

# Crack Growth Laws from Symmetry

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## Abstract

Reviewing work done in collaboration with Jennifer Hodgdon, we derive the most general crack growth law allowed by symmetry for mixed-mode three-dimensional fracture. We do so using the system developed in condensed matter physics to derive new laws that emerge on long length and time scales. In our derivation, we provide a symmetry interpretation for the three modes of fracture, we rederive the law giving the crack growth direction in two dimensional fracture, and we derive a growth law appropriate for three dimensional simulations. We briefly discuss related work subsequent to ours, incorporating disorder, dynamics, and other internal variables.

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In this paper, I review work with Jennifer Hodgdon<sup>1</sup> using symmetry principles to derive the growth law for mixed-mode, curved cracks for isotropic materials in three dimensions. In the process, I'll attempt to illustrate the general approach that condensed-matter physicists have developed to derive new phenomenological laws for macroscale physical phenomena.

The physical laws governing the behavior of many physical systems can be derived using symmetry. Basically, there are five steps, which are not necessarily applied sequentially:

*I. Pick an order parameter field.* The order parameter field is a state variable, which at a point  $\mathbf{x}$  summarizes the current state of the material in the local neighborhood of  $\mathbf{x}$ .

*II. Use the symmetries of the problem.*

*III. Imagine writing the most general possible law.*

*IV. Simplify the theory using small parameters, power counting, etc.* Usually one will need to expand in gradients of the order parameter: the theory then describes only behavior at long wavelengths. In dynamical evolution laws, one will naturally also expand in time derivatives, specializing to low frequencies. One often expands in powers of the order parameter.

*V. Solve the theory.* In many cases, this is straightforward, analytically or computationally. In other cases temperature, dirt, or noise introduce fluctuations that remain important on all length scales (such as near continuous phase transitions), and renormalization-group methods may be needed.

Landau introduced this system to derive the forms of free energies and near-equilibrium dynamics of materials with broken symmetries. His methods have been applied extensively to exotic liquid crystals, superconductors and superfluids, magnetic systems, ferroelectrics, incommensurate systems, martensites, and in spatially extended dynamical systems. In many of these systems, the phenomenological Landau theory was effectively used long before a microscopic mechanism or model was developed. More recently, essentially the same procedure has been used to derive evolution equations in systems which are far from equilibrium.

How shall we proceed to apply these ideas to cracks?

## **I. Pick an order parameter field.**

The region of interest in fracture is the immediate vicinity of the crack front: in general, a curved line in space (figure 1). Let us parameterize this curve by arc length  $s$ , so the curve is  $\vec{X}(s)$ . The geometry of the crack edge demands not only the coordinates of this line,

but also the orientation of the crack surface as it approaches the line. We can define a unit tangent vector  $\hat{t} = d\vec{X}/ds$  to the crack edge, the direction of crack growth  $\hat{n}$  (perpendicular to  $\hat{t}$ ), and the normal to the crack plane  $\hat{b} = \hat{t} \times \hat{n}$ . The functions  $\vec{X}(s)$  and  $\hat{n}(s)$  are enough to determine the geometry of the crack; we use them as the order parameter for the problem.

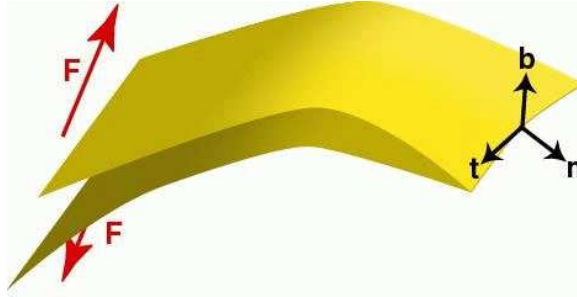


FIG. 1: **Order parameter for crack growth.** The crack is parameterized by arclength  $s$ ; the crack edge is a curve  $\vec{X}(s)$  growing in direction  $\hat{n}(s)$ . The tangent to the crack edge  $\hat{t} = d\vec{X}/ds$  and the normal to the crack plane  $\hat{b} = \hat{t} \times \hat{n}$  are derivable from  $\vec{X}$  and  $\hat{n}$ .

