

Building Planets

Alex Alemi

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You are an interstellar planet constructor. Rich people call up, you build them a planet. Business has been good lately, and you've been getting a lot of new customers by word of mouth. One day, Zorg from the planet Norg calls you up and makes an unusual request. He wants to experience some high gravitational forces. So, he wants you to design a planet whereby you maximize the force due to gravity at some point on the surface. But, he wants you to do it on a budget. You only get so much mass M to work with, and he'll go with the cheapest option, wherein you build the planet out of constant density rock ρ_0 . How do you design such a planet?

As followups: At the point with maximum gravitational acceleration, how does it compare to the same mass spherical planet (assuming they are different shapes)?

Solution to Building Planets

Finding the shape

So, we want to maximize the acceleration due to gravity at some point. To start, let's choose the origin as the point of interest to make things simple. Next, we notice that if we think a bit, we ought to have cylindrical symmetry in the solution. I.e. whatever the planet is, it will be a surface of revolution. This enables us to think in cylindrical coordinates (ϖ, θ, z) , and express our density as dependent on only ϖ, z .

In addition, we know that since we have cylindrical symmetry, we are only interested in the z components of the force due to gravity, as all other components will vanish. So, we begin by computing the acceleration due to gravity at the origin, expressing it as an integral.

$$g = G\rho_0 \int \frac{\cos \theta}{r^2} dV$$

where here, our integral runs over our planet, whatever that shape may be, $\cos \theta$ measures the angle the point of interest makes with the cylindrical axis and r marks the distance from the point of interest to the origin. Using the coordinates we set up, we have

$$d^2 = \varpi^2 + z^2 \quad \cos \theta = \frac{z}{\sqrt{z^2 + \varpi^2}}$$

So we have

$$g = G\rho_0 \int \frac{z}{(z^2 + \varpi^2)^{3/2}} dV$$

Where, again our integral runs over our entire planet.

So, here is where the problem would get complicated if we didn't stop and think for a moment. If we were to blindly proceed, we would have to worry a great deal about the boundaries of integration. However, we are interested in the shape the planet makes. Let's consider what the integrand is telling us here.

What we have done is considered some small chunk of planet located at (r, z) , and the expression

$$\frac{z}{(z^2 + \varpi^2)^{3/2}} \quad (1)$$

tells us the contribution that chunk makes to the acceleration due to gravity along the z axis.

What we are trying to do, is build the planet such that each chunk contributes as much as possible to this integrand. So, if we pause for a second and think about expression (1) as a sort of measure of cost effectiveness, it isn't long before we realize that our planet's surface should correspond to a constant contour of this expression.

In order to make this point more lucid, in figure 1, you see the function described plotted on a contour plot. The darker colors correspond to larger

values of the integrand (ignoring the white in the middle where the computer program breaks down). So if you imagine yourself building the planet, anytime you move a chunk of planet from a lighter color region to a darker one, you increase the gravitational acceleration at the top. So, our planet should have the shape described by these contours, i.e. our planet's surface should correspond to a level curve of Expression 1.

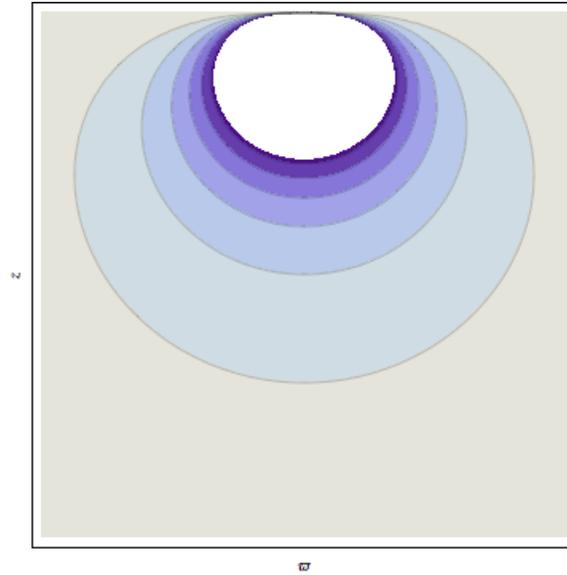


Figure 1: Contour Plot of Expression 1

So, the curve describing the surface of our planet is

$$\frac{z}{(z^2 + w^2)^{3/2}} = C$$

or

$$\boxed{z^2 = C(z^2 + w^2)^3} \quad (2)$$

Which is our answer. Plotted below in figure 2 is a better picture of the shape in question. Remember that the point you are standing is the point at the top.

I tried to find a name for this shape, but couldn't.

Comparing to the Sphere

Now for the bonus. We want to figure out the actual acceleration produced by such a planet and compare it to a sphere. This is a bit trickier. For this we need to actually evaluate the integral:

$$g = G\rho_0 \int \frac{z}{(z^2 + w^2)^{3/2}} dV$$

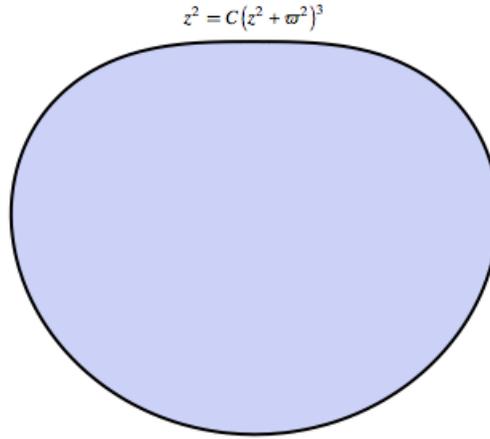


Figure 2: Planet shape with maximal gravitational acceleration at top.

but now know something about the limits of integration since we know the curve creating the surface.

