1 dx_1 \int_{1-x}^1 dx_2 \tag{2.145}$$

So while naively, you may think that any x_2 within the range of 0 to 1 is valid phase space, this turns out not to be the case. However, there is no preference to certain regions of phase space since there is no dependence on x_1 or x_2 . The phase space is shown below,

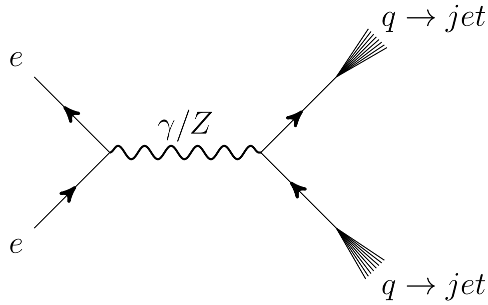


x_2
1
0
0
1
x_1

Chapter 3

QCD

3.1 $e^+e^- \to Hadrons$



The probability of hadronization is 1. If you assume that the scale of hadronization is much smaller then the scale of the parton process thenx,

$$\sigma_{LO}(e^+e^- \to hadrons) = \sigma_{LO}(e^+e^- \to q\bar{q}) \tag{3.1}$$

Hadronization doesn't "interfere" with the short-distance physics. This is actually another instance of factorization.

If $\sqrt{s} \ll M_Z$(just convenient to only study the photon channel but trivial to include the Z) then

$$\sigma_{LO}(e^+e^- \to q\bar{q}) = \underbrace{\frac{4\pi\alpha^2}{3s}}_{\sigma_0 \equiv \sigma_{LO}(e^+e^- \to \mu^+\mu^-)} \sum_q 3Q_q^2 \tag{3.2}$$

where the charge factor arises since the quark vertex contributes a fraction of either 2/3 or 1/3, there are three colors, and we sum over all the possible quarks that can form at the energy threshold, $\sqrt{s}$.

You can now define

$$R = \frac{\sigma(e^+e^- \to had)}{\sigma_0} = \sum_q 3Q_q^2 \tag{3.3}$$

Keep in mind that this formula is only valid away from threshold, since close to threshold there are corrections associated with many gluon emission effects. As an example in the limit that $M_b \ll \sqrt{s} \ll M_Z$ we have

$$R = 3\left(\left(\frac{1}{3}\right)^2 \cdot 3 + \left(\frac{2}{3}\right)^2 \cdot 2\right) = \frac{11}{3} = 3.67 \tag{3.4}$$

However at $\sqrt{s} \approx M_Z$ we have

$$R_Z^{LO} = \frac{3\sum_q \left({g_L^q}^2 + {g_R^q}^2\right)}{{g_L^\mu}^2 + {g_R^\mu}^2} = \frac{\Gamma\left(Z \to hadrons\right)}{\Gamma\left(Z \to \mu^+\mu^-\right)} \tag{3.5}$$

The leading order theory is $R_Z = 20.09$ and the experimental result is 20.79 ± 0.04. These are slightly inconsistent since we haven't included the next to leading order correction.

As before the cross section can be calculated. You need to include the NLO correlations with a real and virtual gluon. The results are the same as for $\mu^-\mu^+$, except for color factors (see equation 2.88),

$$\sigma(e^+e^- \to hadrons) = \sigma^{LO}(e^+e^- \to q\bar{q}) + \sigma^{NLO}(e^+e^- \to q\bar{q}g) + \mathcal{O}(\alpha_s^2) \tag{3.6}$$

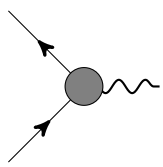
$$= \sigma^{LO}(e^+e^- \to q\bar{q})\left[1 + \frac{\alpha_s}{\pi} + \mathcal{O}(\alpha_s^2)\right] \tag{3.7}$$

where we lost the 3/4 factor due to color.

In perturbation theory,

$$\sigma_{had} = \overbrace{\sigma_{q\bar{q}}}^{\mathcal{O}(1)+\mathcal{O}(\alpha_s)} + \underbrace{\sigma_{q\bar{q}g}}_{\mathcal{O}(\alpha_s)+..} + \overbrace{\sigma_{q\bar{q}gg} + \sigma_{q\bar{q}q\bar{q}}}^{\mathcal{O}(\alpha_s^2)+...} \ldots \tag{3.8}$$

Now of course α_s is not a constant but instead it runs, which we can calculate through the RG equation. e.g. calculating



Such a calculation gives,

$$\frac{d\alpha_s(\mu^2)}{d\log\mu^2} = -b\alpha_s^2\left(1 + b_1\alpha_s + ...\right) \tag{3.9}$$

where

$$b = \frac{33 - 2n_f}{12\pi} \tag{3.10}$$

where n_f is the number of flavors. Note that α_s depends on the renormalization scale so it can't be a direct observable. For any observable O we must have,

$$\frac{dO}{d\log\mu^2} = 0 \tag{3.11}$$

This must be true for any possible choice of μ^2. Since α_s is dependent on μ^2, the equation should hold order-by-order in α_s. In general we get

$$R(s) = R_0(s)\cdot\left[1 + \frac{\alpha_s(\mu^2)}{\pi} + \sum_{n=2}^{\infty} C_n \cdot \left(\frac{\alpha_s(\mu^2)}{\pi}\right)^n\right] \tag{3.12}$$

The dependence of C_n can be deduced as follows. We have two scales in our process, s and our renormalization scale. We must have C_n to be dimensionless. This can only be true if $C_n = C_n\left(\frac{\mu^2}{s}\right)$. We have

$$\frac{dR}{d\log\mu^2} = R_0\left[\frac{1}{\pi}\frac{d\alpha_s}{d\log\mu^2} + \frac{dC_2(\mu^2/s)}{d\log\mu^2}\left(\frac{\alpha_s}{\pi}\right)^2\right] \tag{3.13}$$

By looking at equation 3.9 we see that the lowest order term of $d\alpha_s/d\log\mu^2$ is of order 2 and given by $-b\alpha_s^2$.

So we can write

$$-\frac{b}{\pi}\alpha_s^2 + \frac{dC_2}{d\log\mu^2}\left(\frac{\alpha_s}{\pi}\right)^2 = 0 \tag{3.14}$$

$$\Rightarrow \frac{dC_2}{d\log\mu^2} = \pi b \tag{3.15}$$

This doesn't tell us what C_2 is but just says how it depends on μ^2.

An explicit calculation gives,

$$C_2(\mu^2/s = 1) = 1.986 - 0.115 n_f \tag{3.16}$$

where n_f is the number of flavors in your theory.

Using the running equation (3.15) we find,

$$C_2(\mu^2/s) = C_2(1) - \pi b \log\frac{s}{\mu^2} \tag{3.17}$$

So the running of α_s in fact gives you the running of C_2 as well. The lesson is that even though observables do not depend on your choice of scale, since we do cutoff our perturbation theory at some order we do have some residual error. We have,

$$\frac{dR\big|_{n-loop}}{d\log\mu^2} = \mathcal{O}(\alpha_s^{n+1}) \tag{3.18}$$

The best choice of scale is unknown. One can test examples where two or more terms in perturbation theory are known. One can compore for example NLO answer at various μ^2 to the known NNLO answer which is $\sim \mu^2$ independent. Typically choose

$$\mu = \sqrt{s} \tag{3.19}$$

and use

$$\frac{1}{2}\sqrt{s} \lesssim \mu \lesssim 2\sqrt{s} \tag{3.20}$$

for systematic error.