Of course, the crack growth rates will depend strongly on the applied stresses at the crack front. These depend on the loading far away, as well as the shape of the crack surface behind the growing crack front. In special circumstances (nearly straight crack fronts in mode I<sup>2,3</sup>) we can write closed-form expressions for these stresses in terms of integrals over  $\vec{X}(s)$  and  $\hat{n}(s)$ ; more generally these stresses can be calculated from elastic theory, *e.g.* using finite element analysis<sup>4,5</sup>. We thus assume that the stresses as a function of  $s$  are given.

## II. Use the symmetries of the problem.

We assume that our material is isotropic, and will consider only quasi-static fracture. On length scales short compared to the curvatures of the crack and compared to the gradients of the stresses at the crack front, we have two independent discrete symmetries for the local crack geometry: reflection  $R_b$  in the plane of the crack (taking  $\hat{b}$  to  $-\hat{b}$ ) and reflection  $R_t$  in the plane perpendicular to the crack front  $\hat{t}$ . There is a third symmetry, a 180° rotation about the axis  $\hat{n}$ , which is the product of the two other symmetries.

We can use these symmetries first to simplify the characterization of the quasistatic stresses at the crack tip. As for all linear problems with symmetries, we may decompose a general solution of the linear elastic problem into solutions whose displacement (and strain)

fields are odd or even under the two symmetries  $R_b$  and  $R_t$ . For an uncracked material whose strain is constant along  $\hat{t}$ , a general elastic solution can be decomposed into multipoles: for each power  $n$  of the distance  $r$  to the  $\hat{t}$  axis, there are six elastic solutions whose strains vary as  $r^n$ . (For example, there are six solutions with constant strain  $n = 0$ , corresponding to the six independent coefficients of the elastic strain tensor.) There are three solutions which are even in  $R_b$  and  $R_t$ , and one each of the other three possibilities. For the medium with a crack, three of these elastic solutions are not allowed by the condition that there be no traction at the crack surface. Instead, there are three new classes of solutions with strain fields depending on half-integer powers of  $r$ , with non-zero displacements across the crack surface<sup>6,7</sup>.

What are all these elastic solutions? The solutions which involve strains varying as  $r^n$  for  $n > 0$  are large at the boundaries of the sample but vanish quickly at the crack front. They are important for solving for elastic deformations in complex geometries, but are irrelevant for crack growth except when the sample size is comparable to the size of the nonlinear zone at the crack front. The solutions which have  $n \leq -1$  have diverging total energy at the crack tip. These solutions are important for matching the boundaries of the nonlinear zone to the elastic theory, but will die away at distances comparable to the size of the nonlinear zone. Their magnitudes thus are determined by the local geometry and stress at the crack front, and hence can also be ignored. We are left with three ordinary elastic strains  $n = 0$  and three strain fields with crack opening displacements  $n = -1/2$ . These latter fields are the familiar three modes of fracture (table I).

In our earlier work<sup>1</sup>, we ignored the  $n = 0$  elastic stresses near the crack tip. It's likely that their effects are relatively small, since they aren't intensified near the crack tip, and we will ignore them in this review as well. It would be straightforward to incorporate the overall strains into the equations of motion, for a more complete theory.

There is one more symmetry in our problem: an artificial symmetry which is introduced by our theoretical description. It is not always convenient to use arc-length  $s$  to parameterize the crack. In particular, it makes the growth laws nonlocal: even for a stationary portion of the crack, if a far-away region of the crack with smaller  $s$  changes in length the functions  $\vec{X}(s)$  and  $\hat{n}(s)$  will shift sideways. Of course, we can use any parameterization that's convenient. Changing parameterizations in a formal sense is analogous to changing the choice of gauge in electromagnetism<sup>8</sup>. For roughly straight cracks parallel to an axis  $z$ , we would likely use

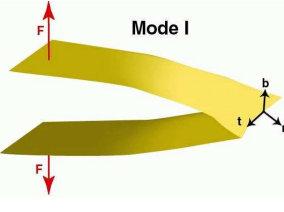


	$R_b$	$R_t$	
Mode I, $K_I$	Even	Even	
Mode II, $K_{II}$	Odd	Even	
Mode III, $K_{III}$	Odd	Odd	
	Even	Odd	No crack opening

TABLE I: Symmetries and Modes of Fracture.