But, we need to ensure that we still fix the mass. For that, we need to evaluate the integral

$$M = \int \rho_0 dV$$

over our volume. Evaluating this in cylindrical coordinates, we have

$$M = \rho_0 \iiint d\varpi \varpi d\theta dz$$

where we need to put in the limits of integration. First off, since we have cylindrical symmetry, we know that we can integrate out θ simply, obtaining

$$M = 2\pi\rho_0 \iint \varpi d\varpi dz$$

Where our limits of integration are determined by ensuring the relation

$$z^2 = \frac{C}{R^4}(z^2 + \varpi^2)^3$$

where I have added the R^4 to ensure that C is dimensionless.

So, let's try and be a bit clever. Looking at the equation for our surface, we have

$$\frac{z}{(z^2 + \varpi^2)^{3/2}} = \frac{C}{R^2}$$

where R will denote the radius of our equivalent sphere and is put in there to make C dimensionless.

It looks a lot like what we get in two dimensions when polar coordinates so lets try the coordinate transformation

$$z = aR \cos \theta \quad \varpi = aR \sin \theta$$

so that our level curves take the form.

$$\frac{\cos \theta}{a^2} = C$$

and we can figure out our integral.

$$\begin{aligned} M &= 2\pi\rho_0 \iiint \varpi \, d\varpi \, dz = 2\pi\rho_0 \iint aR \sin \theta \, aR^2 \, da \, d\theta \\ M &= 2\pi\rho_0 R^3 \iint a^2 \sin \theta \, da \, d\theta = 2\pi\rho_0 R^3 \int_0^{\pi/2} \sin \theta \, d\theta \int_0^{\sqrt{\cos \theta / C}} a^2 \, da \\ M &= 2\pi\rho_0 R^3 \frac{1}{3} C^{-3/2} \int_0^1 d \cos \theta (\cos \theta)^{3/2} \\ M &= \frac{4}{3} \pi \rho_0 R^3 C^{-3/2} \frac{2}{5} \end{aligned}$$

So, using R as the radius of the sphere with equivalent mass, we have

$$\frac{4}{3} \pi \rho_0 R^3 = M = \frac{4}{3} \pi \rho_0 R^3 \left(\frac{1}{5} \right) C^{-3/2}$$

or

$$\left(\frac{1}{5} \right)^{2/3} = C \approx 0.342$$

So we've got that settled. In fact, we can look at the region plot of the area under the level curve and the sphere sharing the same mass, seen below in figure 3.

Next item of business is to calculate the gravitational acceleration at the surface of our new planet. In order to do this, we must compute the integral:

$$g = G\rho_0 \iiint \frac{z}{(z^2 + \varpi^2)^{(3/2)}} \, d\varpi \, \varpi \, d\theta \, dz$$

Integrating out the θ right away and making the same change of coordinates,

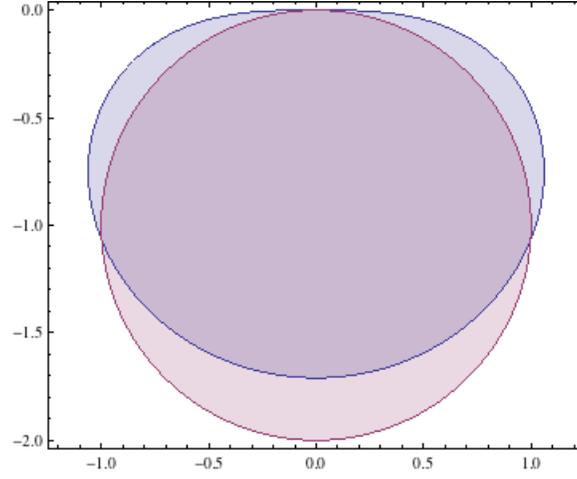


Figure 3: Comparison of our solution and the spherical planet with the same mass.

we obtain

$$\begin{aligned}
 g &= 2\pi G\rho_0 \iint \frac{z}{(z^2 + \varpi^2)^{3/2}} d\varpi \varpi dz \\
 &= 2\pi G\rho_0 \iint \frac{aR \cos \theta}{R^3 a^3} aR \sin \theta aR^2 d\theta da \\
 &= 2\pi G\rho_0 R \int_0^{\pi/2} d\theta \cos \theta \sin \theta \int_0^{\sqrt{\cos \theta/C}} da \\
 &= 2\pi G\rho_0 RC^{-1/2} \int_0^{\pi/2} d\theta \cos \theta \sin \theta \sqrt{\cos \theta} \\
 &= 2\pi G\rho_0 RC^{-1/2} \int_0^1 d \cos \theta (\cos \theta)^{3/2} \\
 &= 2\pi G\rho_0 RC^{-1/2} \frac{2}{5}
 \end{aligned}$$

So, we can compare this to the g due to a spherical planet.

$$g_{\text{sphere}} = \frac{G \frac{4}{3} \pi \rho_0 R^3}{R^2} = \frac{4}{3} G \pi \rho_0 R$$

and we obtain

$$\frac{g}{g_{\text{sphere}}} = \frac{2\pi G\rho_0 R \frac{1}{\sqrt{C}} \frac{2}{5}}{\frac{4}{3} G \pi \rho_0 R} = \frac{3}{5\sqrt{C}} = \frac{3}{5 \left(\frac{1}{5}\right)^{1/3}} = \boxed{\frac{3}{5^{2/3}} \approx 1.026}$$

So, after all that work, in the end of the day, you can only do about 1.03 times better than the sphere if you wanna maximize your gravity. Looks like your client won't get quite the experience he was after.