3.2 Hadronization

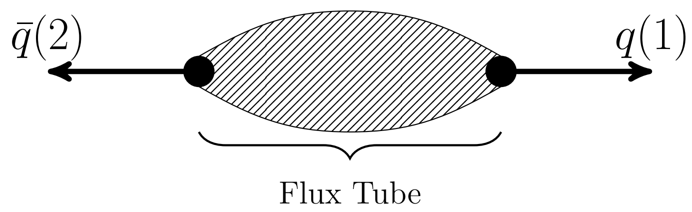
It would be nice to ask some questions beyond just the inclusive cross section. For example,

- What type of hadrons are there?
- What are their energies?

It turns out that the first question is inherently non-perturbative. We can make models but there is no real field theory computation that will give you this. However, the second one can be solved.

Since we cannot know exactly what happens in hadronization, we consider some models. The model we use is known as the "color flux tube model". This model is more then just a handwaving explaination and is actually the engine behind Pythia.

Consider two quarks moving away from each other. They constantly exchange gluons both pertubatively and non-pertabatively. This is important because if they never talked to each other then each of the quarks would stay in their initial colored states of the fundamental and antifundamental representations. Instead, a flux tube builds,



The flux tube has some tension,

$$K \sim \frac{\text{GeV}}{\text{fm}} \tag{3.21}$$

and the tube kind of jiggles around a little bit. You can think of a regular tube that you are stretching. This jiggling gives a transverse momenta given by,

$$p_\perp \sim \Lambda_{QCD} \tag{3.22}$$

An important quantity for this model is the invariant mass of the quarks,

$$M_{inv}(q,\bar{q}) = (p_1 + p_2)^2 = s \tag{3.23}$$

This tube isn't perfectly strong. At one point there will be a breakdown, where the tube splits,

The two new quarks are roughly at rest. The invariant mass of the new meson is

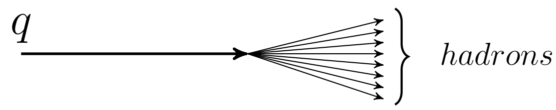
$$M_{inv}^2(q_1\bar{q}) = (p_1 + p_{\bar{q}})^2 = 2\sqrt{s}p_\perp \sim \sqrt{s}\Lambda_{QCD} = s\left(\frac{\Lambda_{QCD}}{\sqrt{s}}\right) \tag{3.24}$$

More breakdown gives $M_{inv} \approx 1$ GeV. The probability of which type of meson forms are input parameters.

The general feature is

$$p_\perp^{had} \sim \Lambda_{QCD} \ll \sqrt{s} \sim p_\parallel^{had} \tag{3.25}$$

So what you end with, regardless of the details, is a collimated beam of hadrons. This collimated set of hadrons is known as jets. This hierarchy between the parallel and perpendicular momenta is model independent. We have,



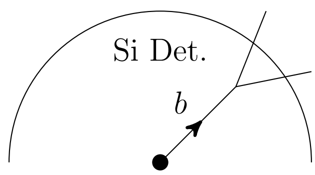
with $\theta \sim \Lambda_{QCD}/\sqrt{s}$.

At high $\sqrt{s}$, can discuss energy/angular distributions of jets but not their composition.

The basic rules of jets are as follows.

1. Jets from $u, \bar{u}, d, \bar{d}, s, \bar{s}$ are essentially indistinguishable[1].
2. Gluon jets look similar to q, but can be distinguished to some extent.
3. Jets from $b/\bar{b}$ and $c/\bar{c}$ can be "tagged".

There are two ways to tag jets. The first way is using the vertex detector.



[1]This isn't completely correct since we can somewhat measure the initial charge.

The decay lengths (forgetting time dialation) for the b and c charms are,

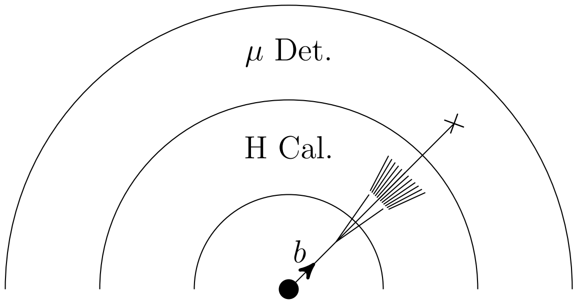
$$c\tau_b \approx 500\mu m \tag{3.26}$$
$$c\tau_c \approx 300\mu m \tag{3.27}$$

The second way to tag jets is to use semileptonic decays. The b quark decays with branching ratio of about 20% by

$$b \to c\ell\nu \tag{3.28}$$

The c quark forms a jet and if the lepton is a muon it can be detected using the muon detector.

A sample process is shown below,

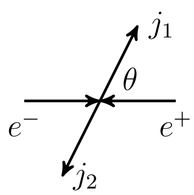


It's important to note that the muon will be inline with the jet since they both come from the same particle. This signiture is quite unique. For any other process, the hadronic jet and the muon are not very unlikley to be inline with each other. This allows another way to tag the b quark. A similar proceduer can be done to tag the c quark. In reality this tagging technique can only tell you that either a c or a b quark was there, it cannot differentiate between the two.

For the LHC the total tagging efficiency is $\sim 50\%$.

3.3 $e^+e^- \to$ Jets

We now begin to talk about jets order by order in perturbation theory. Consider a $2 \to 2$ process,

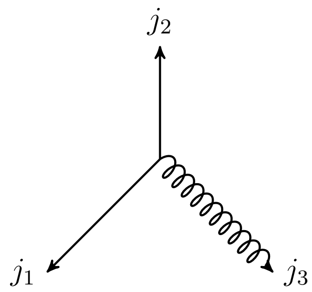


This process is given by

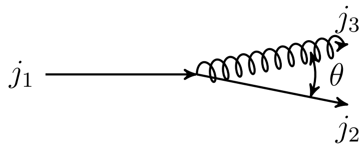
$$\frac{d\sigma(e^+e^- \to 2j)}{d\cos\theta} \propto 1 + \cos^2\theta + (Z\text{-exchange}) \tag{3.29}$$

At NLO, things get interesting. There are basically two kinds of events,

1. Non-singular regions: Every particle hadronizes independently and we have 3 jets.



2. Singular regions: Some particles are collimated ($\theta \to 0$)



In this case two jets can merge into one and you have to do a 2 jet correction.

