Uncertainty, Statistics...

I thought I was interested in uncertainty but now I'm not so sure.

...and why you should care about uncertainty.
What is the difference between Accuracy and Precision?

What is the difference between Error and Uncertainty?
Semantics

Accuracy vs. Precision

Accuracy is the degree to which a measurement agrees with the true value.

Precision is the repeatability of the measurement.

Error vs. Uncertainty

Error is the degree to which a Measurement agrees with the true value.

Uncertainty is an interval around the measurement in which repeated measurements will fall.
Errors, Uncertainty, and inconsistent Language

• **Error** is almost never what we are interested in. In science we typically **do not know the ‘true’ value**.

• Rather we are interested in the **measurement uncertainty**. This is what we need to quantify in any measurement.

• We are often very sloppy and inconsistent in our language and call what is actually an uncertainty an error.

• When we talk about measurement uncertainty,
  • It is not a blunder
  • It is not an accident
  • It is not due to incorrectly handling the equipment
  • **It is not the difference to an accepted value found in the literature**
Uncertainty vs. Error

Example 1:
High fiber diets: A study in 1970 claimed that a high fiber diet reduces polyps forming in the colon, being precursors of cancer. A study in 2000 with more analyzed individuals showed no such effect. The uncertainty in the first study was too large and not properly accounted for. This left people eating lots of fibers for 30 years…

Example 2:
A study in the late 60s found large levels of iron, which is required for red blood cell production, in spinach. Popular comics tried to promote spinach consumption. A study in the 90s showed that the original measurement had a reading error in the decimal point. The iron levels are a factor of 10 lower than claimed. The incorrect reading of the decimal was an error, not due to an uncertainty in the measurement. This left children eating lots of spinach for 30 years…
Why do we quote Measurement Uncertainties?

Let’s assume someone measures the gravitational constant $G$ to be (lab G8)...

(a) $G_N = 6.90 \cdot 10^{-11} \text{ m}^3\text{kg}^{-1}\text{s}^{-2}$. The accepted value is $G_N = (6.6742 \pm 0.0010) \cdot 10^{-11} \text{ m}^3\text{kg}^{-1}\text{s}^{-2}$. What does this result mean?

(b) $G_N = (6.90 \pm 0.25) \cdot 10^{-11} \text{ m}^3\text{kg}^{-1}\text{s}^{-2}$ What does this result mean?

(c) $G_N = (6.90 \pm 0.01) \cdot 10^{-11} \text{ m}^3\text{kg}^{-1}\text{s}^{-2}$ What does this result mean?
Meaning of Measurement Uncertainty

- The **Uncertainty** describes the distance from your measurement result within which your setup has determined that the true value is likely to lie.
- It describes therefore a **property of your measurement procedure**.
- A new measurement following the same procedure will lead to a different measurement result, but usually the same uncertainty.
- Building a good experiment means building an experiment with relatively small uncertainty.
- **Warning**: Let's say you measure $N_A$ in experiment G10 and you obtain the literature value to $10^{-2}$. This does not show that your setup has established that the true value of $N_A$ is likely to be within $10^{-2}$ of your result. The $10^{-2}$ is therefore **not the uncertainty of your experiment**.
Meaning of Measurement Uncertainty

- If we measure for example a voltage $V = 10.2 \pm 0.3$ V, what does this mean?

- In general there are differences in different science disciplines.

- In physics, a 1-sigma uncertainty is generally used. If the measurements are normally distributed (Gaussian), this corresponds to a 68% confidence level (CL) interval. Or that 32% of the time the true value would be outside the quoted uncertainty range.

$\Rightarrow$ Sometimes the true value will be somewhat outside of your quotes uncertainty range!! That’s OK…
Where do uncertainties come from?
(Are there different types?)
Different Types of Uncertainties

Random/Statistical (All)

Always present in a measurement. It is caused by inherently unpredictable fluctuations in the readings of a measurement apparatus or in the experimenter's interpretation of the instrumental reading. Originates in the Poisson distribution.

Systematic (All)

From imperfect calibration of measurement instruments, or imperfect methods of observation, or interference of the environment with the measurement process. Always affect the results of an experiment in a predictable direction.