$z$ -gauge with  $\vec{X}(z)$  and  $\hat{n}(z)$ ; for loops we would likely use  $\theta$ -gauge. Changing the equation of motion from one gauge to another is called a *gauge transformation*. There is thus a *gauge symmetry*: the physical evolution of the crack must be unchanged, or *gauge invariant*, when we change from one parameterization to another.

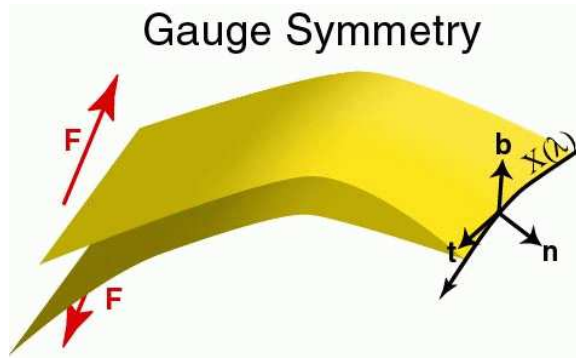


FIG. 2: One must choose a parameterization of the crack edge,  $\lambda$  (choosing a gauge). The form of the equations of motion will change for different choices for the parameterization, but physical quantities like the location of the crack front must be unchanged (gauge invariant).

III. Write the most general possible law.

In two-dimensional fracture, we can immediately write down the most general possible form for the crack velocity and turning rate. The position  $\vec{X}$  of the crack moves forward along  $\hat{n}$  with velocity  $v$  which depends on the stress intensity factors. The velocity does not change sign under  $R_b$  or  $R_t$ , so  $v$  cannot have terms odd in  $K_{II}$  or  $K_{III}$ , giving us the first law

$$\partial\mathbf{X}/\partial t = v(K_I, K_{II}^2, K_{III}^2)\hat{\mathbf{n}}. \quad (1)$$

Similarly, the rate of rotation of the crack surface changes sign under reflection in the plane of the crack  $R_b$ , and doesn't change sign under  $R_t$  so it must have one factor of  $K_{II}$  times an arbitrary function of  $K_I$ ,  $K_{II}^2$ , and  $K_{III}^2$ :

$$\partial\hat{\mathbf{n}}/\partial t = -f(K_I, K_{II}^2, K_{III}^2)K_{II}\hat{\mathbf{b}}. \quad (2)$$

In three dimensions, we must incorporate possible dependences of the equations of motion on the local curvatures of the crack front and the gradients of the stress intensity factors. To linear order in gradients, Hodgdon found

$$\begin{aligned} \partial x/\partial t &= v\hat{n} + w\hat{t} \\ \partial\hat{n}/\partial t &= -\left[\frac{\partial v}{\partial s} + w\frac{\partial\hat{t}}{\partial s} \cdot \hat{n}\right]\hat{t} + \left[-fK_{II} + g_I K_{III}\frac{\partial K_I}{\partial s} + g_{II}K_{II}K_{III}\frac{\partial K_{II}}{\partial s} + g_{III}\frac{\partial K_{III}}{\partial s} + \right. \\ &\quad \left. h_{tb}\frac{\partial\hat{t}}{\partial s} \cdot \hat{b} + h_{nt}K_{II}\frac{\partial\hat{n}}{\partial s} \cdot \hat{t} + (h_{nb}K_{II}K_{III} + w)\frac{\partial\hat{n}}{\partial s} \cdot \hat{b}\right]\hat{b}, \end{aligned} \quad (3)$$

where  $f$ ,  $g_\alpha$ , and  $h_{ij}$  are functions of  $K_I$ ,  $K_{II}^2$ , and  $K_{III}^2$ , and the velocity  $v$  can be a function of these and a number of gradient terms<sup>1</sup>.

#### IV. Simplify the theory using small parameters, power counting, etc.

By focusing on quasi-static fracture, and by assuming that the equations of motion involve only first derivatives in time, we're already making use of the small ratio of the velocity of the crack growth to the natural material velocities (velocity of sound, surface relaxation, etc.) By confining our attention to the external strains that go as  $r^{-1/2}$  and  $r^0$  we made the assumption that the nonlinear zone of the crack is small compared to the system size (so-called K-dominant fracture, since the three stress intensity factors dominate). In three dimensions, by keeping terms to linear order in gradients, we have again assumed the nonlinear zone is small compared to curvatures and changes in stresses.