We now attempt to deal with the singular events. Say $p_\perp^g > Q$ to call an event a "3-jet" event. Then,

$$\sigma_3 = \frac{\alpha_s}{\pi}\left(C_1 \log^2 \frac{s}{Q^2} + C_2 \log \frac{s}{Q^2} + C_3\right) + \mathcal{O}\left(\alpha_s^2, \frac{\Lambda_{QCD}}{Q}\right) \tag{3.30}$$

$$\sigma_2 = \sigma_2^{LO} + \sigma_2^{LO}\frac{\alpha_s}{\pi}\left(-C_1 \log^2 \frac{s}{Q^2} - C_2 \log \frac{s}{Q^2} + C_4\right) \tag{3.31}$$

where the form of σ_2 can be inferred from σ_3 by the condition since we already know the inclusive rate is (we mentioned this earlier),

$$\sigma_2 + \sigma_3 = \sigma_2^{LO}\left(1 + \frac{\alpha_s}{\pi}\right) \tag{3.32}$$

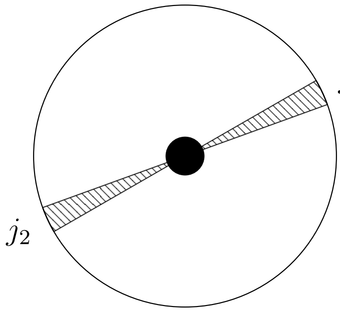
Individually, σ_2 and σ_3 depend sensitivity on the exact criteria used to separate 2-jet and 3-jet events. One should be careful, there are "theory jets" defined in terms of partons and "experimental-jets" defined in terms of hadrons. We're now talking about "theoretical jet algorithms".

3.4 Jet Algorithms

Clealy defining jets is not a trivial task. However, to make the most out of the data at colliders such as the LHC, we need to find algorithms to cluster partilces into jets. Several such algorithsm exist.

3.4.1 Sterman-Weinberg Jets

At the time this algorithm was defined people only conceived of 2-3 jets so this method is not really useful for modern colliders but it is nevertheless instructive.



An event is classified as a 2 jet event if at least a fraction of $1-\epsilon$ of the event's total energy is contained in the two cones of opening half-angle δ. Algebraically,

$$E(\text{within both cones}) \geq (1-\epsilon)E(\text{total}) \tag{3.33}$$

The idea is if you have a real three jet event, then a lot of energy will go into the third jet. So the energy within the first two cones will be larger then the fraction of the total energy.

The parameters here are δ and ϵ and can be anything but typically one should avoid using the extreme values of δ too close to 0 or ϵ too close to 0 or 1.

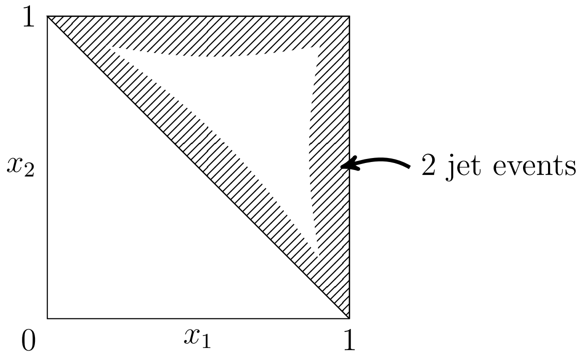
We can make a prediction for how many two jet events we will have. Consider an event with two quarks and a gluon. We denote the fraction of the momentum carried by the quark and antiquark by x_1 and x_2 respectively and the momentum carried by the gluon by x_3. Momentum conservation says that

$$x_3 = 2 - x_1 - x_2 < 1 \tag{3.34}$$

so

$$x_1 + x_2 > 1 \tag{3.35}$$

We have the top right triangle phase space we saw for three body phase space in x_1 and x_2. If x_1 or x_2 is close to 1 we have a singularity and a much larger probability to produce a collinear gluon. In this case the Sterman Weinberg algorithm will see a two jet event since the gluon jet will be part of the quark jet. We have, [Q 5: insert simulated phase space data]



We cannot have events in the bottom left due to the result above. Recall that the cross section is singular when $x_1, x_2 \to 1$. In these regions we would interpret the events to be two jet events.

The filled region is where the 2 jet events will lie. One can calculate the fraction of events that will have 2 jets. We must integrate over the jet energies that obey equation 3.33. It is given by

$$f_2 \equiv \frac{\sigma_{2-jet}}{\sigma_{total}} = 1 - 8C_F \frac{\alpha_s}{2\pi} \left\{ \log\frac{1}{8}\left(\log\left(\frac{1}{2\epsilon} - 1\right) - \frac{3}{4} + 3\epsilon\right) + \right. \tag{3.36}$$

$$\left. \frac{\pi^2}{12} - \frac{7}{16} - \epsilon + \frac{3}{2}\epsilon^2 + \mathcal{O}(\delta^2 \log\epsilon) \right\} \tag{3.37}$$

The particular result is not especially important but we recognize that at leading order all events are 2 jets.

3.4.2 JADE Algorithm

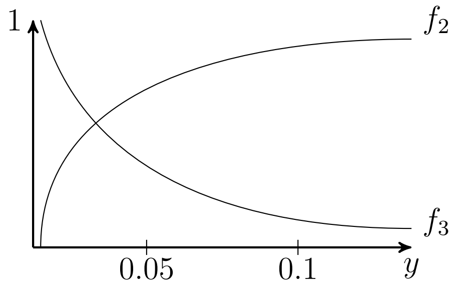
The JADE algorithm defines a 3-jet event as

$$\min\left(p_i + p_j\right)^2 \approx \min\left\{2E_iE_j\left(1 - \cos\theta_{ij}\right)\right\} > y \cdot s \tag{3.38}$$

with $i, j = q, \bar{q}, g$. y is some dimensionless number quantifying how small this invariant mass should be before you call it a 2 jet event(recall that the invariant mass is zero if two particles are collinear). If none of the particles are collinear then the minimum invariant mass will be large, satisfying the JADE algorithm. The smaller the value of y the larger your 3-jet rate is. As before one can calculate the fraction of 3-jet events,

$$f_3 = C_F \frac{\alpha_s}{2\pi}\left[2\log^2\frac{y}{1-y} + (3-6y)\log\frac{y}{1-2y} + \frac{5}{2} - 6y - \frac{9}{2}y^2 + 4Li_2\frac{y}{1-y} - \frac{\pi^2}{3}\right] \tag{3.39}$$

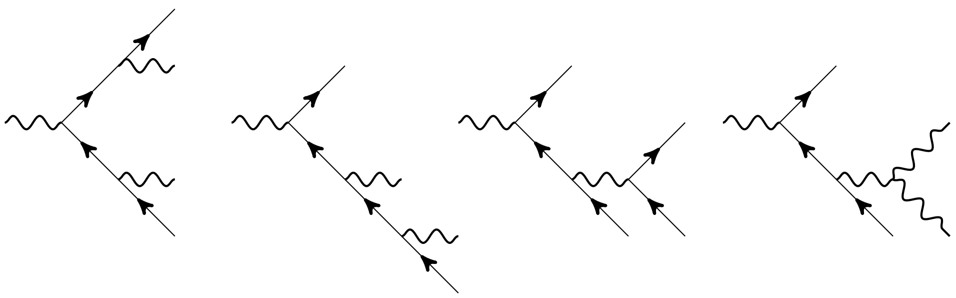
The key point is what you call a 2-jet event vs 3-jet event is a function of the algorithm as well as the parameter that describes your algorithm, in this case y. In principle you can make y very small, but for very small y, $f_3 \geq f_2$ and perturbation theory breaks down ($\alpha_s \log^2 y \sim \mathcal{O}(1)$). These are not QCD innate non-perturbative divergences but can be resummed and eliminated.



For multi-jets: Start with n partons; if

$$\min(p_i + p_j)^2 < ys \tag{3.40}$$

cominbe i and j into a jet with $p_{ij} = p_i + p_j$ and iterate. The interesting part about the 4-jet rate is that it is the first order in which the 3 gluon vertex arises. Sample "right half" of 4-jet diagrams are,



This was the real direct evidence of the triplet-gluon vertex. This algorithm still has a few small issues since there are some combinations of partons which are designated as jets when they really shouldn't be.