- e.g. zero setting error in which the instrument does not read zero when the quantity to be measured is zero

Theory (e.g. N15, N17)

In these experiments you measure the muon lifetime. But there are corrections to the capture rate for muon that come from theory.
Different Types of Uncertainties

It is common to quote these uncertainties separately:
\[ \tau_\mu = (2.19 +/- 0.05_{\text{stat.}} +/- 0.01_{\text{syst.}} +/- 0.02_{\text{th.}}) \mu s \]

Different notations are used for uncertainties, e.g.
\[ \tau_\mu = (2.19(5)_{\text{stat.}} +/- (1)_{\text{syst.}} +/- (2)_{\text{th.}}) \mu s \]
In the following...

• Measurement uncertainties and the value of increasing the number of measurements

• Uncertainty (Error) Propagation

• Counting statistics (labs N2, N4, N5, N17)

• Least square fitting
Measurements

• Let’s assume you measure some given quantity several times.
  – What is the best estimate for the true value?
  – What is the estimate of the precision of a single measurement?
  – What is the uncertainty of combined best estimate value?
Example: What is the width of this room?

1) Guess what the precision of a single measurement might be.

2) Measure the room width several times.

3) What is the standard deviation of your results? Does it agree with your initial estimate?

4) Based on your results, what is the most likely value for the room width?

5) What is your estimate for the uncertainty of this value?

6) Add your final result to the table on the blackboard.
Heights of 928 People (Galton, 1886)
Sample Mean, Standard Deviation

- **Sample mean**: most likely value for true average height (N=sample size=928)
  \[ \bar{x} = \frac{\sum_{i} x_i}{N} = 68.2" \]

- **Standard deviation**: measure of how close to the average value a given measurement is likely to fall = uncertainty of single measurement result
  \[ S \equiv \left[ \frac{\sum_{i} (x_i - \bar{x})^2}{(N-1)} \right]^{1/2} = 2.5" \]
Normal (Gaussian) Distribution of Measurements

The normal distribution often describes the actual distribution of the measurement results reasonably well.

\[ F(x - \bar{x}) = \left[ \frac{1}{S \sqrt{2\pi}} \right] \exp \left[ -\frac{(x - \bar{x})^2}{2S^2} \right] \]
Normal Distribution and Probabilities

- $F(x - \bar{x})$ is a probability distribution

$$\int_{x_1}^{x_2} F(x - \bar{x})dx = \text{probability of measuring value between } x_1 \text{ and } x_2$$

$$\int_{-\infty}^{+\infty} F(x - \bar{x})dx = 1$$

- Probability that measurement gives result within $\pm \sigma$ range of average value: 68%

$$\int_{\bar{x}-S}^{\bar{x}+S} F(x - \bar{x})dx = 0.68$$
<table>
<thead>
<tr>
<th>inside probability</th>
<th>1 -probability</th>
<th>outside 1 in ...</th>
</tr>
</thead>
<tbody>
<tr>
<td>±1σ</td>
<td>0.6827</td>
<td>0.3173</td>
</tr>
<tr>
<td>±2σ</td>
<td>0.9545</td>
<td>0.0455</td>
</tr>
<tr>
<td>±3σ</td>
<td>0.9973</td>
<td>0.0027</td>
</tr>
<tr>
<td>±4σ</td>
<td>0.99993666</td>
<td>0.00006334</td>
</tr>
<tr>
<td>±5σ</td>
<td>0.9999994267</td>
<td>0.00000005733</td>
</tr>
<tr>
<td>±6σ</td>
<td>0.999999998027</td>
<td>0.000000001973</td>
</tr>
</tbody>
</table>
Uncertainty of Mean Value

- **Average height:** \( \overline{x} = \frac{\sum x_i}{N} = 68.2" \)

- **Standard deviation of the mean:** tells you how well the average value has been determined
  \[ \Delta \overline{x} = \pm \frac{S_x}{\sqrt{N}} = 0.08" \]

- **Uncertainty of average value** = uncertainty of single measurement / square root of total number of measurement

- **Notice value of increasing the number of measurements!!**

- **Proof:** use “error” propagation for \( \overline{x} \)
Another Example

Suppose that you have 121 voltage measurements with a standard deviation of 0.20 volts that yield a mean value of 43.204 volts. What is the uncertainty in your mean value?