We can use the small size of the nonlinear zone a third time to establish the relative sizes of different terms in our equations of motion. In our two dimensional equations of motion (1) and (2), the functions  $v$  and  $f$  have different units:  $v/Kf$  has units of length, where  $K$  is any one of the stress intensities. It is natural to assume that this length will be set by the size of the nonlinear zone, and in any case it is part of our approach to assume that this length is small.

## V. Solve the theory.

It is well known that cracks loaded with a mixture of mode I and more II will turn abruptly. In our formulation, the function  $f$  governs the rate at which the crack turns. Hodgdon used results of Cotterell and Rice<sup>2</sup> to show that if the angle of the crack differs from the angle that makes  $K_{II} = 0$  by a small amount  $\Delta\theta$ , then  $K_{II} = K_I\Delta\theta/2$ . Combined with the two growth laws above, we find that  $\Delta\theta \sim \exp(-fK_I x/2v)$ , the crack turns to pure mode I exponentially with a material-dependent decay length of  $2v/fK_I$ : precisely twice the characteristic length scale we assumed was small! Thus we derive the *principle of local symmetry*<sup>9</sup>, which says that a mode II fracture will turn abruptly (that is, on a length comparable to the nonlinear zone size) until it is pure mode I.

There were two other rules in the literature for picking the crack growth direction: one maximizing the energy release<sup>10</sup>, and one moved in the direction of minimum strain energy density<sup>11</sup>. Cracks grown by these different rules (move forward by a small step size  $\Delta x$ , recalculate the stress intensity factors, repeat) all gave rather similar predictions for the shapes. Our analysis did not make any assumptions about microscopic mechanisms, so we can apply it to a hypothetical material which behaved according to these other crack growth rules. We conclude that these other rules will end up forcing the crack to turn until  $K_{II} = 0$ , yielding in all cases the principle of local symmetry! Instead of a nonlinear zone size, the turning radius is given by the step size of the algorithm  $\Delta x$ . Our analysis predicts that *all three growth rules are equivalent* in the limit of small stepsize/turning radius.

## Where to go from here?

Hodgdon's analysis is now a decade old. What came next?

First, Hodgdon used the three-dimensional growth law (equation 3) to investigate the stability of crack growth under mixtures of mode III and mode I. All of the new 3D materials

functions  $g_X$ ,  $h_{yz}$ , and  $w$  in equation (3) multiply gradients, so they aren't unusually large like  $f$ . Hodgdon found that mixed-mode fracture can be stable or unstable depending on details: in particular, for  $g_I > 0$  steady-state mode III cracks are unstable to small perturbations. Mode III fracture is unstable experimentally to the formation of a “factory roof” morphology, with ramps followed by cliffs. With the computers at that time, we couldn't get into the nonlinear regime needed to test whether our theory leads to the same jagged solutions as seen experimentally: Hodgdon's linear stability analysis remains unpublished<sup>7</sup>.

Second, real materials are dirty: not only impurities and inclusions, but also the random fracture strengths introduced by polycrystallinity suggest the incorporation of randomness in the evolution equations. Introducing randomness is also motivated by the experimental observation of Bouchaud<sup>13</sup>, who finds roughness on all scales, with power-law height-height correlation functions. Ramanathan and Fisher<sup>3</sup> (see also<sup>12</sup>) studied the effects of incorporating disorder into a dynamics similar to that described here. They discovered that it does indeed predict a fracture surface which is rough on all length scales, but with much weaker randomness than that observed experimentally (logarithmic rather than power law). It seems likely that the experimental roughness, at least in ductile and intergranular fracture, reflects void growth and the presence of grains and inclusions<sup>14</sup>.

Thirdly, many effects of dynamics on crack growth have been ignored in our discussion. The theory as described here is appropriate, perhaps, for fatigue crack growth where mass and inertia are not important. New phenomena arise as the crack speeds up (such as the mirror-mist-hackle morphology transitions<sup>15</sup>). Incorporating the inertial dynamics of the growing front does lead to interesting traveling wave solutions<sup>16</sup>.

Finally, there is the likely possibility that there are other important degrees of freedom that are important, and need to be incorporated into the order parameter describing the current state of the crack tip. There are recent indications that the curvature of the crack tip<sup>17</sup> and the effects of surface tension and diffusion<sup>18,19</sup> may be important.

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