An important variation of the JADE algorithm is known as the k_T algorithm. Here you compute (c.f. 3.38),

$$2\min\left(E_i^2, E_j^2\right)(1 + \cos\theta_{ij}) > ys \tag{3.41}$$

For $\theta_{ij} \ll 1$ we have $1 - \cos\theta_{ij} \approx \frac{1}{2}\theta_{ij}^2$ we have

$$\min\left(E_i^2, E_j^2\right)\theta_{ij}^2 = \min(p_{i,\perp}^2, p_{j,\perp}^2) \tag{3.42}$$

where $\min(p_{i,\perp}^2, p_{j,\perp}^2)$ is the transverse momentum of i relative to j (if i is the softer particle). So what the algorithm does is group partons as different jets if they have have large $p_\perp$ from one another.

There is something interesting you do to measure α_s from f_3. To second order in α_s

$$f_3 = f(y)f(\frac{\alpha_s(\mu^2)}{\pi}) + \left(\frac{\alpha_s(\mu^2)}{\pi}\right)^2 \cdot G(\frac{s}{\mu^2}, y) \tag{3.43}$$

where G is some unknown second order correction to f_3.

But observables must be independent of renormalization scale,

$$\frac{df_3}{d\log\mu^2} = 0 \tag{3.44}$$

This lets you find how G changes with renormalization scale.

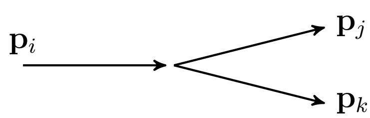
If $\mu^2 \uparrow$ then of course $\alpha_s \downarrow$. This means that $G(\frac{\mu^2}{s}) \uparrow$ which implies that G is a monotonically decreasing function of s. Looking at expression 3.43 we see that the first order correction has no dependence on the energy. Thus we also know that the fraction of 3 jet events will also go down as a function of s,

3.5 Event Shape Variables

We have some collection of hadron and measure their momenta $\mathbf{p}_i$. We don't care how many at this point. We want to find some function that we can measure experimentally and calculate theoretically. Define

$$X \equiv f(\mathbf{p}_i) \tag{3.45}$$

We want "IR safe" observables. These satisfy the following condition. Suppose we have $\mathbf{p}_i \to \mathbf{p}_j + \mathbf{p}_k$,



with $\mathbf{p}_k = A\mathbf{p}_j$ (collinear) or $|\mathbf{p}_k| \ll |\mathbf{p}_j|$ (soft) then an IR safe observable obeys,

$$X \to X + \mathcal{O}\left(|\mathbf{p}_k|^2\right) \tag{3.46}$$

In other words an IR safe observable isn't strongly effected by collinear or soft particles.

Its not hard to construct IR safe observables. This can be done by having linear combinations of momenta. In that case small momenta do not effect the distribution and collinear splittings drop out. One such variable is called the **thrust**,

$$T \equiv \max_{\hat{n}} \frac{\sum_i |\mathbf{p}_i \cdot \hat{n}|}{\sum_i |\mathbf{p}_i|} \tag{3.47}$$

where $\hat{n}$ is some unit vector. The maximum ranges over all possible unit vectors.

To see that thrust is infrared safe, consider a particle emits a soft particle. This soft particle would not show up in the computation of thrust since its momenta is small. Now suppose that a particle momenta splits such that $\mathbf{p} \to (1-\lambda)\mathbf{p}, \lambda\mathbf{p}$. In this case

$$|\mathbf{p}\cdot\hat{n}| \to (1-\lambda)\,|\mathbf{p}\cdot\hat{n}| + \lambda\,|\mathbf{p}\cdot\hat{n}| = |\mathbf{p}\cdot\hat{n}| \tag{3.48}$$

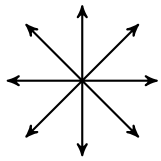
and the same occurs in the denomenator. Thus emitted a collinear or soft particle has no effect on the thrust making it IR safe.

Suppose you have two hadrons going back to back. Then you have,

$$\mathbf{p}_1 \longleftarrow \quad \longrightarrow \mathbf{p}_2$$

which gives $T = 1$ (and the maximizing unit vector is along the direction of motion of one of the particles).

The opposite case is a spherical distribution,



in this case any direction is equally as good at maximizing the sum and we have

$$\frac{\sum_i |\mathbf{p}_i \cdot \hat{n}|}{\sum_i |\mathbf{p}_i|} = \frac{\int d^3p \, |\mathbf{p} \cos\theta|}{\int d^3p \, |\mathbf{p}|} \frac{\sum_i |\cos\theta_i|}{\sum_i 1} \tag{3.49}$$

$$= \frac{\int_0^\pi d\cos\theta \int_0^{2\pi} d\phi \, |\cos\theta|}{4\pi} \tag{3.50}$$

$$= \frac{\int_0^{\pi/2} d\cos\theta \int_0^{2\pi} d\phi \cos\theta}{2\pi} \tag{3.51}$$

$$= \frac{1}{2} \tag{3.52}$$

Hence the thurst tells you how spread out your distribution is with

$$\frac{1}{2} \geq T \geq 1 \tag{3.53}$$

Theory predicts to leading order (where we only have 2 jets) that $T = 1$ since the hadrons will just come out back to back. To next to leading order we have,

$$\frac{d\sigma}{dT} = \int\int dx_1 dx_2 \frac{d\sigma_{q\bar{q}g}}{dx_1 dx_2} \delta\left(T - f\left\{x_i\right\}\right) \tag{3.54}$$

where $f\{x_i\}$ is some function of the energies for such a distribution. The reasoning for this form is as follows. The $\frac{d\sigma_{q\bar{q}g}}{dx_1 dx_2}\delta\left(T - f\left\{x_i\right\}\right)$ term is the contribution to the cross-section with a given thrust, and we integrate over all possible x_i's. One can show that for 3 jets the thrust is, [Q 6: check]

$$f\left\{x_i\right\} = \max\ \left\{x_i\right\} \tag{3.55}$$

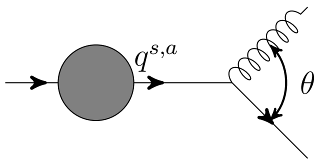
3.6 Parton Branching and Showering

We already saw that $3 - jet/2 - jet$ ratio is enhanced compared to the naive expection $\mathcal{O}(1) \cdot \alpha_s$ by soft/collinear logs. For example in the JADE algorithm,

$$\frac{f_3}{f_2} \sim \frac{\alpha_s}{\pi} \log^2 y \tag{3.56}$$

for small y. One approach is to just keep y large enough so that this is not a problem and we consistently have significantly more 2-jet events then 3-jet events. But this means throwing away a lot of information that's available in the data and potentially useful. Another approach is to try to predict leading-log terms to all orders in perturbation theory. This is possible due to factorization.