The uncertainty in the mean value is approximately (for large N) equal to the uncertainty in a single measurement divided by \( \sqrt{N} \) where N is the total number of independent measurements:

\[
\frac{0.20}{\sqrt{121}} = 0.018, \text{ round to 0.02}
\]

Your result is therefore \( 43.20 = 43.20 \pm 0.02 \) volts.

Note that the number of significant digits (to the right of the decimal) in the reported mean is equal to the number of significant digits in the standard error!
A third Example

Measurements performed with a thermistor in a noisy environment yield the following values for temperature (K):

304.3, 302.1, 310.6, 308.5, 312.6, 306.3 and 305.7.

(a) What is the most likely value of the true temperature?
The mean temperature is the most likely value of the true temperature. T(mean) = 307.2 K

(b) What is your estimate of the precision of this value?
The standard deviation from the mean is:

\[ \sqrt{\left(\frac{(-2.9)^2 + (-5.1)^2 + (3.4)^2 + (1.3)^2 + (5.4)^2 + (-0.9)^2 + (1.5)^2}{7-1}\right)} = \frac{3.6}{\sqrt{7}} = 1.4 \text{ K} \]

(c) If the number of measurements were to be increased to 100, what is your estimate of the precision of the mean compared with your answer in (b)?
The uncertainty in the mean temperature will decrease by a factor \( \sqrt{\frac{7}{100}} = 0.26 \).
Suppose you measure $a=5.0\pm0.4\ m$ and $b=3.0\pm0.3\ m$ and want to calculate $f = a - b$.

What is the uncertainty in the calculated result $f = a - b$?

$$\sigma_f^2 = \sigma_a^2 + \sigma_b^2$$

$$\sigma_f = \sqrt{\sigma_a^2 + \sigma_b^2} = \sqrt{(0.4m)^2 + (0.3m)^2} = 0.5m$$
**Uncertainty (error) Propagation**

- Suppose you wish to calculate a quantity \( f(a,b,c) \) using independently measured quantities \( a, b, c \)... 

- If we know the uncertainties \( \sigma_a, \sigma_b, \sigma_c \) in these experimentally determined quantities, the uncertainty in the calculated result \( \sigma_f \) is given by the fundamental equation of uncertainty propagation:

\[
\sigma_f^2 = [\partial_a f(\bar{a}, \bar{b}, \bar{c})\sigma_a]^2 + [\partial_b f(\bar{a}, \bar{b}, \bar{c})\sigma_b]^2 + [\partial_c f(\bar{a}, \bar{b}, \bar{c})\sigma_c]^2
\]

- Assumes that measurements of \( a, b, c \)... are uncorrelated.
- Assumes that \( \sigma_a / a, \sigma_b / b, \sigma_c / c \)... are small (<~0.1)
Uncertainty (error) Propagation

Proof: \( f = f(a,b) \)

\[
\bar{a} = \langle a \rangle, \quad \delta a = a - \bar{a}, \quad \sigma_a^2 = \langle \delta a^2 \rangle \\
\bar{b} = \langle b \rangle, \quad \delta b = b - \bar{b}, \quad \sigma_b^2 = \langle \delta b^2 \rangle \\
f = f(a,b) \approx f(\bar{a},\bar{b}) + \partial_a f(\bar{a},\bar{b}) \cdot \delta a + \partial_b f(\bar{a},\bar{b}) \cdot \delta b \\
\bar{f} = \left\langle f(\bar{a},\bar{b}) + \partial_a f(\bar{a},\bar{b}) \cdot \delta a + \partial_b f(\bar{a},\bar{b}) \cdot \delta b \right\rangle = f(\bar{a},\bar{b}) \\
\sigma_f^2 = \langle \delta f^2 \rangle \approx \left\langle \left( \partial_a f(\bar{a},\bar{b}) \cdot \delta a + \partial_b f(\bar{a},\bar{b}) \cdot \delta b \right)^2 \right\rangle \\
= \left[ \partial_a f(\bar{a},\bar{b}) \sigma_a \right]^2 + \left[ \partial_b f(\bar{a},\bar{b}) \sigma_b \right]^2 + \partial_a f(\bar{a},\bar{b}) \partial_b f(\bar{a},\bar{b}) \langle \delta a \delta b \rangle \\
\]