Calculating the full cross-section for different events can be very difficult for many jets. This is possible for three jets but not going to be possible for more compilcated cases. However, we can still look at the more singular regions and get approximate answers. We look at "Branching":



We focus on collinear factorization, soft factorization can also be done but a bit more complicated. Thus we work in the $\theta \to 0$ limit.

$$\mathcal{M}_{2\to 3} = \sum_{s,a} \mathcal{M}^{s,a}_{2\to 2} \frac{1}{t} \mathcal{M}_{q^{s,a}\to qg} \tag{3.57}$$

where the invariant mass of the quark prior to branching is

$$t \equiv (p_q + p_g)^2 = 2E_q E_g (1-\cos\theta) \approx E_q E_g \theta^2 \tag{3.58}$$

We have $t \ll$ the initial invariant mass of the collision.

s is the spin of the virtual quark and a is its color. We have our energy fraction definitions,

$$x_q \equiv \frac{2E_q}{\sqrt{t}} \quad , \quad x_g = 1 - x_q \tag{3.59}$$

Now we want to square the ampltidue,

$$\begin{aligned}\overline{\left|\mathcal{M}_{2\to 3}\right|^2} = \frac{1}{t^2} \sum_{s_1,s_2,j,k} &\left(\sum_{s,a} \mathcal{M}^{s,a}_{2\to 2} \mathcal{M}\left(q^{s,a} \to q^{s_1,j} g^{s_2,k}\right)\right) \times \\ &\qquad \left(\sum_{s',b} \mathcal{M}^{s',b}_{2\to 2} \mathcal{M}\left(q^{s',b} \to q^{s_1,j} g^{s_2,k}\right)\right)^* \end{aligned} \tag{3.60}$$

In QED we argued that when a soft photon is emitted the polarization index does not change, $s_1 = s = s'$. This is still true here since QCD preserved parity. Now how do we study the color structure? The color structure is just assocated with the color contribution at the splitting amplitude vertex, which contributes a $\left(T^k\right)^a_{\ j}$ and the complex conjugate which contributes a $\left(T^k\right)^b_{\ j}$,

$$\sum_k \left(T^k\right)^a_{\ j}\left(T^k\right)^b_{\ j} = \delta^a_{\ j}\delta^b_{\ j} - \frac{1}{N}\delta^a_{\ j}\delta^b_{\ j} \tag{3.61}$$

$$= \left(1-\frac{1}{N}\right)\delta^a_{\ j}\delta^b_{\ j} \tag{3.62}$$

$$\xrightarrow{\sum_j} \left(1-\frac{1}{N}\right)\delta^a_{\ b} \tag{3.63}$$

[Q 7: How did we get this relation? $(T^k)^a_{\ j}(T^k)_j^{\ b} = C_2(r)\delta^{ab}$ and I believe $C_2(r) = 3.$‘]

Thus we have

$$\overline{|\mathcal{M}_{2\to3}|^2} = \frac{1}{t^2}\sum_{s,a}|\mathcal{M}^{s,a}_{2\to2}|^2 \sum_{s,s_2,j,k}\left|\mathcal{M}(q^{s,i}\to q^{s,j}g^{s_2,k})\right|^2 \tag{3.64}$$

The splitting contribution is

$$\sum_{s,s_2,j,k}\left|\mathcal{M}(q^{s,i}\to q^{s,j}g^{s_2,k}\right|^2 = t\cdot 4g^2\cdot\mathcal{C}\cdot F^{qg}_s(x_q) \tag{3.65}$$

which gives

$$d\sigma_3 \approx \sum_s d\sigma^s_2\frac{dt}{t}dx_q\frac{d\phi}{2\pi}\frac{\alpha_s}{2\pi}\mathcal{C}F^{qg}_s(x_q) \tag{3.66}$$

Here we have three integration variables. The energy of the incoming particle, t, the energy taken by the outgoing particle, and the angular distribution.

Then you can define the splitting function as

$$P^s_{qq}(x_q) \equiv \int\frac{d\phi}{2\pi}\mathcal{C}F^{qg}_s(x_q) \tag{3.67}$$

For unpolarized production

$$\sum_s d\sigma^s_2 P^s_{qq}(x_q) = d\sigma_2 P_{qq}(x_q) \tag{3.68}$$

where

$$P_{qq} \equiv \frac{1}{N_{pol}}\sum_s P^s_{qq} \tag{3.69}$$

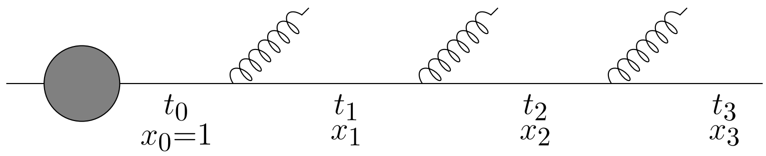
One can show that

$$\hat{P}_{qq}(x) = \frac{4}{3}\frac{1+x^2}{1-x} \tag{3.70}$$

Up to a color factor, this the same splitting function for an electron to split into a photon. We have a probability that the quark will emit a gluon at some particle fraction of the energy. The three particle cross-section is then given by,

$$d\sigma_3 = \frac{dt}{t}\cdot dx_q\cdot\frac{\alpha_s}{2\pi}d\sigma_2 P_{qq}(x_q) \tag{3.71}$$

The important point is that we have 2 singularities, one for small t and another for $x \to 1$. So when you try to integrate these you get logs which can be large. So we need to include multiple splittings. However, this is easy to sum since you can just apply this process again and again.



where remember here t_i denotes the invariant mass going into splitting i and x_i is the quarks energy fraction after splitting. x_i and t_i determine the kinematics completely. The process in a collider is as follows. First we produce particles, then the particles shower. Once the energy of the new particles is below a certain threshold we have hadronization.

This factorization is possible because the time scale to emit an extra gluon is much smaller then the time of the process itself [Q 8: show].

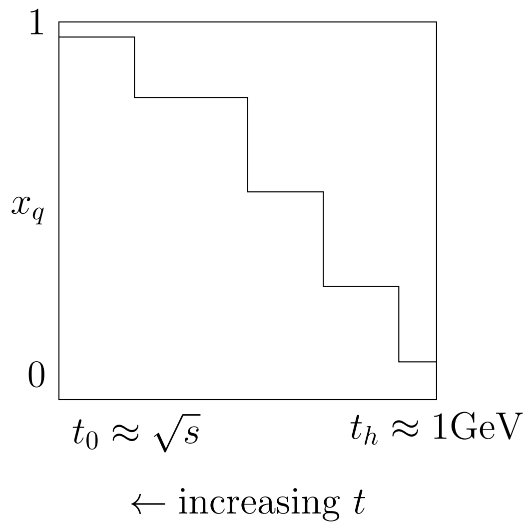
In general you can have many gluon emissions. We don't want to sum the series exactly, we just want to calculation the largest term. By energy conservation you must have,

$$x_0 > x_1 > x_2 > ... > x_N \tag{3.72}$$

We focus on the $\log^N$ enhanced region. Remember that our whole calculation here assumed that the split quark was assumed to be a little bit off-shell. If you take this a little more off-shell and give it a small mass, then the more it increases the invariant mass of the initial quark, t. If you take it too much off-shell then you no longer have a singular region. [Q 9: Understand this argument better and sharpen it.] So we have,

$$t_0 \gg t_1 \gg ... \gg t_N \approx 1\text{GeV} \tag{3.73}$$

We have some sort of "parton evolution" process. We can represent a "path" in $t - x$ space. While t is not time, it is a variable that's monotonically changing. We can draw a plot of the evolution,



where t_h is the hadronization invariant mass. If you know this path then you know that energies of all your gluons and quarks emitted from the shower. Monte Carlo generators compute the Probability of each particles path,

$$P\left(\{t_i, x_i\}\right) \tag{3.74}$$

Now there is a subtely with the value the initial invariant mass. The factorization method discussed above only works if $t \ll \sqrt{s}$. However, in practice what people do is fix the initial t such that $t_0 = \sqrt{s}$. However, it turns out that this doesn't matter since the formula that we will see in a minute will gaurentee that we will see no splitting until we get to lower t values. The probability of splitting is essentially zero until $t \ll \sqrt{s}$. We will see this effect soon.

We now attempt to estimate this. Suppose you start at t_0. We define the probability of no branching until t_1 by

$$p(t_1) \tag{3.75}$$

Suppose now we move from $t \to t + \Delta t$ with $\Delta t < 0$. Then

$$dp(t_1) = p(t_1)\frac{\Delta t}{t}\int dx \frac{\alpha_s}{2\pi} P_{qq}(x_q) \tag{3.76}$$

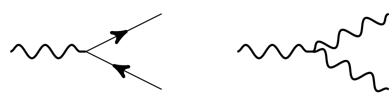
This gives a very simple differential equation for $p(t_1)$. The solution to this equation is,

$$\Delta(t_1) \equiv \exp\left[+\int_{t_h}^{t_1}\frac{dt}{t}\int dx \frac{\alpha_s}{2\pi} P_{qq}(x)\right] \tag{3.77}$$

with $dt < 0$ and $P(t_1) = \Delta(t_0)/\Delta(t_1)$. This is known as the Sudakov form factor.

There are some complication associated with this showering algorithm.

1. There are other breakings. We only included $q \to qg$ but we can also have,



   which adds $P_{qg}, P_{\bar{q}g}$, and P_{gq}.

   Monte Carlo must include relative possibilities of various splittings,

$$\Delta_i(t_1) = \exp\left[+\int_{t_0}^{t_1}\frac{dt}{t}\int dx \frac{\alpha_s}{2\pi}\sum_j P_{ji}(x)\right] \tag{3.78}$$

2. Soft singularities since

$$\int^1 dx P(x) \tag{3.79}$$

is actually divergent! In the context of QCD is it actually clear what is happening. We have,

$$t = x(1-x)\frac{1-\cos\theta}{2} \tag{3.80}$$

so even if you keep θ fixed, taking $x \to 1$ takes $t \to 1$. Emitting a very soft gluon the quark that emits this gluon has to be off-shell by a very small amount. A very soft gluon can only be a emitted by a very slightly offshell quark. But once a gluon gets very soft it is unobservable due to hadronization. We have a natural cut-off for our integral,

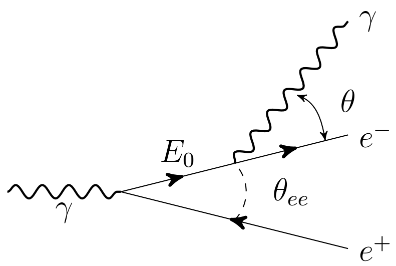
$$t > t_{min} \tag{3.81}$$

or

$$x(1-x) > \frac{t_{min}}{t} \tag{3.82}$$

This is still not the end of the story since in our discussion we assumed that the emitted gluon is just collinear. However, there is also a soft singularity. These soft singularities need to be handeled seperately. We will not discuss this complication further.

3. Coherence effects. Consider first QED. We can have,



with an invariant mass of the emitting photon and electron of,

$$t \approx E_\gamma E_p E_\gamma \theta^2 \approx z E_0^2 \theta^2 \tag{3.83}$$

in the limit that $z \to 0$ and $\theta \to 0$.

It turns out when $\theta < \theta_{ee}$ this cross section is heaviliy suppressed. This is called the Chudakov effect. We now show this in space-times terms. First how long does it take for the electron to emit a photon? We can find this using the propagator,

$$\frac{1}{t} \approx \frac{1}{zE_0^2\theta^2} \approx \frac{1}{p_0^2} = \frac{1}{E_0^2 - \mathbf{p}_0^2} \approx \frac{1}{2E_0\left(E_0 - |\mathbf{p}_0|\right)} \tag{3.84}$$

where we used the fact that $E_0 \approx |\mathbf{p}_0|$. Fourier transforming this propagator gives the time between emission,

$$\Delta t \sim \frac{1}{E_0 - |\mathbf{p}_0|} \approx \frac{1}{zE_0\theta^2} \tag{3.85}$$

We can estimate the physical separation of e^+e^- at the moment of the photon emission,

$$\Delta b = \Delta t \cdot \theta_{ee} = \frac{\theta_{ee}}{zE_0\theta^2} = \frac{\theta_{ee}}{E_\gamma\theta^2} \tag{3.86}$$

If

$$p_\perp^\gamma < \frac{1}{\Delta b} \tag{3.87}$$

then the photon is radiated off e^+ and e^- add coherently. The probability of emission is proportional to net charge which is zero. Suppressed at,

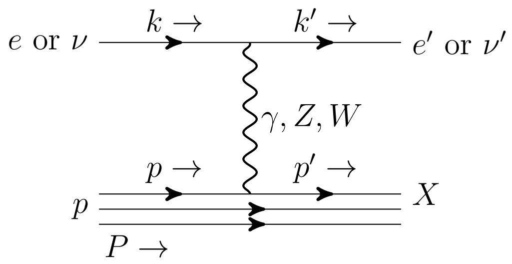
$$E_\gamma\theta < \frac{E_\gamma\theta^2}{\theta_{ee}} \tag{3.88}$$

which implies that $\theta > \theta_{ee}$.

In QCD we have many splittings. In every splitting we tend to have subsequent splittings with small angles. We force "angular ordering" and the emission angle in each subsequent splitting must be less than in the previous splitting.

Chapter 4

Deep Inelastic Scattering



There are two possible types of momenta transfers: neutral and charged. For a neutral current interaction we have one possible interaction: $e^- p \rightarrow e^- X$. The charged current has two possibilities: $e^- p \rightarrow \nu X$, $\nu p \rightarrow e^- X$, where X can be "anything hadronic".

The momentum transfer is $q \equiv k - k'$. Note that

$$q^2 = -2k \cdot k' = -2E_k E_{k'}(1 - \cos\theta) < 0 \tag{4.1}$$

Since q^2 is negative it is convenient to define $Q^2 = -q^2 > 0$. We work in the regime that

$$Q^2 \gg \Lambda_{QCD} \tag{4.2}$$

this is called the "parton picture" and the interaction occurs with one parton in the proton and the rest are spectators. In the cm frame

$$\sqrt{s} = \sqrt{(k+P)^2} \gg M_{proton} \tag{4.3}$$

this implies that we can write,

$$P \approx E_p(1, 0, 0, 1) \tag{4.4}$$

the partons have

$$p_z \approx E_p \tag{4.5}$$

and $p_\perp \approx \Lambda_{QCD}$. We ignore this value. The momenta of the quark is then given by

$$p = \xi P \tag{4.6}$$

where $\xi \in (0,1)$.