Assumption of uncorrelated errors: \( \langle \delta a \delta b \rangle = \langle \delta b \rangle \langle \delta a \rangle = 0 \)
Errors in variable a vary independently of those in variable b.

\[
\sigma_f^2 = \left[ \partial_a f(\bar{a},\bar{b}) \sigma_a \right]^2 + \left[ \partial_b f(\bar{a},\bar{b}) \sigma_b \right]^2 
\]
Example: Uncertainty of the Mean Value

\[ \bar{x} = \left( \frac{1}{N} \right) \left( x_1 + x_2 + x_3 + \ldots + x_N \right) \]

\[ \partial \bar{x}/x_1 = \partial \bar{x}/x_2 = \ldots = \partial \bar{x}/x_N = 1/N \]

Use uncertainty propagation equation:

\[ (\sigma_{\bar{x}})^2 = \left[ \left( \frac{1}{N} \right) \sigma_{x_1} \right]^2 + \left[ \left( \frac{1}{N} \right) \sigma_{x_2} \right]^2 + \ldots + \left[ \left( \frac{1}{N} \right) \sigma_{x_3} \right]^2 = N \left[ \left( \frac{1}{N} \right) \sigma_x \right]^2 = \left( \frac{1}{N} \right) \left( \sigma_x \right)^2 \]

\[ \Rightarrow \quad \sigma_{\bar{x}} = \frac{\sigma_x}{\sqrt{N}} \]
Counting Statistics (Labs N2, N4, N5, N17)

- Imagine a situation where a number of events are recorded for a fixed period of time.
- Assume that the recorded number of events is independent of when the period of time begins; only the length of the period is fixed.
- Example: In N2, a counting experiment is repeated 10 times, which of the 3 outcomes below would you expect?

<table>
<thead>
<tr>
<th>Outcome</th>
<th>Mean</th>
<th>RMS</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a)</td>
<td>99.8</td>
<td>1.0</td>
</tr>
<tr>
<td>(b)</td>
<td>99.2</td>
<td>10.4</td>
</tr>
<tr>
<td>(c)</td>
<td>101.8</td>
<td>31.4</td>
</tr>
</tbody>
</table>
The Poisson probability distribution expresses the likelihood that a given number of events will be recorded during a sampling period of given length:

- For large \( <N> \), the Poisson distribution approaches a normal distribution.
Poisson Distribution

\[ P(N, \mu) = \frac{\mu^N}{N!} e^{-\mu} \]  
Probability that exactly \( N \) events occur when the mean number of events is \( \mu \)

\[ \sum_{N=0}^{\infty} P(N, \mu) = 1 \]
\[ \langle N \rangle = \sum_{N=0}^{\infty} N \; P(N, \mu) = \sum_{N=1}^{\infty} N \frac{\mu^N}{N!} e^{-\mu} = \mu \sum_{N=1}^{\infty} \frac{\mu^{N-1}}{(N-1)!} e^{-\mu} = \mu \]

\[ \text{rms}^2 = \langle (N - \langle N \rangle)^2 \rangle = \langle N^2 - 2\langle N \rangle^2 \rangle = \langle N^2 \rangle - \langle N \rangle^2 \]

\[ \langle N^2 \rangle = \sum_{N=0}^{\infty} N^2 \; P(N, \mu) = \sum_{N=1}^{\infty} N^2 \frac{\mu^N}{N!} e^{-\mu} = \mu \sum_{N=1}^{\infty} \frac{\mu^{N-1}}{(N-1)!} e^{-\mu} \]

\[ = \mu \sum_{N=1}^{\infty} [(N-1) \frac{\mu^{N-1}}{(N-1)!} e^{-\mu} + \frac{\mu^{N-1}}{(N-1)!} e^{-\mu}] = \mu(\mu + 1) \]

\[ \text{rms}^2 = \mu \quad \Rightarrow \quad \text{rms} = \sqrt{\mu} \quad \Rightarrow \quad \frac{\text{rms}}{\langle N \rangle} = \frac{1}{\sqrt{\langle N \rangle}} \]

E.g. for 99 counts, we assign the uncertainty of \( 99^{1/2} = 9.9 \) to say that the measurement leads to 99+/−10 counts.
Uncertainty in Surveys Results, ...