The probability to find parton of flavor f and momentum fraction ξ is given by

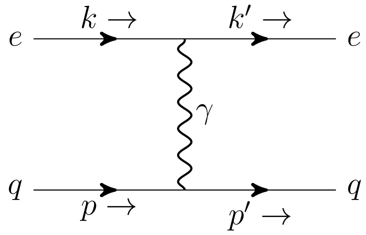
$$f_f(\xi)d\xi \tag{4.7}$$

f_f are called the parton distribution functions (pdf).

In this naive parton model we have,

$$\sigma\left(e^-(k)+p(P)\to e^-(k')+X\right)=\int_0^1 d\xi\sum_f f_f(\xi)\sigma\left(e^-(k)+q_f(\xi P)\to e^-(k')+X\right) \tag{4.8}$$

If we consider the parton level process,



Recall that[1],

$$\overline{|\mathcal{M}|^2}=3q_f^2e^2\left(1+\cos^2\theta\right)=3q_f^2e^2\frac{2(\hat{t}^2+\hat{u}^2)}{\hat{s}^2} \tag{4.9}$$

where we have used $\hat{t}=-\frac{1}{2}\hat{s}(1-\cos\theta), \hat{u}=-\frac{1}{2}\hat{s}\left(1+\cos\theta\right)$. We can use crossing symmetry to find the s channel diagram:

$$\overline{|\mathcal{M}_{s-channel}|^2}=\frac{1}{3}3q_f^2e^2\frac{2(\hat{s}^2+\hat{u}^2)}{\hat{t}^2} \tag{4.10}$$

where, we had to be a little careful with the color factors. We can now use $dt=\frac{1}{2}sd\cos\theta$ to rewrite these expressions:

$$\frac{d\sigma}{d\hat{t}}=\frac{2\pi\alpha^2Q_f^2}{\hat{s}^2}\frac{\hat{s}^2+\hat{u}^2}{\hat{t}^2} \tag{4.11}$$

where we define $Q_f\equiv q_f/e$. The Mandelstam variables are just

$$\hat{t}=q^2=-Q^2 \tag{4.12}$$

$$\hat{s}=(k+\xi P)^2=\xi 2k\cdot P=\xi s \tag{4.13}$$

$$\hat{u}=-\hat{s}-\hat{t}=Q^2-\xi s \tag{4.14}$$

since $\hat{t}=-\frac{1}{2}\hat{s}(1-\cos\theta)$ it implies that

$$Q^2=\frac{1}{2}\xi s\left(1-\cos\theta\right)\in[0,\xi s] \tag{4.15}$$

[1]We denote the parton level Mandelstam variables with a hat and the proton level variables without a hat.

Since $1 - \cos\theta \leq 2$ it implies

$$\xi \geq \frac{Q^2}{s} \tag{4.16}$$

The physics of this is that if Q is very small then you impart very little energy to the electron and there won't be a large momentum fraction [Q 10: Check this!].

The cross-section is given by

$$\frac{d\sigma}{dQ^2} = \frac{2\pi\alpha_s^2}{Q^4}\int_{Q^2/s}^{1} d\xi \sum_f Q_f^2 f_f(\xi)\left[1 + \left(1 - \frac{Q^2}{\xi s}\right)^2\right] \tag{4.17}$$

Another useful observable is

$$x = \frac{Q^2}{2P\cdot q} \tag{4.18}$$

We have,

$$p + k = p' + k' \Rightarrow p + q = p' \Rightarrow (p+q)^2 = 0 \tag{4.19}$$

This implies

$$2p\cdot q = -q^2 = Q^2 \tag{4.20}$$

so

$$x = \frac{2p\cdot q}{2P\cdot q} = \xi \tag{4.21}$$

This "new" quantity isn't really new at all but just the momentum fraction.

There is also one more quantity that is used in the literature denoted by

$$y \equiv \frac{2P\cdot q}{s} = \frac{Q^2}{xs} \in [0,1] \tag{4.22}$$

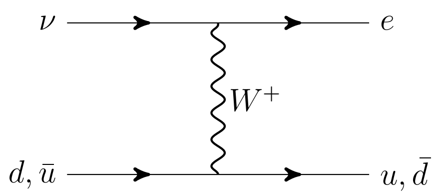
We have,

$$\frac{d^2\sigma}{dx dQ^2} = \frac{2\pi\alpha_s^2}{Q^4}\sum_f Q_f^2 f_f(x)\left[1 + \left(1 - \frac{Q^2}{xs}\right)^2\right] \tag{4.23}$$

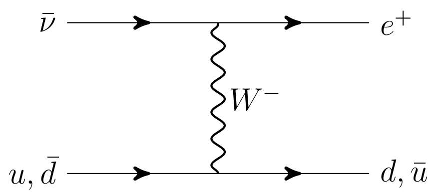
$$\frac{d^2\sigma}{dx dy} = \frac{2\pi\alpha_s^2}{x^2y^2s}\left(\sum_f Q_f^2 f_f(x)\right)\left(1 + (1-y)^2\right) \tag{4.24}$$

The dependence of $1 + (1-y)^2$ is called the Callan-Gross relation and is due to the fermionic nature of quarks. This type of experiment showed that quarks are fermions.

For the other processes one can show that,



$$\frac{d\sigma}{dxdy} = \frac{G_F^2 s}{\pi}\left[xf_d + xf_{\bar{u}}(1-y)^2\right]$$



$$\frac{d\sigma}{dxdy} = \frac{G_F^2 s}{\pi}\left[xf_u(1-y)^2 + xf_{\bar{d}}\right]$$

We have several normalization “sum rules”:

$$\int_0^1 dxx \sum_i f_i(x) = 1 \tag{4.25}$$

and

$$\int_0^1 dx\,[f_u(x) - f_{\bar{u}}(x)] = 2, \quad \int_0^1 dx\,[f_d(x) - f_{\bar{d}}(x)] = 1 \tag{4.26}$$

These are the electric charge sum rules.

This method has its limitations. In particular we can’t go down to arbitrary small x,

$$x_{min} = \frac{Q^2}{s} \tag{4.27}$$

Since $Q^2 \gg \Lambda_{QCD}$ is required to be able to factor short and long distance physics we need high energies to go down and measure pdf’s at small momenta fractions. At HERA for example $s = 320\text{GeV}$ and

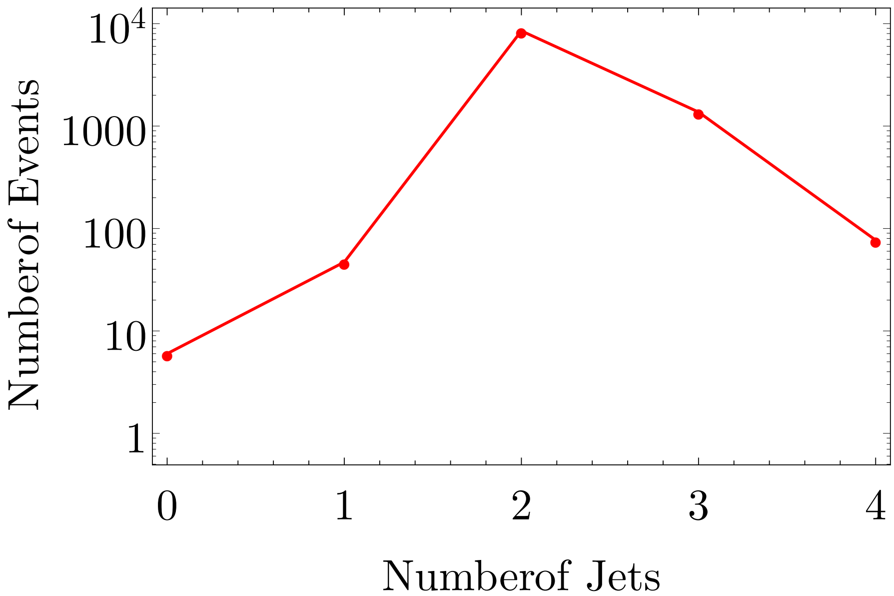
$$x_{min} \sim 10^{-5} - 10^{-4} \tag{4.28}$$