Uncertainty $\sigma = \sqrt{\text{total sample size} \times \text{fraction} \times (1 - \text{fraction})}$

\[
\frac{\text{uncertainty } \sigma}{\text{fraction} \times \text{sample size}} = \sqrt{\frac{(1 - \text{fraction})}{\text{fraction} \times \text{sample size}}}
\]

- Example: “Only 27 percent are happy with his job on the economy, ... That was worse than his previous low of 29 percent approval for handling the economy set in February, and down 4 percentage points from last month”. **What’s wrong with this statement?**
Linear Least Square Fitting

- Suppose we have made measurements of a quantity $y_i$ for known values of a parameter $x_i$ (optical rotation vs sugar concentration, for example, as in O-11). The data suggest a linear relationship between $y_i$ and $x_i$.

- We are therefore led to seek the “best fit” of our data to the linear relationship $y = A + Bx$.

- For the best fit, we must determine the values $A$ and $B$ that minimize the sum of the squared residuals for our measurements $y_i$ (here the residuals are defined as the differences between the measured values $y_i$ and the corresponding “$y$” values given by the best fit. We will call this sum the "sum of squares" given by the expression

$$SS = \sum_i (y_i - A - Bx_i)^2$$
\[ SS = \Sigma_i (y_i - A - Bx_i)^2 \]

To minimize this sum of squared residuals, we take derivatives with respect to \( A \) and \( B \) and set them equal to zero:

\[
\frac{\partial SS}{\partial A} = 0 = \Sigma_i 2(A + Bx_i - y_i)
\]

\[
\frac{\partial SS}{\partial B} = 0 = \Sigma_i 2x_i(A + Bx_i - y_i)
\]

Then solving for \( A \) and \( B \) we find that for a minimum least squares fit:

\[
A = \frac{1}{D}\left[\Sigma(x_i)^2\Sigma y_i - \Sigma x_i y_i \Sigma x_i\right]
\]

\[
B = \frac{1}{D}\left[N\Sigma x_i y_i - \Sigma x_i \Sigma y_i\right]
\]

where the determinant of the 2x2 matrix

\[
\begin{vmatrix}
S & S_x \\
S_x & S_{xx}
\end{vmatrix}
\]

is

\[
D = N\Sigma(x_i)^2 - (\Sigma x_i)^2
\]
## Curve Fitting with Matlab

### Linear Curve Fitting:
**MATLAB Built-in Functions**

<table>
<thead>
<tr>
<th>Function</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>POLYFIT</td>
<td>Fit polynomial to data. POLYFIT(X,Y,N) finds the coefficients of a polynomial P(X) of degree N that fits the data, P(X(I))≈Y(I), in a least-squares sense.</td>
</tr>
<tr>
<td>\</td>
<td>Backslash or matrix left division. If A is a square matrix, A\B is roughly the same as inv(A)*B, except it is computed in a different way.</td>
</tr>
<tr>
<td>POLYVAL</td>
<td>Evaluates a polynomial at given points</td>
</tr>
<tr>
<td>CORRcoef</td>
<td>Compute the correlation of two vectors. This can be used with the POLYFIT and POLYVAL functions to compute the R-Square correlation coefficient between actual data and the output of a fitted curve.</td>
</tr>
</tbody>
</table>

### Nonlinear Curve Fitting:
**MATLAB Built-in Functions**

<table>
<thead>
<tr>
<th>Function</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>FMINBND</td>
<td>Scalar bounded nonlinear function minimization. This function can be used to minimize a function of only one variable on a fixed interval.</td>
</tr>
<tr>
<td>FMINSEARCH</td>
<td>Multidimensional unconstrained nonlinear minimization (Nelder-Mead method). This function finds the minimum of a scalar function of several variables, starting at an initial estimate. This is generally referred to as unconstrained nonlinear optimization.</td>
</tr>
</tbody>
</table>

- Curve fitting tool: cftool
More Information

• In the 510 lab:
• J. Orear
• L. Hand

• Others:
  • http://dcaps.library.cornell.edu/etitles/Frodesen/probabilitystatisticsparticlephysics.pdf