Appendix A

Homework

A.1 Assignment 4

In this assignment we simulate $e^+e^- \to Jets$ with and without Pythia. The simulation is easy enough to run. We work at $\sqrt{s} = M_Z$. We begin by computing the number the number of events with each number of jets. We find:



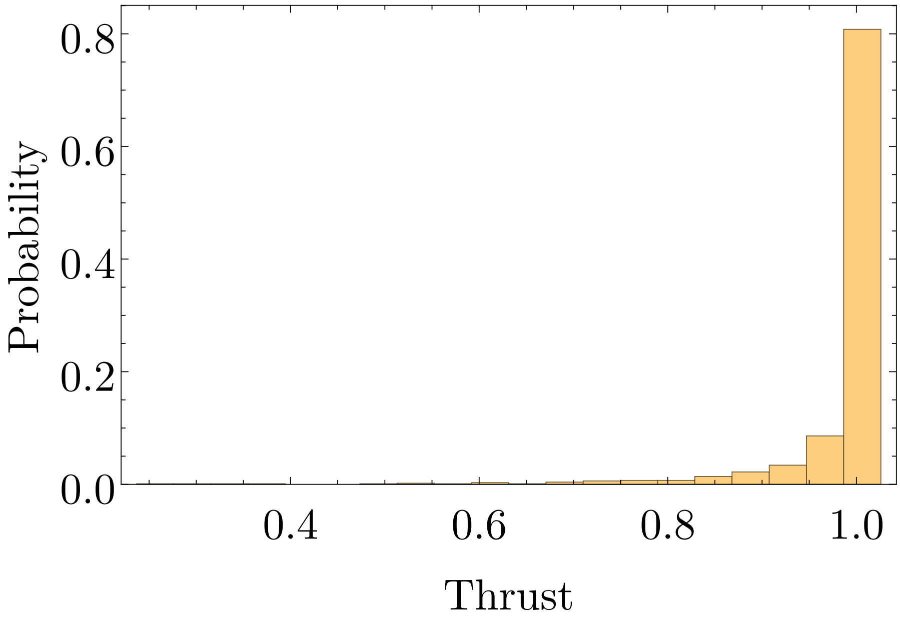
We then go on to measure the thrust for each event. To find the thrust we take the 3 momentum of each event and take its dot product with a radial unit vector,

$$\hat{r} = \{\cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi\} \tag{A.1}$$

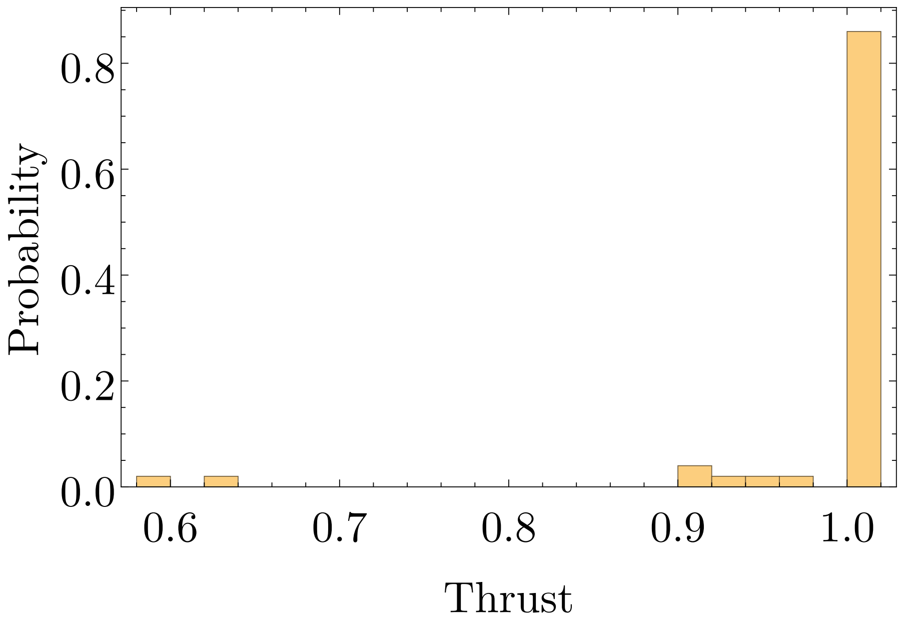
we use Mathematica to maximize

$$T = \frac{\sum_i |\mathbf{p}_i \cdot \hat{r}|}{\sum_i |\mathbf{p}_i|} \tag{A.2}$$

with respect to ϕ, θ. This is a computationally expensive process. Instead of doing this for all events we only do calculate for the first 3000 for Pythia and first 600 for without Pythia. With Pythia we find:



and without:

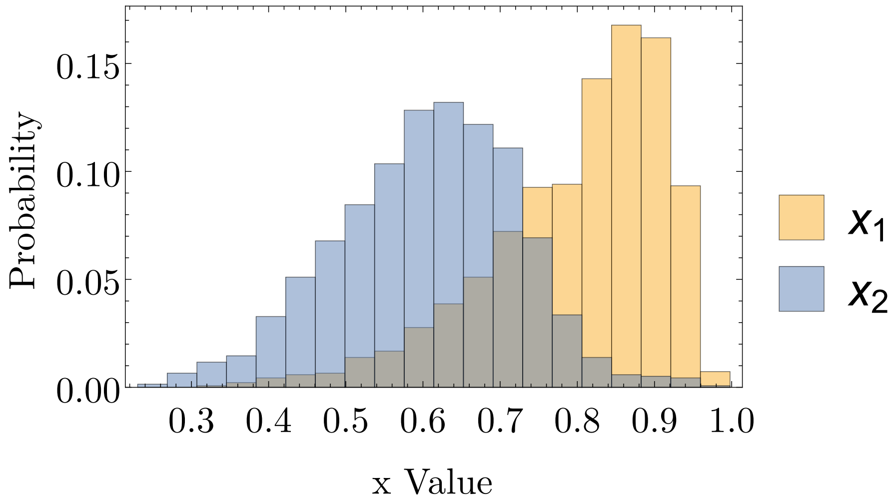


A.1.1 Strange Observations

- Both cases have a similar "lowest thrust" values of about 0.4 (below the allowed minimum for thrust). Possibly this has to do with particles not being detected as they are emitted along the beam line, but I don't quite get how.

A.1.2 x_1 and x_2 Distributions

The x_1 and x_2 distributions in the Pythia 3 jet events is



Bibliography

[1] J. Alwall et al. Madgraph 5: Going beyond. *JHEP*, 1106, 2011.

[2] H. Murayama. Notes on phase space. *hitoshi.berkeley.edu/233B/phasespace.pdf*, 2007.